

Models of Set Theory II

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Abstract

Martin's Axiom and applications, iterated forcing, forcing Martin's axiom

1 Introduction

The method of *forcing* allows to construct models of set theory with interesting or exotic properties. Further results can be obtained by *transfinite iterations* of this technique. More precisely, iterated forcing defines generic extensions, which can be analyzed by an increasing well-ordered tower of intermediate models where successor models are generic extensions of the previous models. Such an analysis is already possible for the Cohen model for $2^{\aleph_0} = \aleph_2$, and we shall indicate some aspects in an introductory chapter. In that model, partially generic filters exist for the standard Cohen forcing $\text{Fn}(\aleph_0, 2, \aleph_0)$. This motivates *forcing axioms* which require the existence of partially generic filters for certain forcings. *Martin's Axiom* MA is a forcing axiom for forcings satisfying the countable antichain condition (ccc). We shall study some consequences of MA and shall then force that axiom by iterated forcing. We shall also study the *Proper Forcing Axiom* PFA for a class of forcings which are *proper*.

Many forcing constructions are mostly directed towards properties of the set \mathbb{R} of real numbers. There are several forcings which adjoin new reals to (ground) models. Different forcings adjoin reals which may be very different with respect to growth behaviour and other aspects. Cardinal characteristics of \mathbb{R} have been introduced to describe such behaviours. They are systematised in CICHON's diagram. Using MA and iterated forcings several constellations of cardinals are realized in CICHON's diagram.

2 Cohen forcing

The most basic forcing construction is the adjunction of a Cohen generic real c to a countable transitive ground model M . The generic extension $M[c]$ is again a countable transitive model of ZFC and it contains the "new" real $c \notin M$. In the previous semester we saw that the adjunction of c has consequences for the set theory within $M[c]$:

Theorem 1. *In the COHEN extension $M[c]$ the set $\mathbb{R} \cap M$ of ground model reals has (Lebesgue) measure zero.*

This implies some (relative) consistency results. We may, e.g., assume that M is a model of the axiom of constructibility $V = L$, i.e., $M = L^M$. Since the class L is absolute between transitive models of set theory of the same ordinal height, $L^{M[c]} = L^M = M$. So:

Theorem 2. *Let M be a ground model of $ZFC + V = L$. Then the COHEN extension $M[c]$ satisfies: the set*

$$\{x \in \mathbb{R} \mid x \in L\}$$

of constructible reals has measure zero.

On the other hand, inside a given model of set theory, the set of reals has positive measure, i.e., does not have measure zero.

Exercise 1. Show that the measure zero sets form a proper ideal on \mathbb{R} which is closed under countable unions.

Exercise 2. Show that the following *Cantor set* of reals has cardinality 2^{\aleph_0} and measure zero:

$$C = \{x \in \mathbb{R} \mid \forall n < \omega x(2n) = x(2n+1)\}.$$

So in the model L the set of constructible reals does not have measure zero:

Theorem 3. *The statement “the set of constructible reals has measure zero” is independent of the axioms of ZFC.*

The set of constructible reals in $M[c]$ can be a set of size \aleph_1 that has measure zero. This leads to the question whether it is (relatively) consistent that *all* sets of reals of size \aleph_1 have measure zero. Of course this necessitates $2^{\aleph_0} > \aleph_1$. It is natural to ask the question about Cohen’s canonical model for $2^{\aleph_0} > \aleph_1$.

Consider adjoining λ COHEN reals to a ground model M where $\lambda = \aleph_2^M$. Define λ -fold COHEN forcing $P = (P, \leq, 1) \in M$ by $P = \text{Fn}(\lambda \times \omega, 2, \aleph_0)$, $\leq = \supseteq$, and $1 = \emptyset$. Let G be M -generic on P . Let $F = \bigcup G: \lambda \times \omega \rightarrow 2$ and extract a sequence $(c_\beta \mid \alpha < \lambda)$ of Cohen reals $c_\beta: \omega \rightarrow 2$ from F by:

$$c_\beta(n) = F(\beta, n).$$

Then the generic extension is generated by the sequence of Cohen reals:

$$M[G] = M[(c_\beta \mid \beta < \lambda)].$$

It is natural to construe $M[G]$ as a limit of the models $M[(c_\beta \mid \beta < \alpha)]$ when α goes towards λ : Fix $\alpha \leq \lambda$. Let $P_\alpha = \text{Fn}(\alpha \times \omega, 2, \aleph_0)$ and $R_\alpha = \text{Fn}((\lambda \setminus \alpha) \times \omega, 2, \aleph_0)$, partially ordered by reverse inclusion. The isomorphisms

$$P \cong P_\alpha \times R_\alpha \text{ and } P_{\alpha+1} \cong P_\alpha \times Q$$

imply that $G_\alpha = G \cap P_\alpha$ is M -generic on P_α and that

$$H_\alpha = \{q \in Q \mid \{((\alpha, n), i) \mid (n, i) \in q\} \in G_{\alpha+1}\}$$

is $M[G_\alpha]$ -generic on Q . Let $M_\alpha = M[G_\alpha]$ be the α -th model in this construction. Then

$$M_{\alpha+1} = M[G_{\alpha+1}] = M[G_\alpha][H_\alpha] = M_\alpha[H_\alpha].$$

It is straightforward to check that $c_\alpha = \bigcup H_\alpha$. So the model $M[G] = M_\lambda$ is obtained by a sequence of models $(M_\alpha \mid \alpha \leq \lambda)$ where each successor step is a Cohen extension of the previous step. The whole construction is held together by the “long” generic set G which dictates the sequence of the construction and also the behaviour at limit stages.

Consider a real $x \in M[G]$. Identifying characteristic functions with sets we can view x as a subset of ω . In the previous course we had seen that there is a name $\dot{x} \in M$, $\dot{x}^G = x$ of the form

$$\dot{x} = \{(\check{n}, q) \mid n < \omega \wedge q \in A_n\},$$

where every A_n is an antichain in P . Since P satisfies the countable chain condition, there is $\alpha < \lambda$ such that $A_n \subseteq P_\alpha$ for every $n < \omega$. Then

$$x = \dot{x}^G = \dot{x}^{(G \cap P_\alpha)} = \dot{x}^{G_\alpha} \in M[G_\alpha]$$

In $M[G]$ consider a set $B = \{x_i \mid i < \aleph_1\}$ of reals of size \aleph_1 . One can view B as a subset of \aleph_1^M . As in the above argument, there is an $\alpha < \lambda$ such that $B \in M_\alpha$. By our previous Lemma, $B \subseteq \mathbb{R} \cap M_\alpha$ has measure zero in the Cohen generic extension $M[c_\alpha]$. So B has measure zero in $M[G]$. The model $M[G]$ establishes:

Theorem 4. *If ZFC is consistent then ZFC + “every set of reals of size $\leq \aleph_1$ has Lebesgue measure zero” is consistent.*

Together with models of the Continuum Hypothesis this shows that the statement “every set of reals of size $\leq \aleph_1$ has Lebesgue measure zero” is independent of the axioms of ZFC.

One can ask for further properties of Lebesgue measure in connection with the uncountable. Is it consistent that every union of an \aleph_1 -sequence of measure zero sets has again measure zero?

Exercise 3.

- a) Show that in the model $M[G] = M[(c_\beta \mid \beta < \lambda)]$ there is an \aleph_1 -sequence of measure zero sets whose union is \mathbb{R} .
- b) Show that $\{c_\beta \mid \beta < \lambda\}$ has measure zero in $M[G]$.

Exercise 4. Define forcing with sets of reals of *positive measure* (i.e., sets which do not have measure zero).

We shall later construct forcing extensions $M[G]$ which are obtained by iterations of forcing notions similar to the above example. We shall require that in the iteration $M_{\alpha+1}$ is a generic extension of M_α by some forcing $Q_\alpha \in M_\alpha = M[G_\alpha]$; the forcing is in general only given by a *name* $\dot{Q}_\alpha \in M$ such that $Q_\alpha = \dot{Q}_\alpha^{G_\alpha}$. To ensure that this is always a partial order we also require that $1_{P_\alpha} \Vdash \dot{Q}_\alpha$ is a partial order. Technical details will be given later.

