Simplified Constructibility Theory

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Prologue

These lecture notes were written for a lecture course on constructibility and fine structure theory at the University of Bonn in the summer term of 2007.

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Noch einfügen: Other approaches to the constructible sets, recursive truth

predicate, Ordinal Turing Machine, Ordinal Register Machine.

Chapter 1
Introduction

After the bround-breaking independence results of PAUL J. COHEN, foundational set theory is *axiomatic set theory* - the study of various axiomatic systems for sets and of models of these systems. The ZERMELO-FRAENKEL axioms (ZF) are usually taken as a base system adequate for most mathematical purposes. By the GÖDEL *incompleteness theorems*, this system cannot decide certain questions. GÖDEL and COHEN showed that the incompleteness phenomenon is not restricted to artificial, e.g., self-referential statement, but it extends to core mathematical properties like the *axiom of choice* (AC) and CANTOR's *continuum hypothesis* (CH), in spite of the general mathematical strength of ZF.

One of the most important results of axiomatic set theory is KURT GÖDEL's *proof* of the *unprovability* of the negation of the continuum hypothesis, i.e., its (*relative*) *consistency*, in notes and articles published between 1938 and 1940 [6], [8], [7], [9]. GÖDEL presented his results in various forms which we can subsume as follows: there is a set-theoretic term L, called the *constructible universe* or the class of *constructible sets*, such that

$$\mathrm{ZF} \vdash \text{“}(L, \in) \vDash \mathrm{ZF} + \text{the axiom of choice (AC)} + \mathrm{CH}\text{”}.$$

So the ZERMELO-FRAENKEL axiomatic system ZF "sees" a model for the stronger theory ZF + AC + CH. If the system ZF is consistent, then so is ZF+AC+CH. The potentially problematic axioms AC and CH may be assumed without increasing the danger of contradictions. Later, PAUL COHEN has complemented these results by showing the relative consistencies of ZF + AC + ¬CH and ZF + ¬ AC. The axioms AC and CH are thus *independent* of ZF in way which is similar to the independence of the parallel postulate from the basic geometric axioms.

In ZF, the constructible universe has many remarkable properties. L is the $\subseteq$-minimal inner model of ZF, i.e., the $\subseteq$-smallest model of ZF which is transitive and contains the class Ord of ordinals. It has a very uniform structure and heuristically it appears that "ordinary" mathematical statements φ are *decided* in L, i.e.,

$$\mathrm{ZF} \vdash \text{“}(L, \in) \vDash \varphi\text{” or } \mathrm{ZF} \vdash \text{“}(L, \in) \vDash \neg\varphi\text{”}.$$

L satisfies strong combinatorial principles which can be used for the construction of specific structures. RONALD JENSEN developed a *fine structural* analysis of L to prove novel principles like $\diamondsuit$ or $\square$. The study of L also has global consequences outside the model L. The famous JENSEN *covering theorem* states that if the set theoretical universe deviates much from L then there must be inner models with *large cardinals*.

The construction of the constructible universe is motivated by the idea of recursively constructing a minimal model of ZF. The central ZF-axiom is ZERMELO's *comprehension schema* (*axiom of subsets*): for every $\in$-formula $\varphi(v, \vec{w})$ postulate

$$\forall x \forall \vec{p}\ \{v \in x | \varphi(v, \vec{p})\} \in V.$$

This requires closing up under operations like

$$(x, \vec{p}) \longmapsto \{v \in x | \varphi(v, \vec{p})\},$$

but with the difficulty where to evaluate the formula φ. This instance of comprehension has to be satisfied in the model to be built eventually, and the quantifiers of φ may have to range about sets which have not yet been included in the construction. To avoid circularities, one evaluates the formula in the sets already constructed and considers modified definability operations

$$(x, \vec{p}) \longmapsto \{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}.$$

The set $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ is determined by the parameters $x, \varphi, \vec{p}$. We shall later view $\{v \in x | (x, \in) \vDash \varphi(v, \vec{p})\}$ as the *interpretation* of the *name* $(x, \varphi, \vec{p})$ and base our fine structure theory on this perspective.

These lecture notes present a comprehensive constructibility theory from the definition of the *constructible hierarchy* to advanced combinatorial principles like $\square$ and *morass* and some applications. We follow the further development of constructibility theory via the JENSEN *covering theorem* up to the *core model* of TONY DODD and JENSEN. The treatment is based on the *hyperfine structure theory* of SY FRIEDMAN and the present author. Roughly speaking, hyperfine structure theory refines GÖDEL's constructible hierarchy so that it can be used like a SILVER *machine*. Such machines were defined by JACK SILVER to simplify applications of JENSEN's fine structure theory; SILVER machines suffer, however, from their abstract nature which is only connected to the constructible universe through rather awkward coding. In hyperfine structure theory the uniform combinatorics of SILVER machines is used in the intuitive context of GÖDEL's canonical hierarchy.

This book owes a great deal to previous presentations of constructibility theory:

- Keith Devlin. Constructibility. Springer
- Sy Friedman and Peter Koepke. An elementary approach to the fine structure of L. Bulletin on Symbolic Logic, 1997
- Kurt Gödel. The consistency of the axiom of choice and of the generalized continuum-hypothesis. *Proc. of the Nat. Acad. of Sciences USA*, 24:556–557, 1938.
- Ronald Jensen. The fine structure of the constructible hierarchy. *Annals of Mathematical Logic*, 1972

Chapter 2
The Language of Set Theory

The intuitive notion of set was described by Georg Cantor:

> Unter einer Menge verstehen wir jede Zusammenfassung M von bestimmten, wohlunterschiedenen Objekten m unsrer Anschauung oder unseres Denkens (welche die ,,Elemente" von M genannt werden) zu einem Ganzen. [Cantor, S. 282; *By a set we understand every collection M of definite, distinguished objects m of our perceptions or thoughts (which are called the "elements" of M) into a whole.*]

The idea of a *collection* ("Zusammenfassung") is usually formalized by *class terms*:

$$M = \{m \,|\, \varphi(m)\}.$$

M is the *class* of all m which satisfy the (mathematical) property φ. The transfer from the definining property φ to the corresponding collection $M = \{m \,|\, \varphi(m)\}$ corresponds to the view of working with abstract "objects", namely *classes*, instead of "immaterial" *properties*. How such classes, including the problematic Russell class $\{m \,|\, m \notin m\}$ can reasonably and consistently be treated as mathematical objects is a matter of set theoretical and foundational concern. It is usually answered by the Zermelo-Fraenkel axioms of set theory which we shall introduce in the next chapter.

