CLASS FORCING

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We have seen how to modify the continuum function $\kappa \mapsto 2^\kappa$ by forcing both CH and $\neg\mathsf{CH}$. It would be interesting to force generalizations of this such as GCH or forcing $2^\kappa = \kappa^{++}$ for every regular cardinal. However, this is impossible, since every forcing $\mathbb{P}$ satisfies the $|\mathbb{P}|^+$-cc and therefore does not modify the continuum function above κ. To achieve this, we have to generalize forcing and allow *proper class* partial orders.

1. Gödel-Bernays set theory

Since we will now deal with classes, we will work in a second-order context. We work in a two-sorted version of $\mathcal{L}_\in$ with (lowercase letter) variables for sets and (capital letter) variables for classes. More precisely, the formulae are given by

- atomic formulae of the form $v_i \in v_j, v_i = v_j, v_i \in W_j, W_i = W_j$
- if φ, ψ are formulae then so are $\neg\varphi, \varphi \wedge \psi$
- if φ is a formula then so are $\exists v_i \varphi, \exists W_i \varphi$.

By *first-order formulae* we denote the formulae which have no class quantifiers (but classes may appear in atomic subformulae).

The theory GB consists of the *set axioms* given by ZF (where in the schemes of Separation and Replacement formulae are allowed to have class parameters[1]) and the *class axioms*

- $\forall X, Y(X = Y \leftrightarrow \forall z(z \in X \leftrightarrow z \in Y)$ (*Extensionality*)
- $\forall X(X \neq \emptyset \to \exists y \in X \forall z \in y(z \notin X))$ (*Foundation*)
- For every formula φ without class quantifiers,

$$\forall X_0, \dots, X_{n-1} \exists Y \forall z[z \in Y \leftrightarrow \varphi(z, X_1, \dots, X_{n-1})] \quad (\textit{first-order Class Comprehension})$$

Furthermore, we denote by GBC the theory of GB enhanced by *global choice*, i.e.

$$\exists F[F \text{ function } \wedge \forall x \neq \emptyset(x \in \mathrm{dom}(F) \wedge F(x) \in x)].$$

As in the case of ZFC, we write $\mathsf{GB(C)}^-$ for the theory $\mathsf{GB(C)}$ without the power set axiom.

A model of GBC (or one of the weaker subtheories) is of the form $\mathbb{M} = \langle M, \mathcal{C} \rangle$ where M contains the sets and $\mathcal{C}$ contains the classes.

Remark 1.1. If M is a transitive model of ZFC and $\mathrm{Def}(M)$ denotes the collection of classes which are definable over M then $\langle M, \mathrm{Def}(M) \rangle \models \mathsf{GB}$. This shows that GBC is *conservative* over ZFC, i.e. every statement that GBC proves about sets can also be proven in ZFC.

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[1] e.g. for Separation this means: Whenever $\varphi(v_0, v_1, p_0, \dots, p_{m-1}, C_0, \dots, C_{n-1})$ is an $\mathcal{L}_\in$-formula with set parameters $p_0, \dots, p_{m-1}$ and class parameters $C_0, \dots, C_{n-1}$ then $\{y \in x \mid \varphi(x, y, p_0, \dots, p_{m-1}, C_0, \dots, C_{m-1})\}$ is a set.

2. Class forcing

Let $\mathbb{M} = \langle M, \mathcal{C}\rangle$ be a model of GB.

Definition 2.1. A *class forcing* for $\mathbb{M}$ is a triple of the form $\mathbb{P} = \langle P, \leq_{\mathbb{P}}, \mathbb{1}_{\mathbb{P}}\rangle$ where $P, \leq_{\mathbb{P}} \in \mathcal{C}$[2]. A class $D \subseteq \mathbb{P}$ is said to be

- *dense*, if for every $p \in \mathbb{P}$ there is $q \leq_{\mathbb{P}} p$ with $q \in D$.
- *dense below* p for some $p \in \mathbb{P}$, if for every $q \leq_{\mathbb{P}} p$ there is $r \leq_{\mathbb{P}} q$ with $r \in D$.
- *predense*, if for every $p \in \mathbb{P}$ there is $q \in D$ such that p and q are compatible.
- *predense below* p for some $p \in \mathbb{P}$, if for every $q \leq_{\mathbb{P}} p$ there is $r \in D$ which is compatible with q.

A filter $G \subseteq \mathbb{P}$ is $\mathbb{M}$-*generic for* $\mathbb{P}$, if $G \cap D \neq \emptyset$ for each dense class $D \in \mathcal{C}$.

Lemma 2.2. *Let $\mathbb{P}$ be a class forcing for $\mathbb{M}$ and $G \subseteq \mathbb{P}$ a filter. Then the following statements are equivalent:*

(1) G is $\mathbb{M}$-generic for $\mathbb{P}$.
(2) $G \cap D \neq \emptyset$ for every class $D \subseteq \mathbb{P}$ in $\mathcal{C}$ which is dense below som ep $\in \mathbb{P}$.
(3) $G \cap D \neq \emptyset$ for every predense class $D \subseteq \mathbb{P}$ which is in $\mathcal{C}$.
(4) $G \cap D \neq \emptyset$ for every class $D \subseteq \mathbb{P}$ in $\mathcal{C}$ which is predense below some $p \in G$.

Proof. The equivalence of (1) and (2) is shown in Problem 11, Models of Set Theory I. We show first that (1) $\Rightarrow$ (3). Let $D \subseteq \mathbb{P}$ be a predense class in $\mathcal{C}$. Then consider

$$\bar{D} = \{p \in \mathbb{P} \mid \exists q \in D(p \leq_{\mathbb{P}} q)\}.$$

We check that $\bar{D}$ is dense. Let $p \in \mathbb{P}$. Then there is $q \in D$ which is compatible with p. Take $r \leq_{\mathbb{P}} p, q$. Then $r \in \bar{D}$, so $\bar{D}$ is dense. Pick $p \in G \cap \bar{D}$ and $q \in D$ with $p \leq_{\mathbb{P}} q$. Then $q \in G \cap D$.

Suppose now that (3) holds and let $p \in G$ and $D \subseteq \mathbb{P}$ be predense below p. Then

$$\bar{D} = D \cup \{q \in \mathbb{P} \mid q \perp_{\mathbb{P}} p\}$$

is predense, so we can pick $q \in G \cap \bar{D}$. But then p, q are compatible, so $q \in G \cap D$.