A principal idea is to let \dot{Q}_α to be some canonical name for a partial order forcing a certain property to hold, like making the set of reals constructed so far a measure zero set. A central concern for such iterations, like for many forcings, is the preservation of cardinals.

3 Forcing axioms

The argument that the set $\mathbb{R} \cap M$ of ground model reals has measure zero in the standard Cohen extension $M[H] = M[c]$ by the Cohen partial order Q rests, like most forcing arguments, on density considerations. For a given $\varepsilon = 2^{-i}$, a sequence I_0, I_1, I_2, \dots of real intervals such that $\sum_{n < \omega} \text{length}(I_n) \leq \varepsilon$ is extracted from the Cohen real c . It remains to show that $X \subseteq \bigcup_{n < \omega} I_n$. For $x \in \mathbb{R} \cap M$ a dense set D_x is defined so that $H \cap D_x \neq \emptyset$ implies that $x \in \bigcup_{n < \omega} I_n$. To cover the real x requires a “partially generic filter” which intersects D_x . This approach is captured by the following definition:

Definition 5. Let $(Q, \leq, 1_Q)$ be a forcing, \mathcal{D} be any set, and κ a cardinal.

- a) $H \subseteq Q$ is a filter on Q if $\forall p \in H \forall q \in Q (p \leq q \rightarrow q \in H)$ and $\forall p, q \in H \exists r \in H (r \leq p \wedge r \leq q)$.
- b) A filter H on Q is \mathcal{D} -generic if $D \cap H \neq \emptyset$ for every $D \in \mathcal{D}$ which is dense in Q .
- c) The forcing axiom $\text{FA}_\kappa(Q)$ postulates that there exists a \mathcal{D} -generic filter on Q for any \mathcal{D} of cardinality $\leq \kappa$.

For any countable \mathcal{D} we obtain the existence of generic filters just like in the case of ground models.

Theorem 6. (Rasiowa-Sikorski) $\text{FA}_{\aleph_0}(Q)$ holds for any partial order Q .

Proof. Let \mathcal{D} be countable. Take an enumeration $(D_n | n < \omega)$ of all $D \in \mathcal{D}$ which are dense in Q . Define an ω -sequence $q = q_0 \geq q_1 \geq q_2 \geq \dots$ recursively, using the axiom of choice:

$$\text{choose } q_{n+1} \text{ such that } q_{n+1} \leq q_n \text{ and } q_{n+1} \in D_n.$$

Then $H = \{q \in Q | \exists n < \omega q_n \leq q\}$ is as desired. □

Exercise 5. Show that $\text{FA}_\kappa(Q)$ holds for any κ -closed partial order Q .

The results of the previous chapter now read as follows:

Theorem 7. Let $Q = \text{Fn}(\omega, 2, \aleph_0)$ be the Cohen partial order and assume $\text{FA}_{\aleph_1}(Q)$. Then every set of reals of cardinality $\leq \aleph_1$ has measure zero.

Theorem 8. Let $M[G]$ be a generic extension of the ground model M by λ -fold Cohen forcing $P = (P, \leq, 1) = \text{Fn}(\lambda \times \omega, 2, \aleph_0)$ where $\lambda = \aleph_2^M$. Then in $M[G]$, $\text{FA}_{\aleph_1}(Q)$ holds.

Proof. We may assume that every $D \in \mathcal{D}$ is a dense subset of Q . Then \mathcal{D} can be coded as a subset of \aleph_1^M . There is $\alpha < \lambda$ such that $\mathcal{D} \in M[G_\alpha]$. The filter H_α corresponding to the α -th Cohen real in the construction is $M[G_\alpha]$ -generic on Q . Since $\mathcal{D} \subseteq M[G_\alpha]$, H_α is \mathcal{D} -generic on Q . □

So for the Cohen forcing Q we have a strengthening of the Rasiowa-Sikorski Lemma from countable to cardinality $\leq \aleph_1$. This is not possible for all forcings:

Lemma 9. *Let $P = \text{Fn}(\aleph_0, \aleph_1, \aleph_0)$ be the canonical forcing for adding a surjection from \aleph_0 onto \aleph_1 . Then $\text{FA}_{\aleph_1}(P)$ is false.*

Proof. For $\alpha < \aleph_1$ define the set

$$D_\alpha = \{p \in P \mid \alpha \in \text{ran}(p)\}$$

which is dense in P . Let $D = \{D_\alpha \mid \alpha < \aleph_1\}$. Assume for a contradiction that H is a \mathcal{D} -generic filter on P . Then $\bigcup H$ is a partial function from \aleph_0 to \aleph_1 .

(1) $\bigcup H$ is onto \aleph_1 .

Proof. Let $\alpha < \aleph_1$. Since H is a \mathcal{D} -generic, $H \cap D_\alpha \neq \emptyset$. Take $p \in H \cap D_\alpha$. Then

$$\alpha \in \text{ran}(p) \subseteq \text{ran}\left(\bigcup H\right)$$

qed.

But this is a contradiction since \aleph_1 is a cardinal. □

Exercise 6. Show that $\text{FA}_{2^{\aleph_0}}(\text{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

So we cannot have an uncountable generalization of the Rasiowa-Sikorski Lemma for forcings which collapse the cardinal \aleph_1 . Since countable chain condition (ccc) forcing does not collapse cardinals, this suggests the following axiom:

Definition 10.

- a) *Let κ be a cardinal. Then MARTIN's axiom MA_κ is the property: for every ccc partial order $(P, \leq, 1_P)$, $\text{FA}_\kappa(P)$ holds.*
- b) *MARTIN's axiom MA postulates that MA_κ holds for every $\kappa < 2^{\aleph_0}$.*

MA_{\aleph_0} holds by Theorem 6. Thus the continuum hypothesis $2^{\aleph_0} = \aleph_1$ trivially implies MA . We shall later see by an iterated forcing construction that $2^{\aleph_0} = \aleph_2$ and MA are relatively consistent with ZFC.

4 Consequences of $\text{MA}_+ \neg \text{CH}$

4.1 Lebesgue measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t \mid t: \text{dom}(t) \rightarrow 2 \wedge \text{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} \mid s \subseteq x\} \subseteq \mathbb{R}$$

with $\text{length}(I_s) = 2^{-\text{dom}(s)}$. Note that $I_s = I_{s \cup \{(\text{dom}(s), 0)\}} \cup I_{s \cup \{(\text{dom}(s), 1)\}}$, $\text{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\text{length}(I_{s \cup \{(\text{dom}(s), 0)\}}) = \text{length}(I_{s \cup \{(\text{dom}(s), 1)\}}) = \frac{1}{2} \text{length}(I_s)$.

Definition 11. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has measure $< \varepsilon$ if there exists a sequence $(I_n \mid n < \omega)$ of intervals in \mathbb{R} such that $X \subseteq \bigcup_{n < \omega} I_n$ and $\sum_{n < \omega} \text{length}(I_n) \leq \varepsilon$. A set $X \subseteq \mathbb{R}$ has measure zero if it has measure $< \varepsilon$ for every $\varepsilon > 0$.*

The measure zero sets form a countably complete ideal on \mathbb{R} . It is easy to see that a countable union of measure zero sets is again measure zero. To strengthen this theorem in the context of MA we need some more topological and measure theoretic notions. The (standard) topology on \mathbb{R} is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n < \omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.

Theorem 12. *Assume MA_κ and let $(X_i | i < \kappa)$ be a family of measure zero sets. Then $X = \bigcup_{i < \kappa} X_i$ has measure zero.*

Proof. Fix $\varepsilon > 0$. We show that $X = \bigcup_{i < \kappa} X_i$ has measure $< 2\varepsilon$. Let

$$\mathcal{I} = \{(a, b) | a, b \in \mathbb{Q}, a < b\}$$

the countable set of rational intervals $(a, b) = \{c \in \mathbb{R} | a < c < b\}$ in \mathbb{R} . The *length* of (a, b) is simply $\text{length}((a, b)) = b - a$. We shall apply MARTIN's axiom to the following forcing $P = (P, \supseteq, \emptyset)$ where

$$P = \{p \subseteq \mathcal{I} | \sum_{I \in p} \text{length}(I) < \varepsilon\}.$$

(1) P is ccc.