One can develop a *class theory*, describing properties of class terms and define complex terms from given ones. Set theory takes the view that sets are "small" classes. The language of class terms is thus also the language of set theory — or even of mathematics, if we think of all of mathematics as formalized within set theory.

2.1 Class Terms

Classes or collections may be queried for certain elements: m is an element of $M = \{m \,|\, \varphi(m)\}$ if it satisfies the defining property φ. In symbols:

$$m \in M \text{ if and only if } \varphi(m).$$

So in $m \in \{m \,|\, \varphi(m)\}$ the class term may be eliminated by just writing the property or *formula* φ. Carrying out this kind of elimination throughout mathematics shows that all mathematical terms and properties may be reduced to basic formulas without class terms. The basic language can be chosen extremely small, but we may also work in a very rich language employing class terms.

The set theoretic analysis of mathematics shows that the following basic language is indeed sufficient:

Definition 2.1. *The (basic)* ***language of set theory*** *has variables* v_0, v_1, ... *The* ***atomic formulas*** *of the language are the formulas* $x = y$ *("x equals y") and* $x \in y$ *("x is an element of y") where x and y are variables. The collection of* ***formulas*** *of the language is the smallest collection* $L(\in)$ *which contains the atomic formulas and is closed under the following rules:*

- *if* φ *is a formula then* $\neg\varphi$ *("not φ") is a formula;*
- *if* φ *and* ψ *are formulas then* $\varphi \vee \psi$ *("φ or ψ") is a formula;*
- *if* φ *is a formula and x is a variable then* $\exists x\,\varphi$ *("there is x such that φ") is a formula.*

A formula is also called an $\in$*-formula. As usual we understand other propositional operators or quantifiers as abbreviations. So* $\varphi \wedge \psi$, $\varphi \rightarrow \psi$, $\varphi \leftrightarrow \psi$ *and* $\forall x\varphi$ *stand for* $\neg(\neg\varphi \vee \neg\psi)$, $\neg\varphi \vee \psi$, $(\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)$ *and* $\neg\exists x\neg\varphi$ *respectively. Also the formula* $\varphi\frac{\vec{y}}{\vec{x}}$ *is obtained from* φ *by* ***substituting*** *the variables* $\vec{x}$ *by* $\vec{y}$.

This is a first-order language. If an $\in$-formula φ is a *logical consequence* of a collection Φ of $\in$-formulas ($\Phi \vdash \varphi$) then there is a finite formal *derivation* of φ from Φ in a first-order *calculus*. Such a calculus may consist of equality rules, propositional rule and substitution rules together with quantifier rules for the introduction and elimination of existential quantifiers.

We now introduce a rich language involving class terms.

Definition 2.2. *A* ***class term*** *is a symbol sequence of the form*

$$\{x \mid \varphi\} \text{ ("the class of } x \text{ such that } \varphi\text{")}$$

where x is one of the variables v_0, v_1, ... *and* φ *is an* $\in$*-formula. A* ***term*** *is a variable or a class term. We now allow arbitrary terms to be used in (atomic) formulas. A* ***generalized atomic formula*** *is a formula of the form* $s = t$ *or* $s \in t$ *where s and t are terms. Form the* ***generalized formulas*** *from the generalized atomic formulas by the same rules as in the previous definition.*

Generalized formulas can be translated into strict $\in$-formulas according to the above intuition of class and collection. It suffices to define the elimination of class terms for generalized atomic formulas. So we recursively translate

$$\begin{aligned}
y \in \{x \mid \varphi\} &\text{ into } \varphi\frac{y}{x},\\
\{x \mid \varphi\} = \{y \mid \psi\} &\text{ into } \forall z(z \in \{x \mid \varphi\} \leftrightarrow z \in \{y \mid \psi\}),\\
x = \{y \mid \psi\} &\text{ into } \forall z(z \in x \leftrightarrow z \in \{y \mid \psi\}),\\
\{y \mid \psi\} = x &\text{ into } \forall z(z \in \{y \mid \psi\} \leftrightarrow z \in x),\\
\{x \mid \varphi\} \in \{y \mid \psi\} &\text{ into } \exists z(\psi\frac{z}{y} \wedge z = \{x \mid \varphi\}),\\
\{x \mid \varphi\} \in y &\text{ into } \exists z(z \in y \wedge z = \{x \mid \varphi\}).
\end{aligned}$$

The translation of the equalities corresponds to the intuition that a class is determined by its extent rather by the specific formula defining it. If at least one of s and t is a class term, then by the elimination procedure

$$s=t \text{ iff } \forall z\,(z\in s\leftrightarrow z\in t).$$

We also have $x=\{y\,|\,y\in x\}$ where we assume a reasonale choice of variables. In this case this means that x and y are different variables. Under the natural assumption that our term calculus satisfies the usual laws of equality, we get

$$\begin{aligned} x=y \;\; &\text{iff} \;\; \{v\,|\,v\in x\}=\{v\,|\,v\in y\} \\ &\text{iff} \;\; \forall z\,(z\in\{v\,|\,v\in x\}\leftrightarrow z\in\{v\,|\,v\in y\}) \\ &\text{iff} \;\; \forall z\,(z\in x\leftrightarrow z\in y). \end{aligned}$$

This is the *axiom of extensionality* for sets, which will later be part of the set-theoretical axioms. We have obtained it here assuming that $=$ for class terms is transitive. In our later development of set theory from the Zermelo-Fraenkel axioms one would rather have to show these axioms imply the equality laws for class terms.