The implication (4) $\Rightarrow$ (1) is obvious. $\square$

Definition 2.3. Let $\mathbb{P}$ be a class forcing for $\mathbb{M}$. A $\mathbb{P}$-*name* is a class whose elements are of the form $\langle \sigma, p\rangle$ where σ is a $\mathbb{P}$-name and $p \in \mathbb{P}$. We define $M^{\mathbb{P}}$ to be the class of all $\mathbb{P}$-names which are elements of M and $\mathcal{C}^{\mathbb{P}}$ the collection of $\mathbb{P}$-names which are in $\mathcal{C}$ (denoted *class* $\mathbb{P}$-*names*).

Given an $\mathbb{M}$-generic filter G for $\mathbb{P}$ and $\sigma \in M^{\mathbb{P}}$, we define

$$\sigma^G = \{\tau^G \mid \exists p \in G(\langle \tau, p\rangle \in \sigma)\}$$

and similarly we define Γ^G for $\Gamma \in \mathcal{C}^{\mathbb{P}}$. Furthermore, let

$$M[G] = \{\sigma^G \mid \sigma \in M^{\mathbb{P}}\}$$
$$\mathcal{C}[G] = \{\Gamma^G \mid \Gamma \in \mathcal{C}^{\mathbb{P}}\}$$

and $\mathbb{M}[G] = \langle M[G], \mathcal{C}[G]\rangle$.

Definition 2.4. If $\varphi \equiv \varphi(v_0, \dots, v_{m-1}, \Gamma_0, \dots, \Gamma_{n-1})$ is an $\mathcal{L}_\in$-formula with class name parameters $\Gamma_0, \dots, \Gamma_{n-1} \in \mathcal{C}^{\mathbb{P}}$, $p \in \mathbb{P}$ and $\sigma_0, \dots, \sigma_{m-1} \in M^{\mathbb{P}}$, we write

$$p \Vdash_{\mathbb{P}}^{\mathbb{M}} \varphi(\sigma_0, \dots, \sigma_{m-1}, \Gamma_0, \dots, \Gamma_{n-1})$$

if for $\mathbb{M}$-generic filter G for $\mathbb{P}$ with $p \in G$,

$$\mathbb{M}[G] \models \varphi(\sigma_0^G, \dots, \sigma_{m-1}^G, \Gamma_0^G, \dots, \Gamma_{n-1}^G).$$

[2]We will usually identify $\mathbb{P}$ with its domain P.

Definition 2.5. Let $\mathbb{M}$ be a model of GB. Let $\varphi \equiv \varphi(v_0, \ldots, v_{m-1}, \Gamma_0, \ldots, \Gamma_{n-1})$ be an $\mathcal{L}_\in$-formula with class name parameters $\Gamma_0, \ldots, \Gamma_{n-1} \in \mathcal{C}^{\mathbb{P}}$.

(1) We say that $\mathbb{P}$ *satisfies the definability lemma for* φ *over* $\mathbb{M}$ if

$$\{\langle p, \sigma_0, \ldots, \sigma_{m-1}\rangle \in P \times M^{\mathbb{P}} \times \ldots \times M^{\mathbb{P}} \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \Gamma_0, \ldots, \Gamma_{n-1})\} \in \mathcal{C}.$$

(2) We say that $\mathbb{P}$ *satisfies the truth lemma for* φ *over* $\mathbb{M}$ if for all $\sigma_0, \ldots, \sigma_{m-1} \in M^{\mathbb{P}}$, $\vec{\Gamma} \in (\mathcal{C}^{\mathbb{P}})^n$ and every filter G which is $\mathbb{P}$-generic over $\mathbb{M}$ with

$$\mathbb{M}[G] \models \varphi(\sigma_0^G, \ldots, \sigma_{m-1}^G, \Gamma_0^G, \ldots, \Gamma_{n-1}^G),$$

there is $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma_0, \ldots, \sigma_{m-1}, \Gamma_0, \ldots, \Gamma_{n-1})$.

(3) We say that $\mathbb{P}$ *satisfies the forcing theorem for* φ *over* $\mathbb{M}$ if $\mathbb{P}$ satisfies both the definability lemma and the truth lemma for φ over $\mathbb{M}$.

Question 2.6. *Is it true that if $M \models \mathsf{GB}$ and $\mathbb{P}$ is a class forcing for M then $M[G] \models \mathsf{GB}$ whenever G is $\mathbb{M}$-generic for $\mathbb{P}$? A related question is: Does every class forcing satisfy the forcing theorem?*

The answer is that, in general, the axioms of set theory are not preserved under class forcing (see e.g. Problem 29). Moreover, there are class forcings for which the forcing theorem fails. Before we proceed to more technical results, let us consider an easy application of class forcing.

Definition 2.7. A class forcing $\mathbb{P}$ is $<\mathrm{Ord}$-*closed*, if for every cardinal κ and for every descending sequence $\langle p_\alpha \mid \alpha < \kappa\rangle$ there is $p \in \mathbb{P}$ with $p \leq_{\mathbb{P}} p_\alpha$ for each $\alpha < \kappa$.

Lemma 2.8. *Let $\mathbb{P}$ be a class forcing which is $<\mathrm{Ord}$-closed. Then the following statements hold:*

(1) If G is $\mathbb{M}$-generic for $\mathbb{P}$ then $M[G] = M$.
(2) $\mathbb{P}$ preserves the axioms of GB (resp. $\mathsf{GB} + \mathsf{AC}$ or GBC).

Proof. This follows from Problem 28. □

Definition 2.9. A *global well-order* for M is a class well-order $\prec \in \mathcal{C}$ of M. Note that by Problem 27 the existence of a global well-order for M is equivalent to *global choice*.

Proposition 2.10. *There is a class forcing $\mathbb{P}$ which adds a global well-order for M. More precisely, if $\mathbb{M} \models \mathsf{GB} + \mathsf{AC}$ and G is $\mathbb{M}$-generic for $\mathbb{P}$ then $\mathbb{M}[G] \models \mathsf{GBC}$.*

Proof. Let $\mathbb{P} = \{\prec \in M \mid \exists x \in M(\prec \text{ is a well-order of } x)\}$, ordered by end-extension, i.e. if $p, q \in \mathbb{P}$ then $p \leq_{\mathbb{P}} q$ if and only if $\mathrm{dom}(p) \supseteq \mathrm{dom}(q)$ and $p \upharpoonright \mathrm{dom}(q) \times \mathrm{dom}(q) = q$, where $\mathrm{dom}(p)$ is the set well-ordered by p. Note that $\mathbb{P}$ is $<\mathrm{Ord}$-closed, so $M[G] = M$ and $\mathbb{M}[G] \models \mathsf{GB}$.