Proof. Let $\{p_i | i < \omega_1\} \subseteq P$. For every $i < \omega_1$ there is $n_i < \omega$ such that p_i has measure $< \varepsilon - \frac{1}{n_i}$. By a pigeonhole principle we may assume that all n_i are equal to a common value $n < \omega$. For every p_i we have

$$\sum_{I \in p_i} \text{length}(I) < \varepsilon - \frac{1}{n}.$$

For every $i < \omega_1$ take a finite set $\bar{p}_i \subseteq p_i$ such that

$$\sum_{I \in p_i \setminus \bar{p}_i} \text{length}(I) < \frac{1}{n}.$$

There are only countably many such set \bar{p}_i , and again by a pigeonhole argument we may assume that for all $i < \omega_1$

$$\bar{p}_i = \bar{p}$$

takes a fixed value. Now consider $i < j < \omega_1$. Then

$$\begin{aligned} \sum_{I \in p_i \cup p_j} \text{length}(I) &\leq \sum_{I \in p_i} \text{length}(I) + \sum_{I \in p_j \setminus \bar{p}} \text{length}(I) \\ &< \varepsilon - \frac{1}{n} + \frac{1}{n} \\ &= \varepsilon \end{aligned}$$

Hence $p_i \cup p_j \in P$ and $p_i \cup p_j \leq p_i, p_j$, and so $\{p_i | i < \omega_1\}$ is not an antichain in P . *qed*(1)

For $i < \kappa$ define

$$D_i = \{p \in P | X_i \subseteq \bigcup p\}.$$

(2) D_i is dense in P .

Proof. Let $q \in P$. Take $n < \omega$ such that

$$\sum_{I \in q} \text{length}(I) < \varepsilon - \frac{1}{n}.$$

Since X_i has measure zero, take $r \subseteq \mathcal{I}$ such that $X_i \subseteq \bigcup p$ and $\sum_{I \in r} \text{length}(I) \leq \frac{1}{n}$. Then

$$X_i \subseteq \bigcup (q \cup r) \text{ and } \sum_{I \in q \cup r} \text{length}(I) \leq \sum_{I \in q} \text{length}(I) + \sum_{I \in r} \text{length}(I) < \varepsilon - \frac{1}{n} + \frac{1}{n} = \varepsilon.$$

Hence $p = q \cup r \in P$, $p \supseteq q$, and $p \in D_i$. *qed*(2)

By MA_κ take a filter G on P which is $\{D_i | i < \kappa\}$ -generic. Let $U = \bigcup G \subseteq \mathcal{I}$.

(3) $X = \bigcup_{i < \kappa} X_i \subseteq \bigcup_{I \in U} I$.

Proof. Let $i < \kappa$. By the genericity of G take $p \in G \cap D_i$. Then

$$X_i \subseteq \bigcup p \subseteq \bigcup U$$

qed(3)

(4) $\sum_{I \in U} \text{length}(I) \leq \varepsilon$.

Proof. Assume for a contradiction that $\sum_{I \in U} \text{length}(I) > \varepsilon$. Then take a finite set $\bar{U} \subseteq U$ such that $\sum_{I \in \bar{U}} \text{length}(I) > \varepsilon$. Let $\bar{B} = \{I_0, \dots, I_{k-1}\}$. For every $I_j \in \bar{U}$ take $p_j \in G$ such that $I_j \in p_j$. Since all elements of G are compatible within G there is a condition $p \in G$ such that $p \supseteq p_0, \dots, p_{k-1}$. Hence $\bar{U} \subseteq p$. But, since $p \in P$, we get a contradiction:

$$\varepsilon < \sum_{I \in \bar{U}} \text{length}(I) \leq \sum_{I \in p} \text{length}(I) < \varepsilon.$$

□

Two easy consequences are:

Corollary 13. *Assume MA_κ and let $X \subseteq \mathbb{R}$ with $\text{card}(X) \leq \kappa$. Then X has measure zero.*

Theorem 14. *Assume MA . Then 2^{\aleph_0} is regular.*

Proof. Assume instead that $\mathbb{R} = \bigcup_{i < \kappa} X_i$ for some $\kappa < 2^{\aleph_0}$, where $\text{card}(X_i) < 2^{\aleph_0}$ for every $i < \kappa$. Every singleton $\{r\}$ has measure zero. By Theorem 12, each X_i has measure zero. Again by Theorem, $\mathbb{R} = \bigcup_{i < \kappa} X_i$ has measure zero. But measure theory (and also intuition) shows that \mathbb{R} does not have measure zero. □

4.2 Almost disjoint forcing

We intend to code subsets of κ by subsets of ω . If such a coding is possible then we shall have

$$2^{\aleph_0} \leq 2^\kappa \leq 2^{\aleph_0}, \text{ i.e. } 2^\kappa = 2^{\aleph_0}.$$

We shall employ almost disjoint coding.

Definition 15. A sequence $(x_i | i \in I)$ is almost disjoint if

- a) x_i is infinite
- b) $i \neq j < \kappa$ implies that $x_i \cap x_j$ is finite

Lemma 16. There is an almost disjoint sequence $(x_i | i < 2^{\aleph_0})$ of subsets of ω .

Proof. For $u \in {}^\omega 2$ let $x_u = \{u \upharpoonright m \mid m < \omega\}$. x_u is infinite. Consider $u \neq v$ from ${}^\omega 2$. Let $n < \omega$ be minimal such that $u \upharpoonright n \neq v \upharpoonright n$. Then

$$x_u \cap x_v = \{u \upharpoonright m \mid m < \omega\} \cap \{v \upharpoonright m \mid m < \omega\} = \{u \upharpoonright m \mid m < n\}$$

is finite. Thus $(x_u | u \in {}^\omega 2)$ is almost disjoint. Using bijections $\omega \leftrightarrow <^\omega 2$ and $2^{\aleph_0} \leftrightarrow {}^\omega 2$ one can turn this into an almost disjoint sequence $(x_i | i < 2^{\aleph_0})$ of subsets of ω . \square

Theorem 17. Assume MA_κ . Then $2^\kappa = 2^{\aleph_0}$.

Proof. By a previous example, $\kappa < 2^{\aleph_0}$. By the lemma, fix an almost disjoint sequence $(x_i | i < \kappa)$ of subsets of ω . Define a map $c: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\kappa)$ by

$$c(x) = \{i < \kappa \mid x \cap x_i \text{ is infinite}\}.$$

We say that x codes $c(x)$. We want to show that every subset of κ can be coded as some $c(x)$. We show this by proving that $c: \mathcal{P}(\omega) \rightarrow \mathcal{P}(\kappa)$ is surjective.

Let $A \subseteq \kappa$ be given. We use the following forcing $(P, \leq, 1)$ to code A :

$$P = \{(a, z) \mid a \subseteq \omega, z \subseteq \kappa, \text{card}(a) < \aleph_0, \text{card}(z) < \aleph_0\},$$

partially ordered by

$$(a', z') \leq (a, z) \text{ iff } a' \supseteq a, z' \supseteq z, i \in z \cap (\kappa \setminus A) \rightarrow a' \cap x_i = a \cap x_i.$$

The weakest element of P is $1 = (\emptyset, \emptyset)$.

The idea of the forcing is to keep the intersection of the first component with x_i fixed, provided $i \notin A$ has entered the second component. This will allow the almost disjoint coding of A by the finite/infinite method.

(1) $(P, \leq, 1)$ satisfies ccc.

Proof. Conditions (a, y) and (a, z) with equal first components are compatible, since $(a, y \cup z) \leq (a, y)$ and $(a, y \cup z) \leq (a, z)$. Incompatible conditions have different first components. Since there are only countably many first components, an antichain in P can be at most countable. *qed*(1)

The outcome of a forcing construction results from an interplay between the partial order and some dense set arguments. We now define dense sets for our requirements.

For $i < \kappa$ let $D_i = \{(a, z) \in P \mid i \in z\}$. D_i is obviously dense in P . For $i \in A$ and $n \in \omega$ let $D_{i,n} = \{(a, z) \in P \mid \exists m > n: m \in a \cap x_i\}$.

(2) If $i \in A$ and $n \in \omega$ then $D_{i,n}$ is dense in P .