2.2 Extending the Language

We introduce special names and symbols for important class terms and formulas. Naming and symbols follow traditions and natural intuitions. In principle, all mathematical notions could be interpreted this way, but we restrict our attention to set theoretical notions. We use also many usual notations and conventions, like $x\neq x$ instead of $\neg x=x$.

Definition 2.3. *Define the following class terms and formulas:*

a) $\emptyset:=\{x\,|\,x\neq x\}$ *is the* ***empty class****;*

b) $x\subseteq y:=\forall z\,(z\in x\rightarrow z\in y)$ *denotes that x is a* ***subclass*** *of y;*

c) $\{x\}:=\{y\,|\,y=x\}$ *is the* ***singleton*** *of x;*

d) $\{x,y\}:=\{z\,|\,z=x\vee z=y\}$ *is the* ***unordered pair*** *of x and y;*

e) $(x,y):=\{\{x\},\{x,y\}\}$ *is the* ***(ordered) pair*** *of x and y;*

f) $\{x_0,\ldots,x_{n-1}\}:=\{y\,|\,y=x_0\vee\ldots\vee y=x_{n-1}\}$*;*

g) $x\cap y:=\{z\,|\,z\in x\wedge z\in y\}$ *is the* ***intersection*** *of x and y;*

h) $x\cup y:=\{z\,|\,z\in x\vee z\in y\}$ *is the* ***union*** *of x and y;*

i) $x\setminus y:=\{z\,|\,z\in x\wedge z\notin y\}$ *is the* ***difference*** *of x and y;*

j) $\overline{x}:=\{y\,|\,y\in x\}$ *is the* ***complement*** *of x;*

k) $\bigcap x:=\{z\,|\,\forall y\,(y\in x\rightarrow z\in y)\}$ *is the* ***intersection*** *of x;*

l) $\bigcup x:=\{z\,|\,\exists y\,(y\in x\wedge z\in y)\}$ *is the* ***union*** *of x;*

m) $\mathcal{P}(x) := \{y \mid y \subseteq x\}$ *is the* ***power*** *of* x;

n) $V := \{x \mid x = x\}$ *is the* ***universe*** *or the* ***class of all sets***;

o) x ***is a set*** $:= x \in V$.

Strictly speaking, these notions are just syntactical objects. Nevertheless they correspond to certain intuitive expectations, and the notation has been chosen accordingly. The axioms of ZERMELO-FRAENKEL set theory will later ensure, that the notions do have the expected properties.

Note that we have now formally introduced the notion of *set*. The variables of our language range over sets, terms which are equal to some variable are sets. If t is a term then

$$t \text{ is a set iff } t \in V \text{ iff } \exists x\, (x = x \wedge x = t) \text{ iff } \exists x\, x = t.$$

Here we have inserted the term t into the formula "x is a set". In general, the substitution of terms into formulas is understood as follows: the formula is translated into a basic $\in$-formula and then the term is substituted for the appropriate variable. In a similar way, terms t_0, ..., t_{n-1} may be substituted into another terms $t(x_0, ..., x_{n-1})$: let $t(x_0, ..., x_{n-1})$ be the class term $\{x \mid \varphi(x, x_0, ..., x_{n-1})\}$; then

$$t(t_0, ..., t_{n-1}) = \{x \mid \varphi(x, t_0, ..., t_{n-1})\}$$

where the right-hand side substitution is carried out as before. This allows to work with complex terms and formulas like

$$\{\emptyset\}, \{\emptyset, \{\emptyset\}\}, x \cup (y \cup z), x \cap y \subseteq x \cup y, \emptyset \text{ is a set.}$$

A few natural properties can be checked already on the basis of the laws of first-order logic. We give some examples:

Theorem 2.4. *a*) *For terms* t *we have* $\emptyset \subseteq t$ *and* $t \subseteq V$.

b) *For terms* s, t, r *with* $s \subseteq t$ *and* $t \subseteq r$ *we have* $s \subseteq r$.

c) *For terms* s, t *we have* $s \cap t = t \cap s$ *and* $s \cup t = t \cup s$.

Proof. b) Assume $s \subseteq t$ and $t \subseteq r$. Let $z \in s$. Then $z \in t$, since $s \subseteq t$. $z \in r$, since $t \subseteq r$. Thus $\forall z\, (z \in s \rightarrow z \in r)$, i.e., $s \subseteq r$.

The other properties are just as easy. □

RUSSELL's antinomy is also just a consequence of logic:

Theorem 2.5. *The class* $\{x \mid x \notin x\}$ *is not a set.*

Proof. Assume for a contradiction that $\{x \mid x \notin x\} \in V = \{x \mid x = x\}$. This translates into $\exists z\, (z = z \wedge z = \{x \mid x \notin x\})$. Take z such that $z = \{x \mid x \notin x\}$. Then

$$z \in z \leftrightarrow (x \notin x)\frac{z}{x} \leftrightarrow z \notin z.$$

Contradiction. □

2.3 Relations and Functions

Apart from sets, *relations* and *functions* are the main building blocks of mathematics. As usual, relations are construed as sets of ordered pairs. Again we note that the subsequent notions only attain all their intended properties under the assumption of sufficiently many set theoretical axioms.

Definition 2.6. *Let t be a term and φ be a formula, where $\vec{x}$ is the sequence of variables which are both free in t and in φ. Then write the generalized class term*

$$\{t \,|\, \varphi\} \text{ instead of } \{y \,|\, \exists \vec{x}\, (y = t(\vec{x}\,) \wedge \varphi(\vec{x}\,))\}.$$

Definition 2.7.

a) $x \times y = \{(u, v) | u \in x \wedge v \in y\}$ *is the* ***(cartesian) product*** *of x and y.*

b) *x is a* ***relation*** $:= x \subseteq V \times V$.

c) *x is a* ***relation on*** y $:= x \subseteq y \times y$.