Let G be $\mathbb{M}$-generic for $\mathbb{P}$ and consider $\prec = \bigcup G$. For each $x \in M$ consider

$$D_x = \{p \in \mathbb{P} \mid x \subseteq \mathrm{dom}(p)\}.$$

We show that D_x is dense. Let $p \in \mathbb{P}$ and $y = \mathrm{dom}(p)$. Let $z = x \setminus y$ and let q be a well-order of z. Then there is a well-order $r \in \mathbb{P}$ of $x \cup y$ given by

$$\langle a, b\rangle \in r \iff (a, b \in y \wedge \langle a, b\rangle \in p) \vee (a, b \in z \wedge \langle a, b\rangle \in q) \vee (a \in y \wedge b \in z).$$

Then $r \leq_{\mathbb{P}} p$ and $r \in D_x$.

To see that $\prec$ is an ordering of M, let $x \in M$ be arbitary and pick $p \in D_{\{x\}}$. Hence $x \in \mathrm{dom}(p) \subseteq \mathrm{dom}(\prec)$. We show that it well-orders M. To see that it is a linear ordering, let $x, y \in M$. Take $p \in G \cap D_{\{x,y\}}$. Then either $\langle x, y\rangle \in p$ or $\langle y, x\rangle \in p$. In the first case, $x \prec y$ and in the second case, $y \prec x$. Moreover, $\prec$ is a well-order: We have to check that for every $x \in M$, x has a $\prec$-least element. Pick $p \in G \cap D_x$. Then $\prec\upharpoonright \mathrm{dom}(p) \times \mathrm{dom}(p) = p$ is a well-order on $\mathrm{dom}(p) \supseteq x$. But then x has a p-least element y and in particular y is also a $\prec$-least element of x. □

3. Pretameness

Since fundamental theorems of set forcing such as the forcing theorem and the preservation of the axioms of set theory can fail for class forcing, we are interested in finding a condition which guarantees that at least GB^- is preserved and the forcing theorem holds.

Definition 3.1. Let $\mathbb{P}$ be a class forcing for $\mathbb{M}$. We say that $\mathbb{P}$ is *pretame*, if for every sequence of dense (below p) classes $\langle D_\alpha \mid \alpha < \kappa\rangle$ in $\mathcal{C}$ (i.e. $\{\langle p,\alpha\rangle \mid p \in D_\alpha\} \in \mathcal{C}$) and for every $p \in \mathbb{P}$ there is a sequence $\langle d_\alpha \mid \alpha < \kappa\rangle \in M$ and $q \leq_{\mathbb{P}} p$ such that each $d_\alpha \subseteq D_\alpha$ is predense below q.

Remark 3.2. Note that if $M \models \mathsf{AC}$ then we can always assume I to be a cardinal in the above definition.

Proposition 3.3 (S. Friedman)**.** *If $\mathbb{P}$ is pretame then $\mathbb{P}$ satisfies the forcing theorem.*

Proof. We have to show that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$ and $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$ are definable, since the definability of non-atomic formulae follows by induction on the formula complexity. We will define a class function $F : \mathbb{P} \times M^{\mathbb{P}} \times M^{\mathbb{P}} \times 2 \to 2 \times M \setminus \{\emptyset\}$.

We will prove by induction that $F(p,\sigma,\tau,0) = \langle i,d\rangle$ for some $d \subseteq \{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}$ such that either

1. $i = 1$ and for every $q \in d$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$, or
2. $i = 0$ and for every $q \in d$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \notin \tau$;

and similarly $F(p,\sigma,\tau,1) = \langle i,d\rangle$ for some $d \subseteq \{q \in \mathbb{P} \mid q \leq_{\mathbb{P}} p\}$ such that either

3. $i = 1$ and for every $q \in d$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau$, or
4. $i = 0$ and for every $q \in d$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \neq \tau$.

Given such a function, we can define the forcing relation by

$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau \Longleftrightarrow \forall q \leq_{\mathbb{P}} p\, \exists d \in M(F(q,\sigma,\tau,0) = \langle 1,d\rangle)$$
$$p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau \Longleftrightarrow \forall q \leq_{\mathbb{P}} p\, \exists d \in M(F(q,\sigma,\tau,1) = \langle 1,d\rangle).$$

We are left with defining such a function F by induction on the name rank. We start with defining $F(p,\sigma,\tau,0)$. By induction, we may assume that for all $\pi \in \operatorname{dom}(\tau)$ and for all $q \in \mathbb{P}$, $F(q,\sigma,\pi,1)$ has already been defined. There are two cases:

Case 1. There exist $\langle \pi, r\rangle \in \tau$ and $q \leq_{\mathbb{P}} p, r$ such that $F(q,\sigma,\pi,1) = \langle 1,d\rangle$ for some $d \in M$. Let $\alpha \in \mathrm{Ord}$ be minimal such that there is such a $d \in \mathsf{V}_\alpha$. Then put $F(p,\sigma,\tau,0) = \langle 1,e\rangle$ where

$$e = \bigcup\{d \in \mathsf{V}_\alpha \mid \exists\langle\pi,r\rangle \in \tau\, \exists q \leq_{\mathbb{P}} p,r\, F(q,\sigma,\pi,1) = \langle 1,d\rangle\}.$$

Case 2. Suppose that Case 1 fails. For each $\langle \pi,r\rangle \in \tau$, consider

$$D_{\pi,r} = \bigcup\{d \in M \mid \exists q \leq_{\mathbb{P}} r\, F(q,\sigma,\pi,1) = \langle 0,d\rangle\} \cup \{q \leq_{\mathbb{P}} p \mid q \perp_{\mathbb{P}} r\}.$$