Proof. Consider $(a, z) \in P$. For $j \in z$, $j \neq i$ is the intersection $x_i \cap x_j$ finite. Take some $m \in x_i$, $m > n$ such that $m \notin x_i \cap x_j$ for $j \in z$, $j \neq i$. Then

$$(a \cup \{m\}, z) \leq (a, z) \text{ and } (a \cup \{m\}, z) \in D_{i,n}.$$

qed(2)

By MA_κ take a filter G on P which is generic for the dense sets in

$$\{D_i | i < \kappa\} \cup \{D_{i,n} | i \in A, n \in \omega\}.$$

Let

$$x = \bigcup \{a | (a, y) \in G\} \subseteq \omega.$$

(3) Let $i \in A$. Then $x \cap x_i$ is infinite.

Proof. Let $n < \omega$. By genericity take $(a, y) \in G \cap D_{i,n}$. By the definition of $D_{i,n}$ take $m > n$ such that $m \in a \cap x_i$. Then $m \in x \cap x_i$, and so $x \cap x_i$ is cofinal in ω . qed(3)

(4) Let $i \in \kappa \setminus A$. Then $x \cap x_i$ is finite.

Proof. By genericity take $(a, y) \in G \cap D_i$. Then $i \in y$. We show that $x \cap x_i \subseteq a \cap x_i$. Consider $n \in x \cap x_i$. Take $(b, z) \in G$ such that $n \in b$. By the filter properties of G take $(a', y') \in P$ such that $(a', y') \leq (a, y)$ and $(a', y') \leq (b, z)$. Then $n \in a'$, and by the definition of \leq , $a' \cap x_i = a \cap x_i$. Thus $n \in a \cap x_i$. qed(4)

So

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\} = A \in \text{range}(c).$$

□

5 Iterated forcing

MARTIN's axiom postulates that for every ccc partial order $(P, \leq, 1_P)$ and \mathcal{D} with $\text{card}(\mathcal{D}) < 2^{\aleph_0}$ there is a \mathcal{D} -generic filter G on P . Syntactically this axiom has a $\forall\exists$ -form: $\forall P \forall \mathcal{D} \exists G \dots$. $\forall\exists$ -properties are often realised through chain constructions: build a chain

$$M = M_0 \subseteq M_1 \subseteq \dots \subseteq M_\alpha \subseteq \dots \subseteq M_\beta \subseteq \dots$$

of models such that for any $P, \mathcal{D} \in M_\alpha$ there is some $\beta \geq \alpha$ such that M_β contains a generic G as required. Then the “union” or limit of the chain should contain appropriate G 's for all P 's and \mathcal{D} 's.

Such chain constructions are wellknown from algebra. To satisfy closure under square roots ($\forall x \exists y: yy = x$) one can e.g. start with a countable field M_0 and along a chain $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ adjoin square roots for all elements of M_n . Then $\bigcup_{n < \omega} M_n$ satisfies the closure property.

In set theory there is a difficulty that unions of models of set theory usually do not satisfy the theory ZF: assume that $M_0 \subseteq M_1 \subseteq M_2 \subseteq \dots$ is an ascending chain of transitive models of ZF such that $(M_{n+1} \setminus M_n) \cap \mathcal{P}(\omega) \neq \emptyset$ for all $n < \omega$. Let $M_\omega = \bigcup_{n < \omega} M_n$. Then $\mathcal{P}(\omega) \cap M_\omega \notin M_\omega$. Indeed, if one had $\mathcal{P}(\omega) \cap M_\omega \in M_\omega$ then $\mathcal{P}(\omega) \cap M_\omega \in M_n$ for some $n < \omega$ and $\mathcal{P}(\omega) \cap M_{n+1} \in M_n$ contradicts the initial assumption. So a “limit” model of models of ZF has to be more complicated, and it will itself be constructed by some limit forcing which is called *iterated forcing*.

Exercise 7. Check which axioms of set theory hold in $M_\omega = \bigcup_{n < \omega} M_n$ where $(M_n)_{n < \omega}$ is an ascending sequence of transitive models of ZF(C).

Since we want to obtain the limit by forcing over a ground model M the construction must be visible in the ground model. This means that the sequence of forcings to be employed to pass from M_α to $M_{\alpha+1}$ has to exist as a sequence $(\dot{Q}_\beta | \beta < \kappa)$ of names in the ground model. The initial sequence $(\dot{Q}_\beta | \beta < \alpha)$ already determines a forcing P_α and \dot{Q}_α is intended to be a P_α -name. If G_α is M -generic over P_α then furthermore $Q_\alpha = (\dot{Q}_\alpha)^{G_\alpha}$ is intended to be a forcing in the model $M_\alpha = M[G_\alpha]$, and $M_{\alpha+1}$ is a generic extension of M_α by forcing with Q_α . The following iteration theorem says that any sequence $(\dot{Q}_\beta | \beta < \kappa) \in M$ gives rise to an iteration of forcing extensions. In applications the sequence has to be chosen carefully to ensure that some $\forall\exists$ -property holds in the final model M_κ . Without loss of generality we only consider forcings Q_α whose maximal element is \emptyset .

Theorem 18. *Let M be a ground model, and let $((\dot{Q}_\beta, \dot{\leq}_\beta) | \beta < \kappa) \in M$ with the property that $\forall \beta < \kappa: \emptyset \in \text{dom}(\dot{Q}_\beta)$. Then there is a uniquely determined sequence $((P_\alpha, \leq_\alpha, 1_\alpha) | \alpha \leq \kappa) \in M$ such that*

- a) $(P_\alpha, \leq_\alpha, 1_\alpha)$ is a partial order which consists of α -sequences;
- b) $P_0 = \{\emptyset\}$, $\leq_0 = \{(\emptyset, \emptyset)\}$, $1_0 = \emptyset$;
- c) If $\lambda \leq \kappa$ is a limit ordinal then the forcing P_λ is defined by:

$$\begin{aligned} P_\lambda &= \{p: \lambda \rightarrow V \mid (\forall \gamma < \lambda: p \restriction \gamma \in P_\gamma) \wedge \exists \gamma < \lambda \forall \beta \in [\gamma, \lambda) p(\beta) = \emptyset\} \\ p \leq_\lambda q &\text{ iff } \forall \gamma < \lambda: p \restriction \gamma \leq_\gamma q \restriction \gamma \\ 1_\lambda &= (\emptyset \mid \gamma < \lambda) \end{aligned}$$

- d) If $\alpha < \kappa$ and $1_\alpha \Vdash_{P_\alpha} “(\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset)$ is a forcing”, then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{aligned} P_{\alpha+1} &= \{p: \alpha + 1 \rightarrow V \mid p \restriction \alpha \in P_\alpha \wedge p(\alpha) \in \text{dom}(\dot{Q}_\alpha) \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha\} \\ p \leq_{\alpha+1} q &\text{ iff } p \restriction \alpha \leq_\alpha q \restriction \alpha \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leq}_\alpha q(\alpha) \\ 1_{\alpha+1} &= (\emptyset \mid \gamma < \alpha + 1) \end{aligned}$$

- e) If $\alpha < \kappa$ and not $1_\alpha \Vdash_{P_\alpha} “(\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset)$ is a forcing”, then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{aligned} P_{\alpha+1} &= \{p: \alpha + 1 \rightarrow V \mid p \restriction \alpha \in P_\alpha \wedge p(\alpha) = \emptyset\} \\ p \leq_{\alpha+1} q &\text{ iff } p \restriction \alpha \leq_\alpha q \restriction \alpha \\ 1_{\alpha+1} &= (\emptyset \mid \gamma < \alpha + 1) \end{aligned}$$

$((P_\alpha, \leq_\alpha, 1_\alpha) | \alpha \leq \kappa)$, and in particular P_κ are called the (*finite support*) iteration of the sequence $((\dot{Q}_\beta, \dot{\leq}_\beta) | \beta < \kappa)$.

Proof. To justify the above recursive definition of the sequence $((P_\alpha, \leq_\alpha, 1_\alpha) | \alpha \leq \kappa)$ it suffices to show recursively that every P_α is a forcing.

Obviously, P_0 is a trivial one-element forcing.