d) $x r y := (x, y) \in r$ *is the usual* ***infix*** *notation for relations.*

e) $\mathrm{dom}(r) := \{x \,|\, \exists y\, x r y\}$ *is the* ***domain*** *of r.*

f) $\mathrm{ran}(r) := \{y \,|\, \exists x\, x r y\}$ *is the* ***range*** *of r.*

g) $\mathrm{field}(r) := \mathrm{dom}(r) \cup \mathrm{ran}(r)$ *is the* ***field*** *of r.*

h) $r \restriction a := \{(x, y) | (x, y) \in r \wedge x \in a\}$ *is the* ***restriction*** *of r to a.*

i) $r[a] := \{y \,|\, \exists x (x \in a \wedge (x, y) \in r\}$ *is the* ***image*** *of a under r.*

j) $r^{-1}[b] := \{x \,|\, \exists y (y \in b \wedge (x, y) \in r\}$ *is the* ***preimage*** *of b under r.*

k) $r \circ s := \{(x, z) \,|\, \exists y (x r y \wedge y s z)\}$ *is the* ***composition*** *of r and s.*

l) $r^{-1} := \{(y, x) \,|\, (x, y) \in r\}$ *is the* ***inverse*** *of r.*

Definition 2.8. *a)* *f is a* ***function*** $:=$ *f is a relation* $\wedge$ $\forall x \forall y \forall z (x f y \wedge x f z \rightarrow y = z)$.

b) $f(x) = \bigcup \{y \,|\, x f y\}$ *is the* ***value*** *of f at x.*

c) *f is a* ***function*** *from a into b* $:= f \colon a \rightarrow b :=$ *f is a function* $\wedge\, \mathrm{dom}(f) = a \wedge \mathrm{ran}(f) \subseteq b$.

d) ${}^a b := \{f \,|\, f \colon a \rightarrow b\}$ *is the space of all functions from a into b.*

e) $\bigtimes g := \{f \,|\, \mathrm{dom}(f) = \mathrm{dom}(g) \wedge \forall x (x \in \mathrm{dom}(g) \rightarrow f(x) \in g(x))\}$ *is the* ***(cartesian) product*** *of g.*

Note that the product of g consists of *choice functions* f, where for every argument $x \in \mathrm{dom}(g)$ the value $f(x)$ chooses an element of $g(x)$.

Chapter 3
The Zermelo-Fraenkel Axioms

Russell's antinomy can be seen as a motivation for the axiomatization of set theory: not *all* classes can be sets, but we want *many* classes to be sets. We formulate axioms in the term language introduced above. Most of them are *set existence axioms* of the form $t \in V$. In writing the axioms we omit all initial universal quantifiers, i.e., φ stands for $\forall \vec{x}\, \varphi$ where $\{\vec{x}\}$ is the set of free variables of φ.

Definition 3.1.

1. ***Axiom of extensionality***: $x \subseteq y \wedge y \subseteq x \rightarrow x = y$.
2. ***Pairing axiom***: $\{x, y\} \in V$.
3. ***Union axiom***: $\bigcup x \in V$.
4. ***Axiom of infinity***: $\exists x\, (\emptyset \in x \wedge \forall y\, (y \in x \rightarrow y \cup \{y\} \in x))$.
5. ***Axiom (schema) of subsets***: *for all terms A postulate:* $x \cap A \in V$.
6. ***Axiom (schema) of replacement***: *for all terms F postulate:* F *is a function* $\rightarrow F[x] \in V$.
7. ***Axiom (schema) of foundation***: *for all terms A postulate:* $A \neq \emptyset \rightarrow \exists x\, (x \in A \wedge x \cap A = \emptyset)$.
8. ***Powerset axiom***: $\mathcal{P}(x) \in V$.
9. ***Axiom of choice*** *(AC):* f *is a function* $\wedge \forall x\, (x \in \mathrm{dom}(f) \rightarrow f(x) \neq \emptyset) \rightarrow \times f \neq \emptyset$.
10. *The* **Zermelo-Fraenkel** ***axiom system*** ZF *consists of the axioms 1 - 8.*
11. *The axiom system* ZF^- *consists of the axioms 1 - 7.*
12. *The axiom system* ZFC *consists of the axioms 1 - 9.*

Remarkably, virtually all of mathematics can be developed naturally in the axiom system ZFC: one formalizes the systems of natural, integer, rational, and real numbers; all further notions of mathematics can be expressed by set operations and properties. This is usually presented in introductory texts on set theory.

Note that the set theoretical axioms possess very different characters. There are seemingly week axioms like the pairing or union axiom which postulate the existence of concretely specified sets. On the other hand, a powerset seems to be a vast object which is hard to specify other than by its general definition. The theory ZF^- avoids the problematic powerset axiom as well as the axiom of choice. We shall carry out most of our initial development within ZF^-. Assume ZF^- for the following exercises.

Exercise 3.1. Show: a) $\bigcup V = V$. b) $\bigcap V = \emptyset$. c) $\bigcup \emptyset = \emptyset$. d) $\bigcap \emptyset = V$.

Exercise 3.2. Show that the term (x, y) has the fundamental properties of an ordered pair:

a) $\forall x \forall y \exists z\, z = (x, y)$;

b) $(x, y) = (x', y') \rightarrow x = x' \wedge y = y'$.

Exercise 3.3.

a) Show that $\langle x,\ y\rangle := \{\{x,\ \emptyset\},\ \{y,\ \{\emptyset\}\}\}$ also satisfies the fundamental property of ordered pairs (F. HAUSDORFF).

b) Can $\{x, \{y, \emptyset\}\}$ be used as an ordered pair?

Exercise 3.4. Give a set-theoretical formalization of an ordered-*triple* operation.

Exercise 3.5. Define a relation $\sim$ on V by

$$x \sim y \longleftrightarrow \exists f\, f: x \leftrightarrow y\,.$$

One say that x and y are *equinumerous* or *equipollent*. Show that $\sim$ is an equivalence relation on V. What is the equivalence class of $\emptyset$? What is the equivalence class of $\{\emptyset\}$?