We show that $D_{\pi,r}$ is dense below p. Let $q \leq_{\mathbb{P}} p$. If $q \perp_{\mathbb{P}} r$ then we are done. Otherwise take $s \leq_{\mathbb{P}} q, r$. Since Case 1 fails, $F(s,\sigma,\pi,1) = \langle 0,d\rangle$ for some $d \in M \setminus \{\emptyset\}$. Since d is nonempty, take $t \in d$. Then $t \in D_{\pi,r}$ and $t \leq_{\mathbb{P}} s \leq_{\mathbb{P}} q$. By pretameness there are conditions $q \leq_{\mathbb{P}} p$ and $\langle d_{\pi,r} \mid \langle\pi,r\rangle \in \tau\rangle \in M$ such that each $d_{\pi,r}$ is a subset of $D_{\pi,r}$ which is predense below q. Now let $\alpha \in \mathrm{Ord}$ be minimal such that there is such q in V_α. Then put $F(p,\sigma,\tau,0) = \langle 0,e\rangle$ where

$$e = \{q \in \mathsf{V}_\alpha \cap \mathbb{P} \mid \exists \langle d_{\pi,r} \mid \langle \pi,r\rangle \in \tau\rangle \in M(d_{\pi,r} \text{ predense subset of } D_{\pi,r} \text{ below } q)\}.$$

We are left with checking (1) and (2). Suppose that $F(p,\sigma,\tau,0) = \langle 1,e\rangle$. We have to check that for every $q \in e$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \tau$. Take $q \in d$ and an $\mathbb{M}$-generic filter G with $q \in G$. Since we are in Case 1, there is $\langle \pi,r\rangle \in \tau$ and $s \leq_{\mathbb{P}} p, r$ with $F(s,\sigma,\pi,1) = \langle 1,d\rangle$ for some

d and $q \in d$. Then $q \leq_{\mathbb{P}} s$ and so $s \in G$. But by induction, since $\operatorname{rank}(\pi) < \operatorname{rank}(\tau)$, $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \pi$ and so $\sigma^G = \pi^G \in \tau^G$.

Secondly, assume that $F(p,\sigma,\tau,0) = \langle 0, e\rangle$ and let $q \in e$ and G an $\mathbb{M}$-generic filter for $\mathbb{P}$. Now by Case 2 there is a sequence $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \tau\rangle$ of sets $d_{\pi,r} \subseteq D_{\pi,r}$ which are predense below q. Suppose for a contradiction that $M[G] \models \sigma^G \in \tau^G$. Then there is $\langle \pi, r\rangle \in \tau$ with $r \in G$ and $\sigma^G = \pi^G$. Since $d_{\pi,r}$ is predense below q there is $s \in d_{\pi,r} \cap G$. Then s is compatible with r and so there are $d \in M$ and $t \leq_{\mathbb{P}} r$ with $F(t,\sigma,\pi,1) = \langle 0, d\rangle$ and $s \in d$. By induction, $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \neq \pi$, contradicting that $\sigma^G = \pi^G$.

Now we define $F(p,\sigma,\tau,1)$. Again, we may assume that for every $\pi \in \operatorname{dom}(\sigma \cup \tau)$ and for every $q \in \mathbb{P}$, $F(q,\pi,\sigma,0)$ and $F(q,\pi,\tau,0)$ have already been defined. As above, me make a case distinction:

Case 1. There exist $\langle \pi, r\rangle \in \sigma \cup \tau$, $q \leq_{\mathbb{P}} p, r, i \in 2, d, e \in M$ and $s \in d$ such that $F(q,\pi,\sigma,0) = \langle i, d\rangle$ and $F(s,\pi,\tau,0) = \langle 1-i, e\rangle$. Then let $\alpha \in \mathrm{Ord}$ be minimal with the property that such e as above exists in V_α. Then put $F(p,\sigma,\tau,1) = \langle 0, f\rangle$, where

$$f = \bigcup\{e \in \mathsf{V}_\alpha \mid \exists \langle \pi, r\rangle \in \sigma\cup\tau\, \exists q \leq_{\mathbb{P}} p, r\, \exists i \in 2\, \exists d \in M\, \exists s \in d$$
$$(F(q,\pi,\sigma,0) = \langle i, d\rangle \wedge F(s,\pi,\tau,0) = \langle 1-i, e\rangle)\}.$$

Case 2. Suppose that Case 1 fails. For each $\langle \pi, r\rangle \in \sigma \cup \tau$ let

$$D_{\pi,r} = \bigcup\{e \mid \exists q \leq_{\mathbb{P}} r\, \exists i \in 2\, \exists d\, \exists s \in d(F(q,\pi,\sigma,0) = \langle i, d\rangle \wedge F(s,\pi,\tau,0) = \langle i, e\rangle\}$$
$$\cup \{q \in \mathbb{P} \mid q \perp_{\mathbb{P}} r\}.$$

Since Case 1 fails, each $D_{\pi,r}$ is dense below p. By pretameness there are conditions $q \leq_{\mathbb{P}} p$ and $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma\cup\tau\rangle \in M$ such that each $d_{\pi,r}$ is a subset of $D_{\pi,r}$ which is predense below q. Now let $\alpha \in \mathrm{Ord}$ be minimal such that there is such q in V_α. Then put $F(p,\sigma,\tau,0) = \langle 1, f\rangle$ where

$$f = \{q \in \mathsf{V}_\alpha \cap \mathbb{P} \mid \exists \langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma\cup\tau\rangle \in M(d_{\pi,r} \text{ predense subset of } D_{\pi,r} \text{ below } q)\}.$$

It remains to check (3) and (4). Suppose first that $F(p,\sigma,\tau,1) = \langle 1, f\rangle$ and let $q \in f$. Since we are in Case 2, this means that there is a sequence $\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma \cup \tau\rangle \in M$ such that each $d_{\pi,r}$ is a subset of $D_{\pi,r}$ which is predense below q. Now let G be $\mathbb{M}$-generic for $\mathbb{P}$ with $q \in G$. Suppose that $\sigma^G \neq \tau^G$. Then there is $\langle \pi, r\rangle \in \sigma \cup \tau$ such that $r \in G$ and either $\pi^G \in \sigma^G \setminus \tau^G$ or $\pi^G \in \tau^G \setminus \sigma^G$. Without loss of generality, we may assume the former. Now by predensity of $d_{\pi,r}$ there is $s \in d_{\pi,r} \cap G \subseteq D_{\pi,r} \cap G$. But then by definition of $D_{\pi,r}$ there are $d, e \in M$, $i \in 2$ and $t \leq_{\mathbb{P}} r$ and $u \in d$ such that $F(t,\pi,\sigma,0) = \langle i, d\rangle$ and $F(u,\pi,\tau,0) = \langle i, e\rangle$ and $s \in e$. If $i = 0$ then, since $s \in e$, $s \Vdash^{\mathbb{M}}_{\mathbb{P}} \pi \notin \sigma$ and so $\pi^G \notin \sigma^G$, a contradiction. Otherwise, since $s \in e$, $s \leq_{\mathbb{P}} u$ and so $u \in G$. But $u \in d$ and so $u \leq_{\mathbb{P}} t$ and in particular, $t \in G$. Moreover, by induction we have $t \Vdash^{\mathbb{M}}_{\mathbb{P}} \pi \in \tau$ and therefore $\pi^G \in \tau^G$ contradicting our assumption.