Consider a limit $\lambda \leq \kappa$ and assume that P_γ is a forcing for $\gamma < \alpha$. We have to show that the relation \leq_λ is transitive with maximal element 1_λ . Consider $p \leq_\lambda q \leq_\lambda r$. Then $\forall \gamma < \lambda: p \restriction \gamma \leq_\gamma q \restriction \gamma$ and $\forall \gamma < \lambda: q \restriction \gamma \leq_\gamma r \restriction \gamma$. Since all \leq_γ with $\gamma < \lambda$ are transitive relations, $\forall \gamma < \lambda: p \restriction \gamma \leq_\gamma r \restriction \gamma$ and so $p \leq_\lambda r$. Now consider $p \in P_\lambda$. Then $\forall \gamma < \lambda: p \restriction \gamma \in P_\gamma$. By the inductive assumption, $\forall \gamma < \lambda: p \restriction \gamma \leq_\gamma 1_\gamma = 1_\lambda \restriction \gamma$ and so $p \leq_\lambda 1_\lambda$.

For the successor step assume that $\alpha < \kappa$ and that P_α is a forcing.

Case 1. $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leq_{\alpha+1}$ consider $p \leq_{\alpha+1} q \leq_{\alpha+1} r$. Then $p \restriction \alpha \leq_\alpha q \restriction \alpha \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leq}_\alpha q(\alpha)$ and $q \restriction \alpha \leq_\alpha r \restriction \alpha \wedge q \restriction \alpha \Vdash_{P_\alpha} q(\alpha) \dot{\leq}_\alpha r(\alpha)$. By the transitivity of \leq_α : $p \restriction \alpha \leq_\alpha r \restriction \alpha$. Moreover $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leq}_\alpha q(\alpha)$, $p \restriction \alpha \Vdash_{P_\alpha} q(\alpha) \dot{\leq}_\alpha r(\alpha)$ and $p \restriction \alpha \Vdash_{P_\alpha}$ “ $\dot{\leq}_\alpha$ is transitive”. This implies $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leq}_\alpha r(\alpha)$ and together that $p \leq_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p \in P_{\alpha+1}$. Then $p \restriction \alpha \in P_\alpha \wedge p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Then $p \restriction \alpha \leq_\alpha 1_\alpha = 1_{\alpha+1} \restriction \alpha$. Moreover $p \restriction \alpha \Vdash_{P_\alpha}$ “ \emptyset is maximal in $\dot{\leq}_\alpha$ ” implies that $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leq}_\alpha \emptyset = 1_{\alpha+1}(\alpha)$. Hence $p \leq_{\alpha+1} 1_{\alpha+1}$.

Case 2. It is not the case that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leq_{\alpha+1}$ consider $p \leq_{\alpha+1} q \leq_{\alpha+1} r$. Then $p \restriction \alpha \leq_\alpha q \restriction \alpha$ and $q \restriction \alpha \leq_\alpha r \restriction \alpha$. By the transitivity of \leq_α : $p \restriction \alpha \leq_\alpha r \restriction \alpha$ and so $p \leq_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p \in P_{\alpha+1}$. Then $p \restriction \alpha \in P_\alpha$. By induction, $p \restriction \alpha \leq_\alpha 1_\alpha$ and so $p \leq_{\alpha+1} 1_{\alpha+1}$. \square

The term “finite support iteration” is justified by the following

Lemma 19. *In the above situation let $p \in P_\kappa$. Then*

$$\text{supp}(p) = \{\alpha < \kappa \mid p(\alpha) \neq \emptyset\}$$

is finite.

Proof. Prove by induction on $\alpha \leq \kappa$ that $\text{supp}(p)$ is finite for every $q \in P_\alpha$. The crucial property is the definition of P_λ at limit λ in the above iteration theorem. \square

Let us fix a ground model M and the iteration $((\dot{Q}_\beta, \dot{\leq}_\beta) \mid \beta < \kappa) \in M$ and $((P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha \leq \kappa) \in M$ as above. Let G_κ be M -generic for P_κ . We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M = M_0 \subseteq M_1 = M[G_1] = M_0[H_0] \subseteq M_2 = M[G_2] = M_1[H_1] \subseteq \dots \subseteq M_\alpha = M[G_\alpha] \subseteq \dots \subseteq M_\kappa$$

of generic extensions.

Let us first note some relations within the tower $(P_\alpha)_{\alpha \leq \kappa}$ of forcings.

Lemma 20.

a) *Let $\alpha \leq \kappa$ and $p, q \in P_\alpha$.*

Then $p \leq_\alpha q$ iff $\forall \gamma \in \text{supp}(p) \cup \text{supp}(q): p \restriction \gamma \Vdash_{P_\gamma} p(\gamma) \dot{\leq}_\gamma q(\gamma)$.

b) *Let $\alpha \leq \beta \leq \kappa$ and $p \in P_\beta$. Then $p \restriction \alpha \in P_\alpha$.*

c) *Let $\alpha \leq \beta \leq \kappa$ and $p \leq_\beta q$. Then $p \restriction \alpha \leq_\alpha q \restriction \alpha$.*

d) *Let $\alpha \leq \beta \leq \kappa$, $q \in P_\beta$, $\bar{p} \leq_\alpha q \restriction \alpha$. Then $\bar{p} \cup (q(\gamma) \mid \alpha \leq \gamma < \beta) \in P_\beta$ and $\bar{p} \cup (q(\gamma) \mid \alpha \leq \gamma < \beta) \leq_\beta q$.*

Proof. a) By a straightforward induction on $\alpha \leq \kappa$. Now b) – d) follow immediately. \square

For $\alpha \leq \kappa$ define $G_\alpha = \{p \restriction \alpha \mid p \in G_\kappa\}$.

(1) G_α is M -generic for P_α .

Proof. By (a), $G_\alpha \subseteq P_\alpha$. Consider $p \restriction \alpha, q \restriction \alpha \in G_\alpha$ with $p, q \in G_\kappa$. Take $r \in G_\kappa$ such that $r \leq_\kappa p, q$. By (a), $r \restriction \alpha \leq_\alpha p \restriction \alpha, q \restriction \alpha$. Thus all elements of G_α are compatible in P_α .

Consider $p \restriction \alpha \in G_\alpha$ with $p \in G_\kappa$ and $\bar{q} \in P_\alpha$ with $p \restriction \alpha \leq_\alpha \bar{q}$. By (a),

$$q = \bar{q} \cup (\emptyset \restriction \alpha \leq \gamma < \kappa)$$

is an element of P_κ and $p \leq_\kappa q$. Since G_κ is a filter, $q \in G_\kappa$, and so $\bar{q} = q \restriction \alpha \in G_\alpha$. Thus G_α is upwards closed.

For the genericity consider a set $\bar{D} \in M$ which is dense in P_α . We claim that the set

$$D = \{d \in P_\kappa \mid d \restriction \alpha \in \bar{D}\} \in M$$

is dense in P_κ : let $p \in P_\kappa$. Then $p \restriction \alpha \in P_\alpha$. Take $\bar{d} \in \bar{D}$ such that $\bar{d} \leq_\alpha p \restriction \alpha$. By (c,d),

$$d = \bar{d} \cup (p(\gamma) \restriction \alpha \leq \gamma < \kappa) \in P_\kappa$$

and $d \leq_\kappa p$.

By the genericity of G_κ take $p \in D \cap G_\kappa$. Then $p \restriction \alpha \in \bar{D} \cap G_\alpha \neq \emptyset$. *qed*(1)

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α .

(2) Let $\alpha < \beta \leq \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.

Proof. $G_\alpha = \{p \restriction \alpha \mid p \in G_\kappa\} = \{(p \restriction \beta) \restriction \alpha \mid p \in G_\kappa\} = \{q \restriction \alpha \mid q \in G_\beta\} \in M[G_\beta]$. *qed*(2)

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leq^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leq}_\alpha^{G_\alpha}, \emptyset), & \text{if } 1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset) \text{ is a forcing”} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), & \text{else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} \mid p \in G_\kappa\}.$$

(3) H_α is M_α -generic for Q_α .

Proof. If it is not the case that $1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset) \text{ is a forcing”}$, then $(Q_\alpha, \leq^{Q_\alpha}, \emptyset) = (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset)$ and $H_\alpha = \{\emptyset\}$ is trivially M_α -generic. So assume that $1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leq}_\alpha, \emptyset) \text{ is a forcing”}$.