Exercise 3.6. Consider functions $F: A \rightarrow B$ and $F': A \rightarrow B$. Show that

$$F = F' \text{ iff } \forall a \in A\; F(a) = F'(a).$$

Chapter 4

Induction, recursion, and ordinals

4.1 $\in$-induction

We work in the theory ZF^-. Let us first introduce some notation:

Definition 4.1. *Write*

$$\exists x \in s\, \varphi \text{ instead of } \exists x\, (x \in s \wedge \varphi),$$

$$\forall x \in s\, \varphi \text{ instead of } \forall x\, (x \in s \rightarrow \varphi),$$

and

$$\{x \in s \,|\, \varphi\} \text{ instead of } \{x \,|\, x \in s \wedge \varphi\}.$$

These notations use x as a ***bounded variable****, the quantifiers $\exists x \in s$ and $\forall x \in s$ are called* ***bounded quantifiers****.*

The axiom of foundation is equivalent to an induction schema for the $\in$-relation: if a property is inherited from the $\in$-predecessors, it holds everywhere.

Theorem 4.2. *Let $\varphi(x, \vec{y}\,)$ be an $\in$-formula such that*

$$\forall x\, (\forall z \in x\, \varphi(z, \vec{y}\,) \rightarrow \varphi(x, \vec{y}\,)).$$

Then

$$\forall x\, \varphi(x, \vec{y}\,).$$

Proof. Assume not. Then $A := \{x \,|\, \neg\varphi(x, \vec{y}\,)\} \neq \emptyset$. By the foundation schema for A take some $x \in A$ such that $x \cap A = \emptyset$, i.e., $\forall z \in x\; x \notin A$. By the definition of A

$$\neg\varphi(x, \vec{y}\,) \text{ and } \forall z \in x\, \varphi(z, \vec{y}\,).$$

This contradicts the assumption of the theorem. □

4.2 Transitive Sets and Classes

Definition 4.3. *The class s is* ***transitive*** *iff $\forall x \in s \forall y \in x\; y \in s$. We write* $\mathrm{Trans}(s)$ *if s is transitive.*

Theorem 4.4. *s is transitive iff $\forall x \in s\ x \subseteq s$ iff $\forall x \in s\ x = x \cap s$.*

A transitive class is an $\in$-initial segment of the class of all sets.

Theorem 4.5. *a) $\emptyset$ and V are transitive.*

b) If $\forall x \in A\, \mathrm{Trans}(x)$ then $\bigcap A$ and $\bigcup A$ are transitive.

c) If x is transitive then $x \cup \{x\}$ is transitive.

Proof. *Exercise.* □

Exercise 4.1. Show that arbitrary unions and intersections of transitive sets are again transitive.

4.3 $\in$-recursion

We prove a recursion principle which corresponds to the principle of $\in$-induction.

Theorem 4.6. *Let $G: V \to V$. Then there is a class term F such that*

$$F: V \to V \text{ and } \forall x\, F(x) = G(F \restriction x).$$

The function F is uniquely determined: if F': $V \to F$ and $\forall x\, F'(x) = G(F' \restriction x)$. Then

$$F = F'.$$

The term F is defined explicitely in the subsequent proof and is called the ***canonical term defined by $\in$-recursion by*** *$F(x) = G(F \restriction x)$.*

Proof. We construct F as a union of approximations to F. Call a function $f \in V$ a *G-approximation* if

- $f: \mathrm{dom}(f) \to V$;
- $\mathrm{dom}(f)$ is transitive;
- $\forall x\, f(x) = G(f \restriction x)$.

We prove some structural properties for the class of G-approximations:
(1) If f and f' are G-approximations then $\forall x \in \mathrm{dom}(f) \cap \mathrm{dom}(f')\ f(x) = f'(x)$.
Proof. Assume not and let $x \in \mathrm{dom}(f) \cap \mathrm{dom}(f')$ be $\in$-minimal with $f(x) \neq f'(x)$. Since $\mathrm{dom}(f) \cap \mathrm{dom}(f')$ is transitive, $x \subseteq \mathrm{dom}(f) \cap \mathrm{dom}(f')$. By the $\in$-minimality of x, $f \restriction x = f' \restriction x$. Then

$$f(x) = G(f \restriction x) = G(f' \restriction x) = f'(x),$$

contradiction. *qed*(1)
(2) $\forall x \exists f\, (f$ is a G-approximation $\wedge\, x \in \mathrm{dom}(f))$.
Proof. Assume not and let x be an $\in$-minimal counterexample. For $y \in x$ define

$$f_y = \bigcap \{f \mid f \text{ is a } G\text{-approximation} \wedge y \in \mathrm{dom}(f)\}.$$

By the minimality of x, there at least one f such that

$$f \text{ is a } G\text{-approximation} \wedge y \in \operatorname{dom}(f).$$

The intersection of such approximations is an approximation itself, so that

$$f_y \text{ is a } G\text{-approximation} \wedge y \in \operatorname{dom}(f_y).$$

Then define

$$f_x = (\bigcup_{y \in x} f_y) \cup \{(x, G((\bigcup_{y \in x} f_y) \upharpoonright x)).$$

One can now check that f_x is a G-approximation with $x \in \operatorname{dom}(f_x)$. Contradiction. *qed*(2)

Now set

$$F = \bigcup \{f \mid f \text{ is a } G\text{-approximation}\}.$$

Then F satisfies the theorem. □

.