For (4), suppose that $F(p,\sigma,\tau,1) = \langle 0, f\rangle$ and $q \in f$. Then we are in Case 1, and so there are $\langle \pi, r \in \sigma \cup \tau, s \leq_{\mathbb{P}} p, r, i \in 2, d, e \in M$ and $t \in d$ such that $F(s,\pi,\sigma,0) = \langle i, d\rangle$ and $F(t,\pi,\tau,0) = \langle 1-i, e\rangle$ and $q \in e$. Let G be $\mathbb{M}$-generic for $\mathbb{P}$ with $q \in G$. Suppose first that $i = 1$. Then $q \Vdash^{\mathbb{M}}_{\mathbb{P}} \pi \notin \tau$ and so $\pi^G \notin \tau^G$. Since $q \in e$, $q \leq_{\mathbb{P}} t$ and so $t \in G$. But $t \in d$ and so $t \Vdash^{\mathbb{M}}_{\mathbb{P}} \pi \in \sigma$, hence $\pi^G \in \sigma^G$. In particular, $\sigma^G \neq \tau^G$ as desired. The case that $i = 0$ is similar.

We now define the forcing relation for atomic formulae containing classes. We claim that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$ if and only if the class

$$D = \{q \leq_{\mathbb{P}} p \mid \exists \langle \tau, r\rangle \in \Gamma(q \leq_{\mathbb{P}} r \wedge q \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma = \tau)\} \in \mathcal{C}$$

is dense below p. Suppose first that $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Gamma$ and let $q \leq_{\mathbb{P}} p$. Take an $\mathbb{M}$-generic filter G for $\mathbb{P}$ with $q \in G$. Then $p \in G$ and so $\sigma^G \in \Gamma^G$. Let $\langle \pi, r\rangle \in \Gamma$ such that $r \in G$ and

$\sigma^G = \pi^G$. Since G is a filter, q and r are compatible and therefore there is $s \leq_{\mathbb{P}} q, r$ with $s \in G$. In particular, $s \in D$. Conversely, suppose that G is $\mathbb{M}$-generic for $\mathbb{P}$ with $p \in G$. By genericity, we can take $q \in D \cap G$. But then there is $\langle \pi, r\rangle \in \Gamma$ such that $q \leq_{\mathbb{P}} r$. Hence $r \in G$ and so $\sigma^G = \tau^G \in \Gamma^G$.

Finally, we define $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \Gamma = \Pi$ for class $\mathbb{P}$-names Γ, Π. But this is equivalent to stipulating that for all $\langle \sigma, s\rangle \in \Gamma$ and for all $\langle \tau, t\rangle \in \Pi$ and for all $q \leq_{\mathbb{P}} p, s$ and for all $r \leq_{\mathbb{P}} r, t$ there are $\bar{q} \leq_{\mathbb{P}} q$ and $\bar{r} \leq_{\mathbb{P}} r$ such that $\bar{q} \Vdash^{\mathbb{M}}_{\mathbb{P}} \sigma \in \Pi$ and $\bar{r} \Vdash^{\mathbb{M}}_{\mathbb{P}} \tau \in \Gamma$.

This shows that the definability lemma holds for atomic formulae. For composite formulae, we can proceed by induction on the formula complexity, since at every step only finitely many new quantifiers are added. The truth lemma follows from the definability lemma as in set forcing. □

Remark 3.4. Problem 29 shows that the converse is false. However, there are class forcings which do not satisfy the forcing theorem. Moreover, if $\mathbb{P}$ is non-pretame then there is a class forcing $\mathbb{Q}$ and a dense embedding $\mathbb{P} \to \mathbb{Q}$ such that $\mathbb{Q}$ does not satisfy the forcing theorem.

Proposition 3.5. *If $\mathbb{P}$ is pretame and G is $\mathbb{M}$-generic for $\mathbb{P}$ then $\mathbb{M}[G] \models \mathsf{GB}^-$. If $\mathbb{M}$ is a model of global choice then so is $\mathbb{M}[G]$.*

Proof. All set axioms except for Replacement, Separation, Union have exactly the same proof as for set forcing. We prove Replacement: Suppose that

$$\mathbb{M}[G] \models \forall x \in \sigma^G \, \exists! y \, \varphi(x, y, \tau^G, \Gamma^G),$$

where τ^G is a set parameter and Γ^G is a class parameter. Let $p \in G$ with $p \Vdash^{\mathbb{M}}_{\mathbb{P}} \forall x \in \sigma \, \exists! y \, \varphi(x, y, \tau, \Gamma)$. Now for each $\langle \pi, r\rangle \in \sigma$, the class

$$D_{\pi,r} = \{q \in \mathbb{P} \mid (q \leq_{\mathbb{P}} p, r \wedge \exists \mu \in M^{\mathbb{P}}(q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\pi, \mu, \tau, \Gamma)) \vee q \perp_{\mathbb{P}} r\}$$

is dense below p. Now for each $\alpha \in \mathrm{Ord}$ let

$$e^{\alpha}_{\pi,r} = \{q \in D_{\pi,r} \cap \mathsf{V}_{\alpha} \mid q \leq_{\mathbb{P}} r \to \exists \mu \in M^{\mathbb{P}} \cap \mathsf{V}_{\alpha}(q \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\pi, \mu, \tau, \Gamma))\} \in M.$$