(a) $H_\alpha \subseteq Q_\alpha$.

Proof. Let $p \in G_\kappa$. Then $p \restriction \alpha + 1 \in P_{\alpha+1}$ and so $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Since $p \restriction \alpha \in G_\alpha$ we have that $p(\alpha)^{G_\alpha} \in \dot{Q}_\alpha^{G_\alpha} = Q_\alpha$. *qed*(a)

(b) H_α is a filter.

Proof. Let $p(\alpha)^{G_\alpha} \in H_\alpha$ and $p(\alpha)^{G_\alpha} \leq^{Q_\alpha} r \in Q_\alpha$.

(e) Let $\bar{D} \in M_\alpha$ be dense in Q_α . Then $\bar{D} \cap H_\alpha \neq \emptyset$.

Proof. Take $\dot{D} \in M$ such that $\bar{D} = \dot{D}^{G_\alpha}$. Take $p \in G_\kappa$ such that

$$p \restriction \alpha \Vdash_{P_\alpha} \dot{D} \text{ is dense in } \dot{Q}_\alpha.$$

Define

$$D = \{d \in P_\kappa \mid d \restriction \alpha \Vdash d(\alpha) \in \dot{D}\} \in M.$$

We show that D is dense in P_κ below p . Let $q \leq_\kappa p$. Then $q \restriction \alpha \leq_\alpha p \restriction \alpha$ and $q \restriction \alpha \Vdash q(\alpha) \leq_\alpha p(\alpha)$. Hence $q \restriction \alpha \Vdash_{P_\alpha} \dot{D}$ is dense in \dot{Q}_α and there is $\bar{d} \leq_\alpha q \restriction \alpha$ and some $d(\alpha) \in \text{dom}(\dot{Q}_\alpha)$ such that

$$\bar{d} \Vdash_{P_\alpha} (d(\alpha) \leq_\alpha q(\alpha) \wedge d(\alpha) \in \dot{D}).$$

Define

$$d = \bar{d} \cup \{(\alpha, d(\alpha))\} \cup (q(\gamma) \mid \alpha < \gamma < \kappa).$$

Then $d \in P_\kappa$, $d \leq_\kappa q$, and $d \in D$.

By the genericity of G_κ take $d \in D \cap G_\kappa$. Then $d(\alpha)^{G_\alpha} \in H_\alpha$, $d \restriction \alpha \in G_\alpha$, and $d(\alpha)^{G_\alpha} \in (\dot{D})^{G_\alpha} = \bar{D}$. Thus $H_\alpha \cap \bar{D} \neq \emptyset$.

(4) $M_{\alpha+1} = M_\alpha[H_\alpha]$.

Proof. \supseteq is straightforward. For the other direction, it suffices to show that $G_{\alpha+1} \in M_\alpha[H_\alpha]$, and indeed we show that

$$G_{\alpha+1} = \{q \in P_{\alpha+1} \mid q \restriction \alpha \in G_\alpha \wedge q(\alpha)^{G_\alpha} \in H_\alpha\}.$$

Let $q \in G_{\alpha+1}$. Take $p \in G_\kappa$ such that $p \restriction \alpha + 1 = q$. Then $q \restriction \alpha = p \restriction \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} = p(\alpha)^{G_\alpha} \in H_\alpha$. For the converse consider $q \in P_{\alpha+1}$ such that $q \restriction \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} \in H_\alpha$. Take $p_1, p_2 \in G_\kappa$ such that $q \restriction \alpha = p_1 \restriction \alpha$ and $q(\alpha)^{G_\alpha} = p_2(\alpha)^{G_\alpha}$. Take $p \in G_\kappa$ such that $p \leq_\kappa p_1, p_2$. We also may assume that $p \restriction \alpha \Vdash q(\alpha) = p_2(\alpha)$. $p \restriction \alpha \leq_\alpha p_1 \restriction \alpha = q \restriction \alpha$ and $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \leq_\alpha p_2(\alpha) = q(\alpha)$. Hence $p \restriction \alpha + 1 \leq_{\alpha+1} q$. Since $p \restriction \alpha + 1 \in G_{\alpha+1}$ and since $G_{\alpha+1}$ is upward closed, we get $q \in G_{\alpha+1}$.

5.1 Embeddings

In the above construction, $M[G_\alpha] \subseteq M[G_\beta]$ canonically. This corresponds to canonical transformations of names used in the construction of $M[G_\alpha]$ into names used to construct $M[G_\beta]$. Such transformation of names is important for the construction and analysis of iterations. We first reduce our ‘‘name spaces’’ from all of M to more specific P -names.

Definition 21. *Let P be a forcing. Define recursively: \dot{x} is a P -name if every element of \dot{x} is an ordered pair (\dot{y}, p) where \dot{y} is a P -name and $p \in P$. Let V^P be the class or name space of all P -names.*

The generic interpretation of an arbitrary name only depends on ordered pairs whose second component is in P . This observation leads to

Lemma 22. *Let P be a forcing. Define $\tau: V \rightarrow V^P$ recursively by*

$$\tau(\dot{x}) = \{(\tau(\dot{y}), p) \mid (\dot{y}, p) \in \dot{x}\}.$$

Then $\tau(\dot{x})$ is a P -name and

$$1_P \Vdash \dot{x} = \tau(\dot{x}).$$

I.e., $\dot{x}^G = (\tau(\dot{x}))^G$ for every generic filter on P .

Let $\pi: P \rightarrow Q$ be an orderpreserving embedding of forcings. This induces an embedding of name spaces $\pi^*: V^P \rightarrow V^Q$ which is defined recursively:

$$\pi^*(\dot{x}) = \{(\pi^*(\dot{y}), \pi(p)) \mid (\dot{y}, p) \in \dot{x}\}.$$

One can study such embeddings in general. They satisfy “universal properties”, sometimes relying on structural properties of the embedding π .

Exercise 8. Examine, how generic filters are mapped by π and its inverse and how this induces embeddings of generic extensions. Formulate sufficient properties for the original map π .

We restrict our considerations to embeddings connected to iterated forcing. So let $((P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha \leq \kappa)$ be a finite support iteration of the sequence $((\dot{Q}_\alpha, \dot{\leq}_\alpha) \mid \alpha < \kappa)$. In view of the previous lemma we also require in the iteration that every \dot{Q}_α is a P_α -name.

There are canonical maps between the P_α 's. For $\alpha \leq \beta \leq \kappa$ define $\pi_{\alpha\beta}: P_\alpha \rightarrow P_\beta$ by

$$\pi_{\alpha\beta}(p) = p \cup (\emptyset \mid \alpha \leq \gamma < \beta).$$

Also define $\pi_{\beta\alpha}: P_\beta \rightarrow P_\alpha$ by $\pi_{\beta\alpha}(q) = q \upharpoonright \alpha$. $\pi_{\beta\alpha}$ is a left inverse of $\pi_{\alpha\beta}$:

$$\pi_{\beta\alpha} \circ \pi_{\alpha\beta} = \text{id}_{P_\alpha}.$$

Let the previous constructions take place within a ground model M . Let G_κ be M -generic for P_κ and let $M_\alpha = M[G_\alpha]$ for $\alpha \leq \kappa$ be the associated tower of extensions. Let $\alpha \leq \beta \leq \kappa$. The inclusion $M[G_\alpha] \subseteq M[G_\beta]$ corresponds to the following