Definition 4.7. *Let* TC *be the canonical term defined by* ∈*-recursion by*

$$\mathrm{TC}(x) = x \cup \bigcup_{y \in x} \mathrm{TC}(y).$$

TC(x) *is called the* ***transitive closure of*** x*.*

Theorem 4.8. *For all* $x \in V$*:*

a) TC(x) *is transitive and* $\mathrm{TC}(x) \supseteq x$*;*

b) TC(x) *is the* ⊆*-smallest transitive superset of* x*.*

Proof. By ∈-induction. Let $x \in V$ and assume that a) and b) hold for all $z \in x$. Then
(1) $\mathrm{TC}(x) \supseteq x$ is obvious from the recursive equation for TC.
(2) TC(x) is transitive.
Proof. Let $u \in v \in \mathrm{TC}(x)$.
Case 1: $v \in x$. Then

$$u \in v \subseteq \mathrm{TC}(v) \subseteq \bigcup_{y \in x} \mathrm{TC}(y) \subseteq \mathrm{TC}(x).$$

Case 2: $v \notin x$. Then take $y \in x$ such that $v \in \mathrm{TC}(y)$. TC(y) is transitive by hypothesis, hence

$$u \in \mathrm{TC}(y) \subseteq \bigcup_{y \in x} \mathrm{TC}(y) \subseteq \mathrm{TC}(x).$$

qed(2)
b) Let $w \supseteq x$ be transitive. Let $y \in x$. Then $y \in w$, $y \subseteq w$. By hypothesis, TC(y) is the ⊆-minimal superset of y, hence $\mathrm{TC}(y) \subseteq w$. Thus

$$\bigcup_{y \in x} \mathrm{TC}(y) \subseteq w$$

and

$$\mathrm{TC}(x) = x \cup \bigcup_{y \in x} \mathrm{TC}(y) \subseteq w$$ □

4.4 Ordinals

The number system of *ordinal numbers* is particularly adequate for the study of the infinite. We present the theory of VON NEUMANN-ordinals based on the notion of transitivity.

Definition 4.9. *A set x is an **ordinal** if $\mathrm{Trans}(x) \wedge \forall y \in x\, \mathrm{Trans}(y)$. Let*

$$\mathrm{Ord} = \{x \mid x \text{ is an ordinal}\}$$

be the class of all ordinals.

We show that the ordinals are a generalization of the natural numbers into the transfinite.

Theorem 4.10. *The class* Ord *is strictly well-ordered by* $\in$.

Proof. (1) $\in$ is a transitive relation on Ord.
Proof. Let $x, y, z \in \mathrm{Ord}$, $x \in y$, and $y \in z$. Since z is a transitive set, $x \in z$. *qed*(1)
(2) $\in$ is a linear relation on Ord, i.e., $\forall x, y \in \mathrm{Ord}\ (x \in y \vee x = y \vee y \in x)$.
Proof. Assume not. Let x be $\in$-minimal such that

$$\exists y\, (x \notin y \wedge x \neq y \wedge y \notin x).$$

Let y be $\in$-minimal such that

$$x \notin y \wedge x \neq y \wedge y \notin x. \tag{4.1}$$

Let $x' \in x$. Then by the minimality of x we have

$$x' \in y \vee x' = y \vee y \in x'.$$

If $x' = y$ then $y = x' \in x$, contradicting (4.1). If $y \in x'$ then $y \in x' \in x$ and $y \in x$, contradicting (4.1). Thus $x' \in y$. This shows $x \subseteq y$.

Conversely let $y' \in y$. Then by the minimality of y we have

$$x \in y' \vee x = y' \vee y' \in x.$$

If $x \in y'$ then $x \in y' \in y$ and $x \in y$, contradicting (4.1). If $x = y'$ then $x = y' \in y$, contradicting (4.1). Thus $y' \in x$. This shows $y \subseteq x$.

Hence $x = y$, contradicting (4.1). *qed*(2)
(3) $\in$ is an irreflexive relation on Ord, i.e., $\forall x \in \mathrm{Ord}\ x \notin x$.
Proof. Assume for a contradiction that $x \in x$. By the foundation scheme applied to the term $A = \{x\} \neq \emptyset$ let $y \in \{x\}$ with $y \cap \{x\} = \emptyset$. Then $y = x$, $x \in x = y$, $x \in y \cap \{x\}$ which contradicts the choice of y. *qed*(3)
(4) $\in$ is a well-order on Ord, i.e., for every non-empty $A \subseteq \mathrm{Ord}$ there exists $\alpha \in A$ such that $\forall \beta \in \alpha\, \beta \notin A$.
Proof. By the foundation scheme applied to A let $\alpha \in A$ with $\alpha \cap A = \emptyset$. Then $\forall \beta \in \alpha\, \beta \notin A$. □

By this theorem, $\in$ is the canonical order on the ordinal numbers. We use greek letters $\alpha, \beta, \gamma, \ldots$ as variables for ordinals and write $\alpha < \beta$ instead of $\alpha \in \beta$. When we talk about smallest or largest ordinals this is meant with respect to the ordering $<$.

Theorem 4.11. *a)* $\emptyset$ *is the smallest element of* Ord. *We write* 0 *instead of* $\emptyset$ *when* $\emptyset$ *is used as an ordinal.*

b) *If* $\alpha \in \mathrm{Ord}$ *then* $\alpha \cup \{\alpha\}$ *is the smallest element of* Ord *which is larger than* α*, i.e.,* $\alpha \cup \{\alpha\}$ *is the* ***successor*** *of* α*. We write* $\alpha + 1$ *instead of* $\alpha \cup \{\alpha\}$*. Every ordinal of the form* $\alpha + 1$ *is called a* ***successor ordinal****.*

Proof. b) Let $\alpha \in \mathrm{Ord}$.
(1) $\alpha \cup \{\alpha\}$ is transitive.
Proof. Let $u \in v \in \alpha \cup \{\alpha\}$.
Case 1. $v \in \alpha$. Then $u \in \alpha \subseteq \alpha \cup \{\alpha\}$ since α is transitive.
Case 2. $v \in \{\alpha\}$. Then $u \in v = \alpha \subseteq \alpha \cup \{\alpha\}$. *qed*(1)
(2) $\forall y \in \alpha \cup \{\alpha\}\ \mathrm{Trans}(y)$.
Proof. Let $y \in \alpha \cup \{\alpha\}$.
Case 1. $y \in \alpha$. Then $\mathrm{Trans}(y)$, since α is an ordinal.
Case 2. $y \in \{\alpha\}$. Then $y = \alpha$, and $\mathrm{Trans}(y)$, since α is an ordinal. *qed*(2)

So $\alpha \cup \{\alpha\}$ is an ordinal, and $\alpha \cup \{\alpha\} > \alpha$.
(3) $\alpha \cup \{\alpha\}$ is the smallest ordinal $> \alpha$.
Proof. Let $\beta < \alpha \cup \{\alpha\}$. Then $\beta \in \alpha$ or $\beta = \alpha$. Hence $\beta \leqslant \alpha$ and $\beta \not> \alpha$. □