We claim that the class

$$E = \{q \in \mathbb{P} \mid \exists \alpha \in \mathrm{Ord} \, \forall \langle \pi, r\rangle \in \sigma (e^{\alpha}_{\pi,r} \text{ is predense below } q)\}$$

is dense below p. Now let $q \leq_{\mathbb{P}} p$. By pretameness there is $r \leq_{\mathbb{P}} q$ and there are sets $d_{\pi,r} \subseteq D_{\pi,r}$ which are predense below r. Now let $\alpha = \mathrm{rank}(\langle d_{\pi,r} \mid \langle \pi, r\rangle \in \sigma\rangle)$. Then each $d_{\pi,r}$ is a subset of $e_{\pi,r}$ and so $e_{\pi,r}$ is also predense below r. In particular, $r \in E$. By genericity, pick $q \in E \cap G$ and $\alpha \in \mathrm{Ord}$ witnessing that $q \in E$. Then put

$$\nu = \{\langle \mu, s\rangle \mid \mu, s \in \mathsf{V}_{\alpha} \wedge \exists \langle \pi, r\rangle \in \sigma (s \leq_{\mathbb{P}} r \wedge s \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\pi, \mu, \tau, \Gamma))\}.$$

It is easy to check that $\nu^G = \mathrm{ran}(f)$, where $f : \sigma^G \to M[G]$ is the function $x \mapsto y$, where $y \in M[G]$ is unique such that $\mathbb{M}[G] \models \varphi(x, y, \tau^G, \Gamma^G)$.

Separation follows from Replacement, and Union follows from Separation: Let $\sigma \in M^{\mathbb{P}}$. Then $\tau = \bigcup \mathrm{dom}(\sigma) \in M^{\mathbb{P}}$ and so $\bigcup \sigma^G = \{x \in \tau^G \mid \exists y \in \sigma^G (x \in y)\} \in M[G]$ using Separation.

Foundation and extensionality for classes are clear. For class comprehension, note that $\Gamma = \{\langle \sigma, p\rangle \mid p \Vdash^{\mathbb{M}}_{\mathbb{P}} \varphi(\sigma, \Gamma_0, \ldots, \Gamma_{n-1})\}$ is a class name for the class $\{x \mid \varphi(x, \Gamma^G_0, \ldots, \Gamma^G_{n-1})\}$. For global choice, suppose that $\prec$ is a global well-order in $\mathcal{C}$. Then

$$x \lhd y \iff \exists \sigma \in M^{\mathbb{P}}[x = \sigma^G \wedge \forall \tau \in M^{\mathbb{P}}(y = \tau^G \to \sigma \prec \tau)]$$

defines a global-wellorder of $M[G]$ in $\mathcal{C}[G]$. □

Proposition 3.6. *Let $\mathbb{P}$ be a class forcing such that for each $p \in \mathbb{P}$ there is an $\mathbb{M}$-generic filter $G \subseteq \mathbb{P}$ with $p \in G$ such that $\mathbb{M}[G] \models \mathsf{GB}^-$. Then $\mathbb{P}$ is pretame.*

Proof. This is exactly Problem 30. □

Corollary 3.7. *A class forcing $\mathbb{P}$ is pretame if and only if it preserves* GB^-.

4. Easton's Theorem

Definition 4.1. An *Easton index function* is a class function $F : \mathrm{dom}(F) \to \mathrm{Card}$ such that for $\kappa, \lambda \in \mathrm{dom}(F)$,

- κ is a regular cardinal
- $\mathrm{cf}(F(\kappa)) > \kappa$
- if $\kappa < \lambda$ then $F(\kappa) \leq F(\lambda)$.

Definition 4.2. Let F be an Easton index function. The associated *Easton forcing* $\mathbb{P}_F$ be the class forcing with conditions p with $\mathrm{dom}(p) \subseteq \mathrm{dom}(F)$ and for each $\kappa \in \mathrm{dom}(p)$, $p(\kappa) \in \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$ and for all regular cardinals λ,

$$|\mathrm{dom}(p) \cap \lambda| < \lambda.$$

The ordering of $\mathbb{P}_F$ is given by reverse inclusion.

Definition 4.3. Let F be an Easton index function, $p \in \mathbb{P}_F$ and λ a regular cardinal. Then we define

$$p^{\leq\lambda} = p \restriction \lambda^+, \qquad p^{>\lambda} = p \restriction \mathrm{Ord} \setminus \lambda^+.$$

Furthermore, let

$$\mathbb{P}_F^{\leq\lambda} = \{p^{\leq\lambda} \mid p \in \mathbb{P}_F\}, \qquad \mathbb{P}_F^{>\lambda} = \{p^{>\lambda} \mid p \in \mathbb{P}_F\}.$$

Remark 4.4. If $p \in \mathbb{P}_F$ then $p = p^{\leq\lambda} \cup p^{>\lambda}$. In particular, this shows that $\mathbb{P}_F$ is isomorphic to $\mathbb{P}_F^{\leq\lambda} \times \mathbb{P}_F^{>\lambda}$.

Lemma 4.5. *Let λ be a regular cardinal. Then $\mathbb{P}^{>\lambda}$ is λ^+-closed.*

Proof. Let $\langle p_i \mid i < \lambda\rangle$ be a descending sequence in $\mathbb{P}^{>\lambda}$. Let $p = \bigcup_{i<\lambda} p_i$. Then $\mathrm{dom}(p) = \bigcup_{i<\lambda} \mathrm{dom}(p_i) \subseteq \mathrm{Ord} \setminus \lambda^+$. Now if $\kappa \in \mathrm{dom}(p)$ then $\kappa \geq \lambda^+$ and so

$$|\mathrm{dom}(p(\kappa))| = |\bigcup_{i<\lambda} \mathrm{dom}(p_i(\kappa))| < \kappa$$

since κ is regular.