Lemma 23. *Let $\dot{x} \in M^{P_\alpha}$ be a P_α -name and $\ddot{x} = \pi_{\alpha\beta}^*(\dot{x}) \in M^{P_\beta}$ its “lift” to P_β . Then*

$$\dot{x}^{G_\alpha} = \ddot{x}^{G_\beta}.$$

Proof. By induction on \dot{x} :

$$\begin{aligned} \ddot{x}^{G_\beta} &= \{\dot{y}^{G_\beta} \mid \exists q \in G_\beta (\dot{y}, q) \in \ddot{x}\} \\ &= \{\dot{y}^{G_\beta} \mid \exists q (q \in G_\beta \wedge (\dot{y}, q) \in \ddot{x})\} \\ &= \{\dot{y}^{G_\beta} \mid \exists q (q \in G_\beta \wedge \exists (\dot{y}, p) \in \dot{x} ((\pi_{\alpha\beta}^*(\dot{y}), \pi_{\alpha\beta}(p)) \in \ddot{x} \wedge \dot{y} = \pi_{\alpha\beta}^*(\dot{y}) \wedge q = \pi_{\alpha\beta}(p)))\} \\ &= \{\dot{y}^{G_\beta} \mid \exists p \in G_\alpha \exists (\dot{y}, p) \in \dot{x} \dot{y} = \pi_{\alpha\beta}^*(\dot{y})\} \\ &= \{\pi_{\alpha\beta}^*(\dot{y})^{G_\beta} \mid \exists p \in G_\alpha (\dot{y}, p) \in \dot{x}\} \\ &= \{\dot{y}^{G_\alpha} \mid \exists p \in G_\alpha (\dot{y}, p) \in \dot{x}\} \\ &= \dot{x}^{G_\alpha} \end{aligned}$$

□

In the intended applications of iterated forcing we shall usually be confronted at “time” α with several tasks which have to be dealt with “one by one” along the ordinal axis κ : there will be, e.g., two distinct partial orders $R, S \in M[G_\alpha]$ for which we want to adjoin generic filters. These have P_α -names $\dot{R}, \dot{S} \in M^{P_\alpha}$. In the iteration we may set $\dot{Q}_\alpha = \dot{R}$, but then we have to deal with \dot{S} at some later “time” β . This will be possible by lifting \dot{S} to a P_β -name: set $\dot{Q}_\beta = \pi_{\alpha\beta}^*(\dot{S})$. In the construction some “bookkeeping mechanism” will ensure that eventually all tasks will be looked after.

5.2 Two-step iterations

Definition 24. Consider a forcing $(P, \leq_P, 0)$ and names $\dot{Q}, \dot{\leq}$ such that

$$1_P \Vdash (\dot{Q}, \dot{\leq}, 0) \text{ is a forcing.}$$

and $0 \in \text{dom}(\dot{Q})$. Then the two-step iteration $(P * \dot{Q}, \preceq, 1)$ is defined by:

$$\begin{aligned} P * \dot{Q} &= \{(p, \dot{q}) \mid p \in P \wedge \dot{q} \in \text{dom}(\dot{Q}) \wedge p \Vdash_P \dot{q} \in \dot{Q}\} \\ (p', \dot{q}') \preceq (p, \dot{q}) &\text{ iff } p' \leq_P p \wedge p' \Vdash_P \dot{q}' \dot{\leq} \dot{q}' \\ 1 &= (0, 0) \end{aligned}$$

The two-step iteration can be construed as an iteration of a sequence $((\dot{Q}_\beta, \dot{\leq}_\beta) \mid \beta < 2)$ of length 2: Let $\dot{Q}_0 = \check{P}$, $\dot{\leq}_0 = \check{\leq}_P$ where the canonical names \check{P} and $\check{\leq}_P$ are formed with respect to the trivial forcing $P_0 = \{\emptyset\}$, $\leq_0 = \{(\emptyset, \emptyset)\}$, $1_0 = \emptyset$. Then $(P, \leq_P, 0)$ is canonically isomorphic to the induced forcing $(P_1, \leq_1, 1_1)$ by the map $h: p \mapsto \check{p}$. We may assume that \dot{Q} is a P -names in the restricted sense that for every ordered pair $(a, p) \in \text{TC}(\dot{Q})$ $p \in P$. Then define a corresponding P_1 -name \dot{Q}_1 by replacing recursively each $(a, p) \in \text{TC}(\dot{Q})$ by $(\dots, h(p))$. Similarly for $\dot{\leq}_1$.

One can check that the iterated forcing $(P_2, \leq_2, 1_2)$ defined from $((\dot{Q}_\beta, \dot{\leq}_\beta) \mid \beta < 2)$ is canonically isomorphic to $(P * \dot{Q}, \preceq, 1)$.

Such identifications using subtle but canonical isomorphisms occur often in the theory of iterated forcing.

Exercise 9. If G is M -generic for $P * \dot{Q} \in M$ where M is a ground model define

$$\begin{aligned} G_0 &= \{p \in P \mid \exists \dot{q} \in \text{dom}(\dot{Q}): (p, \dot{q}) \in G\} \\ G_1 &= \{\dot{q}^{G_0} \mid \exists p \in P: (p, \dot{q}) \in G\} \end{aligned}$$

Show that G_0 is M -generic for P and that G_1 is M -generic for \dot{Q}^{G_0} .

Conversely let G_0 be M -generic for P and G_1 M -generic for \dot{Q}^{G_0} . Show that

$$G = \{(p, \dot{q}) \mid p \in G_0, \dot{q}^{G_0} \in G_1\}$$

is M -generic for $P * \dot{Q}$.

5.3 Products of partial orders

A special case of a finite support iteration is a finite support *product*. So let M be a ground model, and let $((Q_\beta, \leq_\beta) \mid \beta < \kappa) \in M$ be a sequence of forcings such that \emptyset is a maximal element of every Q_β . Define the *finite support product* $\prod_{\beta < \kappa} Q_\beta$ as the following forcing:

$$\begin{aligned} \prod_{\beta < \kappa} Q_\beta &= \{p: \kappa \rightarrow V \mid \forall \beta < \kappa: p(\beta) \in Q_\beta, \text{supp}(p) \text{ is finite}\} \\ p \preceq q &\text{ iff } \forall \beta < \kappa: p(\beta) \leq_\beta q(\beta) \\ 1_\kappa &= (0 \mid \beta < \kappa) \end{aligned}$$

We want to show that the product corresponds to a simple iteration. Define a sequence

$$((\check{Q}_\beta, \check{\leq}_\beta) \mid \beta < \kappa) \in M$$

where \check{Q}_β is the canonical name for Q_β with respect to a forcing which has the β -sequence $1_\beta = (0 \mid \gamma < \beta)$ as its maximal element. (Note that the definition of $\check{x} = \{(\check{y}, 1_\beta) \mid y \in x\}$ only depends on 1_β .) Let the sequence $((P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha \leq \kappa) \in M$ be defined from the sequence $((\check{Q}_\beta, \check{\leq}_\beta) \mid \beta < \kappa)$ of names as in the iteration theorem.

Then there is a canonical isomorphism

$$\pi: \prod_{\beta < \kappa} Q_\beta \leftrightarrow P_\kappa$$

defined by: $p \mapsto p'$ where

$$p'(\beta) = \widetilde{p(\beta)}$$

with respect to a partial order with maximal element 1_β . It is tedious but straightforward to check that this defines an isomorphism.

6 Iteration theorems

A main concern of forcing is the preservation of cardinals. There are several criteria for ensuring cardinal preservation or at least the preservation of \aleph_1 . Iteration theorems take the form: if every \check{Q}_β in $((\check{Q}_\beta, \check{\leq}_\beta) \mid \beta < \kappa)$ is forced to satisfy the preservation criterion then also P_κ satisfies the criterion.

Theorem 25. *Let λ be a regular cardinal. Consider the two-step iteration $(P * \check{Q}, \preceq, 1)$ of $(P, \leq_P, 0)$ and $(\check{Q}, \check{\leq}, 0)$. Assume that $(P, \leq_P, 0)$ satisfies the λ -c.c. and $0 \Vdash_P \text{“}(\check{Q}, \check{\leq}, 0) \text{ satisfies } \check{\lambda}\text{-c.c.”}$. Then $(P * \check{Q}, \preceq, 1)$ satisfies the λ -c.c.*

Proof. We may assume that the assumptions of the theorem are satisfied in some ground model M . It suffices to prove the theorem in M . Work inside M . Let $((p_\alpha, q_\alpha) \mid \alpha < \lambda)$ be a sequence in $(P * \check{Q}, \preceq, 1)$. It suffices to find two compatible conditions in this sequence.

(1) There is a condition $p \in P$ such that $p \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \check{G}\} = \lambda$ where \check{G} is the canonical name for a generic filter on P .