Theorem 4.12.

a) Ord *is transitive.*

b) $\forall x \in \mathrm{Ord}\ \mathrm{Trans}(x)$*.*

c) $\mathrm{Ord} \notin V$*, i.e.,* Ord *is a proper class.*

Proof. a) Let $x \in y \in \mathrm{Ord}$.
(1) $\mathrm{Trans}(x)$, since every element of the ordinal y is transitive.
(2) $\forall u \in x\ \mathrm{Trans}(u)$.
Proof. Let $u \in x$. Since y is transitive, $u \in y$. Since every element of y is transitive, $\mathrm{Trans}(u)$. *qed*(2)

Thus $x \in \mathrm{Ord}$.
b) is part of the definition of ordinal.
c) Assume $\mathrm{Ord} \in V$. By a) and b), Ord satisfies the definition of an ordinal, and so $\mathrm{Ord} \in \mathrm{Ord}$. This contradicts the foundation scheme. □

Exercise 4.2.

a) Let $A \subseteq \mathrm{Ord}$ be a term, $A \neq \emptyset$. Then $\bigcap A \in \mathrm{Ord}$.

b) Let $x \subseteq \mathrm{Ord}$ be a set. Then $\bigcup A \in \mathrm{Ord}$.

4.5 Natural numbers

One can construe the common natural numbers as those ordinal numbers which can be reached from 0 by the $+1$-operation. Consider the following term:

Definition 4.13. $\omega = \{\alpha \in \mathrm{Ord} \,|\, \forall\beta \in \alpha + 1\, (\beta = 0 \vee \beta \textit{ is a successor ordinal})\}$ *is the class of* ***natural numbers****.*

Theorem 4.14. *ω is transitive.*

Proof. Let $x \in \alpha \in \omega$.
(1) $x \in \mathrm{Ord}$, since Ord is transitive.
(2) $x \subseteq \alpha$, since α is transitive.
(3) $x + 1 \subseteq \alpha \subseteq \alpha + 1$.
(4) $\forall\beta \in x + 1\, (\beta = 0 \vee \beta$ is a successor ordinal$)$, since $\alpha \in \omega$ and $x + 1 \subseteq \alpha + 1$.
Then (1) and (4) imply that $x \in \omega$. □

Theorem 4.15. *$\omega \in V$, i.e., ω is the* ***set of natural numbers****.*

Proof. By the axiom of infinity, take a set x such that

$$(0 \in x \wedge \forall y\, (y \in x \rightarrow y \cup \{y\} \in x)).$$

(1) $\omega \subseteq x$.
Proof. Assume for a contradiction that $\omega \nsubseteq x$. By foundation take $z \in \omega$ $\in$-minimal such that $z \notin x$. By the definition of ω we have $z = 0$ or z is a successor ordinal. The case $z = 0$ is impossible by the choice of x. Hence z is a successor ordinal. Take $y \in \mathrm{Ord}$ such that $z = y + 1$. Then $y \in z \in \omega$ and $y \in \omega$ by the transitivity of ω. By the $\in$-minimal choice of z we have $y \in x$. By the choice of x we have $z = y + 1 = y \cup \{y\} \in x$. This contradicts the choice of z. *qed*(1)

The subset schema implies that $\omega = x \cap \omega \in V$. □

Theorem 4.16. *ω is a* ***limit ordinal****, i.e., an ordinal $\neq 0$ which is not a successor ordinal. Indeed, ω is the smallest limit ordinal:*

$$\omega = \bigcap \{\alpha \,|\, \alpha \textit{ is a limit ordinal}\}.$$

Proof. First note that ω is an ordinal, since it is a transitive set and each of its elements is transitive.

Obviously $0 \in \omega$, hence $\omega \neq 0$. Assume for a contradiction that ω is a successor ordinal. Take some $\alpha \in \mathrm{Ord}$ such that $\omega = \alpha + 1$. Then $\alpha \in \omega$ and

$$\forall\beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal}).$$

$\omega + 1 = (\alpha + 1) \cup \{\omega\}$. Since ω is assumed to be a successor ordinal

$$\forall\beta \in \omega + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal}).$$

Hence $\omega \in \omega$. But this contradicts the foundation schema.

Thus ω is a limit ordinal.

Let γ be (another) limit ordinal. Since all elements of ω are 0 or successor ordinals, we cannot have $\gamma < \omega$. Therefore $\omega \leqslant \gamma$. □

Let us justify this formalization of the set of natural numbers by

Theorem 4.17. *The structure $(\omega, +1, 0)$ satisfies the* PEANO *axioms:*

a) $0 \in \omega$*;*

b) $\forall n \in \omega\, n+1 \in \omega$;

c) $\forall n \in \omega\, n+1 \neq 0$;

d) $\forall m, n \in \omega\, (m+1 = n+1 \rightarrow m = n)$;

e) $\forall x \subseteq \omega\, ((0 \in x \wedge \forall m \in x\, m+1 \in x) \rightarrow x = \omega)$.

Proof. Axioms a) to d) are immediate from the definition of ω or from the general properties of ordinals. For e) consider a set $x \subseteq \omega$ such that

$$0 \in x \wedge \forall m \in x\, m+1 \in x.$$

Assume for a contradiction that $x \neq \omega$. By foundation take $z \in \omega$ $\in$-minimal such that $z \notin x$. By the definition of ω we have $z = 0$ or z is a successor ordinal. The case $z = 0$ is impossible by the properties of x. Hence z is a successor ordinal. Take $y \in \mathrm{Ord}$ such that $z = y + 1$. Then $y \in z \in \omega$ and $y \in \omega$ by the transitivity of ω. By the $\in$-minimal choice of z we have $y \in x$. By the inductive property of x we have $z = y + 1 = y \cup \{y\} \in x$. This contradicts the choice of z. $\square$

Chapter 5

Transitive $\in$-models

Axiomatic set theory studies the axiom systems ZF and ZFC. By the Gödel incompleteness theorem, these systems are incomplete. So one is lead to consider extensions of these systems of the form ZF + φ or ZFC + φ for various φ. Even some simple questions of the arithmetic of infinite cardinals like Cantor's *continuum hypothesis* are not decided by ZFC and present an ongoing challenge to set theoretical research.