Moreover, if $\kappa \geq \lambda^+$ is a regular cardinal, then

$$|\mathrm{dom}(p) \cap \kappa| = |\bigcup_{i<\lambda} \mathrm{dom}(p_i) \cap \kappa| < \kappa.$$

This shows that $p \in \mathbb{P}^{>\lambda}$ and $p \leq_{\mathbb{P}^{>\lambda}} p_i$ for each $i < \lambda$. □

Lemma 4.6. *For every regular cardinal λ, $\mathbb{P}^{\leq\lambda}$ satisfies the λ^+-cc.*

Proof. For $p \in \mathbb{P}^{\leq\lambda}$ let

$$d(p) = \bigcup\{\{\kappa\} \times \mathrm{dom}(p(\kappa)) \mid \kappa \in \mathrm{dom}(p) \cap \lambda^+\}.$$

Then by assumption, $|d(p)| < \lambda$. Now suppose that $A \subseteq \mathbb{P}^{\leq\lambda}$ is an antichain of size λ^+. Since $M \models \mathsf{GCH}$, $\lambda^{<\lambda} = \lambda$ and so we can apply the Δ-System Lemma. There is $B \subseteq A$ of size λ^+ such that

$$\{d(p) \mid p \in B\}$$

forms a Δ-system with root r, i.e. for all $p, q \in B$ with $p \neq q$, $d(p) \cap d(q) = r$. So $|r| < \lambda$ and using the GCH we have $2^{|r|} \leq \lambda$. But then there is $C \subseteq B$ of size λ^+ such that for all $p, q \in C$ and for all $\langle\kappa, x\rangle \in r$, $p(\kappa)(x) = q(\kappa)(x)$. But then all elements of C are compatible, a contradiction. □

We will use a class version of the Product lemma:

Lemma 4.7. *Let $\mathbb{P} = \mathbb{P}_0 \times \mathbb{P}_1$, where $\mathbb{P}_0, \mathbb{P}_1$ are class forcings for $\mathbb{M}$ such that $\mathbb{P}_0$ satisfies the forcing theorem. Then the following statements hold:*

(1) If G_0 is $\mathbb{M}$-generic for $\mathbb{P}_0$ and G_1 is $\mathbb{M}[G_0]$-generic then $G_0 \times G_1$ is $\mathbb{M}$-generic for $\mathbb{P}$.

(2) If G is $\mathbb{M}$-generic for $\mathbb{P}$ then G is of the form $G_0 \times G_1$, where G_0 is $\mathbb{M}$-generic for $\mathbb{P}_0$ and G_1 is $\mathbb{M}[G_0]$-generic for $\mathbb{P}_1$.

Proof. The proof is the same as for set forcing. □

Lemma 4.8. *Let $\mathbb{P}$ and $\mathbb{Q}$ be class forcings such that $\mathbb{P} \times \mathbb{Q}$ satisfies the forcing theorem, $\mathbb{P}$ is λ^+-closed and $\mathbb{Q}$ satisfies the λ^+-cc. If $G \times H$ is $\mathbb{M}$-generic for $\mathbb{P} \times \mathbb{Q}$ then every function $f : \lambda \to M$ in $M[G \times H]$ is in $M[H]$. In particular,*

$$\mathcal{P}^{M[G\times H]}(\lambda) = \mathcal{P}^{M[H]}(\lambda).$$

Proof. Let $\dot{f} \in M^{\mathbb{P}\times\mathbb{Q}}$ be a name such that $\dot{f}^{G\times H} = f$. Without loss of generality, we may assume that $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} \Vdash^{\mathbb{M}}_{\mathbb{P}} \dot{f} : \check{\lambda} \to \check{X}$ for some $X \in M$. For $\alpha < \lambda$, consider

$$D_\alpha = \{p \in \mathbb{P} \mid \exists A \subseteq \mathbb{Q}(A \text{ max. antichain} \wedge \forall q \in A\, \exists x \in X(\langle p, q\rangle \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\alpha}) = \check{x}))\}.$$

Clearly, each D_α is open. It is also dense: Let $p \in \mathbb{P}$. By induction, we construct a descending sequence $\langle p_i \mid i < \delta\rangle$ of conditions in $\mathbb{P}$ and a sequence $\langle q_i \mid i < \delta\rangle$ of conditions in $\mathbb{Q}$ and $\langle x_i \mid i < \delta\rangle$ of sets in X such that $\delta < \lambda^+$ for all $i < \delta$,

$$\langle p_i, q_i\rangle \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\alpha}) = \check{x} \text{ for some } x \in X$$

and $\langle q_i \mid i < \delta\rangle$ enumerates a maximal antichain in $\mathbb{Q}$. Since $\mathbb{1}_{\mathbb{P}\times\mathbb{Q}} \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f} : \check{\lambda} \to \check{X}$ there are $\langle p_0, q_0\rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle p, \mathbb{1}_{\mathbb{Q}}\rangle$ and $x_0 \in X$ with $\langle p_0, q_0\rangle \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\alpha}) = \check{x}_0$.

Suppose that $\langle p_i \mid i < \gamma\rangle, \langle q_i \mid i < \gamma\rangle, \langle x_i \mid i < \gamma\rangle$ have already been defined for some $\gamma < \lambda^+$. Since $\mathbb{P}$ is λ^+-closed there is $\bar{p}_\gamma$ such that $\bar{p}_\gamma \leq_{\mathbb{P}} p_i$ for each $i < \gamma$. If $\{q_i \mid i < \gamma\}$ is a maximal antichain then $\bar{p}_\gamma \in D_\alpha$ and we are done. Otherwise, there is $\bar{q}_\gamma \in \mathbb{Q}$ which is incompatible with each q_i, $i < \gamma$. Now choose $\langle p_\gamma, q_\gamma\rangle \leq_{\mathbb{P}\times\mathbb{Q}} \langle \bar{p}_\gamma, \bar{q}_\gamma\rangle$ with $\langle p_\gamma, q_\gamma\rangle \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\alpha}) = \check{x}_\gamma$ for some $x_\gamma \in X$.

Since $\mathbb{Q}$ satisfies the λ^+-cc, there must be some stage $\delta < \lambda^+$ such that $\{q_i \mid i < \delta\}$ is a maximal antichain. But then $\bar{p}_\delta$ as defined above strenghtens p and lies in D_α.