Proof. If not, then there is a maximal antichain A in P of conditions q for which there is an ordinal $\gamma_q < \kappa$ such that $q \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \check{G}\} = \check{\gamma}_q$. By the κ -c.c., $\text{card}(A) < \lambda$. By the regularity of κ there is $\gamma < \kappa$ such that $\forall q \in A \gamma_q < \gamma$. Since A is a maximal antichain,

$$0 = 1_P \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \check{G}\} \leq \check{\gamma}.$$

But $p_{\gamma+1} \Vdash \check{p}_{\gamma+1} \in \check{G}$ and $p_{\gamma+1} \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \check{G}\} \geq \check{\gamma} + 1$. Contradiction. *qed*(1)

Take an M -generic filter G on P such that $p \in G$ and $p \Vdash \sup \{\alpha \mid \check{p}_\alpha \in \check{G}\} = \lambda$. In $M[G]$ form the sequence $(q_\alpha^G \mid p_\alpha \in G)$; by (1) this sequence has ordertype λ . \check{Q}^G satisfies the λ -c.c. in $M[G]$ and λ is still a regular cardinal in $M[G]$. So there are $\alpha < \beta < \lambda$ such that q_α^G and q_β^G are compatible in \check{Q}^G . Take $r \in G$ and $q \in \text{dom}(\check{Q})$ such that $r \leq p_\alpha, p_\beta$ and $r \Vdash q \leq q_\alpha, q_\beta$. Then $(r, q) \in P * \check{Q}$ and $(r, q) \preceq (p_\alpha, q_\alpha), (p_\beta, q_\beta)$. \square

Theorem 26. *Let $((P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha \leq \kappa)$ be the finite support iteration of the sequence $((\dot{Q}_\beta, \dot{\leq}_\beta) \mid \beta < \kappa)$. Let λ be a regular cardinal and suppose that*

$$P_\beta \Vdash \text{“}\dot{Q}_\beta \text{ is } \check{\lambda}\text{-cc”}$$

for all $\beta < \kappa$. Then every P_α , $\alpha \leq \kappa$ is λ -cc.

Proof. Again it suffices to prove the theorem in some ground model M . Work inside M . We prove the theorem by induction on $\alpha \leq \kappa$. The theorem is trivial for $P_0 = \{\emptyset\}$.

Let $\alpha = \beta + 1$. One can canonically prove that $P_{\beta+1} \cong P_\beta * \dot{Q}_\beta$. Then P_α is λ -cc by the inductive assumption and the previous theorem.

Finally consider a limit ordinal $\alpha \leq \kappa$. Let $A \subseteq P_\alpha$ have cardinality λ . Every condition $p \in A$ has a finite support $\text{supp}(p)$. By the Δ -system lemma, we may suppose that $(\text{supp}(p) \mid p \in A)$ is a Δ -system with some finite kernel d . Take $\beta < \alpha$ such that $d \subseteq \beta$. By the inductive assumption P_β is λ -cc. Take distinct $p, q \in A$ such that $p \restriction \beta, q \restriction \beta$ are compatible in P_β . Take $r \in P_\beta$ such that $r \leq_\beta p \restriction \beta, q \restriction \beta$. We then define a compatibility element $s \leq_\alpha p, q$ by

$$s(i) = \begin{cases} r(i), & \text{for } i < \beta \\ p(i), & \text{for } \beta \leq i < \alpha, i \in \text{supp}(p) \\ q(i), & \text{for } \beta \leq i < \alpha, i \notin \text{supp}(p) \end{cases}$$

□

Although the final model $M[G_\kappa]$ is not the union of the models $M[G_\beta]$ it may behave like a union with respect to “small” sets.

Lemma 27. *In a ground model M let $((P_\alpha, \leq_\alpha, 1_\alpha) \mid \alpha \leq \kappa)$ be the finite support iteration of the sequence $((\dot{Q}_\beta, \dot{\leq}_\beta) \mid \beta < \kappa)$ of limit length κ . Let G_κ be M -generic over P_κ . Consider a sets $S \in M$, $X \in M[G_\kappa]$, $X \subseteq S$ and assume that $M[G_\kappa] \models \text{card}(S) < \text{cof}(\kappa)$. Then there is $\alpha < \kappa$ such that $X \in M[G_\alpha]$ where $G_\alpha = \{p \restriction \alpha \mid p \in G_\kappa\}$.*

Proof. Take $\dot{X} \in M$ and $X = \dot{X}^{G_\kappa}$. Without loss of generality we may assume that $1_\kappa \Vdash \dot{X} \subseteq S$. Work in $M[G_\kappa]$. For all $x \in X$ choose a condition $p_x \in G_\kappa$ such that $p_x \Vdash \check{x} \in \dot{X}$. For every $x \in X$ there is some $\alpha_x < \kappa$ such that $\text{supp}(p_x) \subseteq \alpha_x$. Since $\text{card}(S) < \text{cof}(\kappa)$ take an $\alpha < \kappa$ such that $\alpha_x \leq \alpha$ for all $x \in X$. We claim that

$$(1) X = \{x \in S \mid \exists p \in P_\kappa (p \restriction \alpha \in G_\alpha \wedge \text{supp}(p) \subseteq \alpha \wedge p \Vdash \check{x} \in \dot{X})\}.$$

Proof. If $x \in X$ then p_x satisfies the existential condition on the right. Conversely assume that $p \restriction \alpha \in G_\alpha \wedge \text{supp}(p) \subseteq \alpha \wedge p \Vdash \check{x} \in \dot{X}$. Take $q \in G_\kappa$ such that $p \restriction \alpha = q \restriction \alpha$. Then $\text{supp}(p) \subseteq \alpha$ implies that $q \leq_\kappa p$. Hence $p \in G_\kappa$ and $x \in X$. *qed(1)*

This proves $X \in M[G_\alpha]$. □

Corollary 28. *In the previous lemma let P_κ have the countable chain condition and let κ be an uncountable regular cardinal. Then*

$$\mathcal{P}(\delta) \cap M[G_\kappa] = \mathcal{P}(\delta) \cap \bigcup_{\alpha < \kappa} M[G_\alpha]$$

for all $\delta < \kappa$.

7 Forcing Martin's axiom

Martin's axiom postulates the existence of partially generic sets for *all* ccc forcings. Recalling that every Cohen forcing $\text{Fn}(\lambda, 2, \aleph_0)$ is ccc i.e., this amounts to a proper class of forcings to consider. To reduce the class of requirements to a set that can be dealt with in a set-sized iterated forcing, we show that MA_κ “reflects” down to cardinality κ .

Lemma 29. *For infinite cardinals κ the following are equivalent:*

- a) MA_κ ;
- b) *for every ccc forcing Q whose underlying set is a subset of κ and every $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ with $\text{card}(\mathcal{D}) \leq \kappa$ there exists a \mathcal{D} -generic filter on Q .*

Proof. (a) \rightarrow (b) is obvious. For the converse use a Löwenheim-Skolem downward argument. Let $(P, \leq, 1)$ be a ccc forcing and let the set \mathcal{D} have cardinality $\leq \kappa$. Without loss of generality we may assume that $\mathcal{D} \subseteq \mathcal{P}(P)$ and that every $D \in \mathcal{D}$ is dense in P . Consider the first-order structure

$$(P, \leq, 1, (D)_{D \in \mathcal{D}})$$

with a language of cardinality $\leq \kappa$. By the Löwenheim-Skolem theorem there is an elementary substructure

$$(Q, \leq \cap Q^2, 1, (D \cap Q)_{D \in \mathcal{D}}) \prec (P, \leq, 1, (D)_{D \in \mathcal{D}})$$

such that $\text{card}(Q) \leq \kappa$. By elementarity $(Q, \leq \cap Q^2, 1)$ is a forcing and every $D \cap Q$ is dense in Q . If $A \subseteq Q$ is an antichain in Q then it is an antichain in P . So A is countable and Q is ccc.

We may assume that $Q \subseteq \kappa$. By (b) take a $(D \cap Q)_{D \in \mathcal{D}}$ -generic filter F on Q . We show that

$$G = \{p \in P \mid \exists q \in F q \leq p\}$$

is a \mathcal{D} -generic filter on P . The filter properties are easy. For the \mathcal{D} -genericity consider $D \in \mathcal{D}$. By the $(D \cap Q)_{D \in \mathcal{D}}$ -genericity of F there is $q \in F \cap (D \cap Q)$. Then

$$q \in F \cap (D \cap Q) \subseteq G \cap D \neq \emptyset.$$

□