Moreover, since $\mathbb{P}$ is λ^+-closed, $D = \bigcap_{\alpha<\lambda} D_\alpha$ is dense. By genericity pick $p \in G \cap D$. For each $\alpha < \lambda$ we can choose a maximal antichain $A_\alpha \subseteq \mathbb{Q}$ and a family $\{x^\alpha_{p,q} \mid q \in A_\alpha\} \subseteq X$ such that

$$\langle p, q\rangle \Vdash^{M}_{\mathbb{P}\times\mathbb{Q}} \dot{f}(\check{\alpha}) = \check{x}^\alpha_{p,q}$$

for each $q \in A_\alpha$. By genericity of H, for each $\alpha < \lambda$ there is a unique $q_\alpha \in A_\alpha \cap H$. But then we have

$$f(\alpha) = x^\alpha_{p,q_\alpha}$$

and so $f \in M[H]$. □

Proposition 4.9. *Let $M \models \mathsf{GB} + \mathsf{AC} + \mathsf{GCH}$ and let F be an Easton index function in $\mathcal{C}$. If G is $\mathbb{M}$-generic for $\mathbb{P}_F$ then $\mathbb{M}[G] \models \mathsf{GB} + \mathsf{AC}$ and for each $\kappa \in \mathrm{dom}(F)$, $M[G] \models 2^\kappa = F(\kappa)$. Moreover, $\mathbb{P}_F$ preserves all cardinals and cofinalities.*

Proof. For the sake of notational simplicity, we will write $\mathbb{P}$ for $\mathbb{P}_F$ and correspondingly $\mathbb{P}^{\leq\lambda}$ and $\mathbb{P}^{>\lambda}$ instead of $\mathbb{P}^{\leq\lambda}_F$ and $\mathbb{P}^{>\lambda}_F$. Note that since $\mathbb{P}$ is isomorphic to $\mathbb{P}^{>\lambda} \times \mathbb{P}^{\leq\lambda}$, we can write G as $G^{>\lambda} \times G^{\leq\lambda}$ and we have $M[G] = M[G^{>\lambda}][G^{\leq\lambda}]$.

To see that $\mathbb{P}$ is pretame, let $p \in \mathbb{P}$ and $\langle D_i \mid i < \lambda\rangle$ be a sequence of dense classes below p and suppose that λ is a regular cardinal. Let $\langle q_i \mid i < \lambda\rangle$ be an enumeration of $\mathbb{P}^{\leq\lambda}$ and let $h : \lambda \times \lambda \to \lambda$ be Gödel pairing. We define a descending sequence $\langle p_i \mid i < \lambda\rangle$ of conditions in $\mathbb{P}$ such that $p_i^{\leq\lambda} = p^{\leq\lambda}$ for each $i < \lambda$.

- Let $p_0 = p$.
- Given $\langle p_j \mid j < i\rangle$ let $\bar{p}_i = \bigcup_{j<i} p_i$. Then $\bar{p}_i \in \mathbb{P}$ since $\mathbb{P}^{>\lambda}$ is λ^+-closed. Suppose that $i = h(i_0, i_1)$. Then choose $p_i \leq_{\mathbb{P}} \bar{p}_i$ with $p_i^{\leq\lambda} = p^{\leq\lambda}$ such that there is some $r_i \in D_{i_1}$ with $q_{i_0} \cup r \leq_{\mathbb{P}} r_i$, if possible. Otherwise $p_i = \bar{p}_i$.

Now let $\bar{p} = \bigcup_{i<\lambda} p_i$ and $d_i = \{r_j \mid r_j \in D_i\}$. Then d_i is predense below $\bar{p}$: Suppose that $r \leq_{\mathbb{P}} \bar{p}$. Let $s \leq_{\mathbb{P}} r$ with $s \in D_i$. Let $j < \lambda$ such that $s^{\leq\lambda} = q_j$. Put $k = h(j, i)$. Then $p_k \cup q_j \leq_{\mathbb{P}} r_k$ and $r_k \in D_i$. In particular, $s \leq_{\mathbb{P}} p_k \cup q_j \leq_{\mathbb{P}} r_k$ and $s \leq_{\mathbb{P}} r$, so r is compatible with $r_k \in d_i$.

Furthermore, $\mathbb{P}$ preserves the power set axiom: Note that it is enough to check that for every regular cardinal λ, $\mathcal{P}(\lambda)$ exists. Now observe that by Lemma 4.8,

$$\mathcal{P}^{M[G]}(\lambda) = \mathcal{P}^{M[G^{\leq\lambda}]}(\lambda)$$

but this exists since $\mathbb{P}^{\leq\lambda}$ is a set-sized forcing.

Next, suppose that κ is a regular cardinal in M. We need to check that κ is regular in $M[G]$. Suppose for a contradiction that there is some map $f : \lambda \to \kappa$ in $M[G]$ for some regular M-cardinal $\lambda < \kappa$. By Lemma 4.8, $f \in M[G^{\leq\lambda}]$. But $\mathbb{P}^{\leq\lambda}$ is a set forcing which satisfies the λ^+-cc, so κ is remains regular in $M[G^{\leq\lambda}]$.

We check that for each regular $\kappa \in \mathrm{dom}(F)$, $(2^\kappa)^{M[G]} = F(\kappa)$. Fix $\kappa \in \mathrm{dom}(F)$. Note that by Lemma 4.8, $(2^\kappa)^{M[G]} = (2^\kappa)^{M[G^{\leq\kappa}]}$. Since $G^\kappa = \{p(\kappa) \mid p \in G\} \in M[G^{\leq\kappa}]$ is $\mathbb{M}$-generic for $\mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$ and this forcing adds $F(\kappa)$-many Cohen subsets of κ, $(2^\kappa)^{M[G^{\leq\kappa}]} \geq F(\kappa)$. For the converse, note that in M,

$$|\mathbb{P}^{\leq\kappa}| = 2^{<\kappa} \cdot F(\kappa)^{<\kappa} = F(\kappa)$$

since $M \models \mathsf{GCH}$ and $\mathrm{cof}(F(\kappa)) > \kappa$. But then since $\mathbb{P}^{\leq\kappa}$ has the κ^+-cc, by Theorem 4.2.16 (Models of Set Theory I),

$$(2^\kappa)^{M[G^{\leq\kappa}]} \leq ((|\mathbb{P}^{\leq\kappa}|^{<\kappa^+})^\kappa)^M = F(\kappa)^\kappa = F(\kappa)$$

as desired.

□

Corollary 4.10 (Easton's Theorem)**.** *Let $M \models \mathsf{ZFC} + \mathsf{GCH}$ and let F be a definable Easton index function. If G is $\langle M, \mathrm{Def}(M)\rangle$-generic for $\mathbb{P}_F$ then $M[G] \models \mathsf{ZFC}$ and for each $\kappa \in \mathrm{dom}(F)$, $M[G] \models 2^\kappa = F(\kappa)$.* □

Corollary 4.11. *If ZFC is consistent then so is $\mathsf{ZFC} + \forall\kappa$ regular$(2^\kappa = \kappa^{++})$.* □