To show that a theory like ZFC + φ is consistent one constructs *models* of that theory (making some initial assumptions). Usually these models will be an $\in$-*model* of the form $(M, \in)$, where M is some class.

5.1 Relativizations of Formulas and Terms

Evaluating an $\in$-formula φ in a model $(M, \in)$ amount to bounding the range of quantifiers in φ to M.

Definition 5.1. *Let M be a term. For φ an $\in$-formula define the* ***relativization*** *φ^M of φ to M by recursion on the complexity of φ:*

- $(x \in y)^M := x \in y$
- $(x = y)^M := x = y$
- $(\neg\varphi)^M := \neg(\varphi^M)$
- $(\varphi \vee \psi)^M := \varphi^M \vee \psi^M$
- $(\exists x \varphi)^M := \exists x \in M \varphi^M$

Definition 5.2. *Let M be a term and let Φ be a (metatheoretical) set of formulas. Then the (metatheoretical) set*

$$\Phi^M = \{\varphi^M \mid \varphi \in \Phi\}$$

is the ***relativization*** *of Φ to M.*

The relativizations φ^M and Φ^M correspond to the model-theoretic satisfaction relations $(M, \in) \vDash \varphi$ and $(M, \in) \vDash \Phi$. This is illustrated by

Theorem 5.3. *Let Φ be a finite set of $\in$-formulas and let φ be an $\in$-formula such that $\Phi \vdash \varphi$ in the calculus of first-order logic. Let M be a transitive term, $M \neq \emptyset$, which has no common free variables with Φ or φ. Then*

$$\forall \vec{x} \in M\left(\left(\bigwedge \Phi\right)^M \rightarrow \varphi^M\right),$$

where $\vec{x}$ includes all the free variables of Φ and φ.

Proof. By induction on the lengths of derivations it suffices to prove the theorem for the case that $\Phi \vdash \varphi$ is derivable by a single application of a rule of the first-order calculus. We check this for the various rules.

The theorem is obvious in case φ is an element of Φ.

In case $\varphi = (x = x)$, the relativization $(x = x)^M = (x = x)$ holds in any case.

The theorem is easy to show for all propositional rules and the substitution rule.

So let us now consider the quantifier rules. Assume that $\varphi(x, \vec{y})^M$ where x, $\vec{y} \in M$. Then $\exists x (x \in M \wedge \varphi(x, \vec{y})^M)$ and

$$(\exists x \varphi(x, \vec{y}))^M$$

as required.

For the $\exists$-introduction in the *antecedens* suppose that

$$\forall x, \vec{y} \in M\left(\left(\bigwedge \Phi\right)^M \wedge \psi^M(x, \vec{y}) \rightarrow \varphi^M(\vec{y})\right), \tag{5.1}$$

where the variable x does not occur in Φ or φ. Now let $\vec{y} \in M$ and assume that $(\bigwedge \Phi)^M \wedge (\exists x \psi)^M(\vec{y})$. Then $\exists x \in M \psi^M(x, \vec{y})$. Take $x \in M$ such that $\psi^M(x, \vec{y})$. By (4.1) we get $\varphi^M(\vec{y})$. Hence

$$\forall \vec{y} \in M\left(\left(\bigwedge \Phi\right)^M \wedge (\exists x \psi)^M(\vec{y}) \rightarrow \varphi^M(\vec{y})\right).$$

□

We shall later construct models M such that ZFC^M holds and obtain relative consistency results.

Theorem 5.4. *Assume that the theory* ZF *is consistent. Let M be a class term such that $\mathrm{ZF} \vdash M \neq \emptyset$ and $\mathrm{ZF} \vdash \varphi^M$ for every* ZFC*-axiom φ. Then* ZFC *is consistent.*

Proof. Assume that ZFC is inconsistent, i.e., $\mathrm{ZFC} \vdash x \neq x$. By the finiteness of formal proofs take a *finite* collection Φ of ZFC-axioms such that $\Phi \vdash x \neq x$. By assumption $\mathrm{ZF} \vdash (\bigwedge \Phi)^M$. By the previous theorem

$$\mathrm{ZF} \vdash \forall x \in M\left(\left(\bigwedge \Phi\right)^M \rightarrow (x \neq x)^M\right).$$

Hence $\mathrm{ZF} \vdash \forall \in M\, x \neq x$. Together with $\mathrm{ZF} \vdash M \neq \emptyset$ this leads to $\mathrm{ZF} \vdash x \neq x$, i.e., ZF is inconsistent. □

Definition 5.5. *Let M be a term. For a class term $s = \{x \mid \varphi\}$ define the* ***relativization*** *s^M of s to M by:*

$$s^M := \{x \in M \mid \varphi^M\}.$$

If s is a variable, $s=x$, then let $s^M=s$.

s^M is the term s as evaluated in M. We show that evaluating a formula with class terms (a *generalized* formula) in a *transitive* class M is the same as relativizing the basic formula without class terms and then inserting the relatived class terms. This will make many notions *absolute* between M and V.

Note that the relativization of a *bounded* quantifier $\exists x\in y$ to a transitive class M with $y\in M$ has no effect:

$$\exists x\in y\varphi\leftrightarrow\exists x\in y\cap M\,\varphi.$$

Theorem 5.6. *Let M be a transitive class. Let $\varphi(x_0,\ldots,x_{n-1})$ be a basic formula and $t_0,\ldots,t_{n-1}$ be terms. Then*

$$\forall\vec{w}\in M[(\chi(t_0,\ldots,t_{n-1}))^M\leftrightarrow\chi^M(t_0^M,\ldots,t_{n-1}^M)],$$

where $\{\vec{w}\}$ is the set of free variables of $\chi(t_0,\ldots,t_{n-1})$.

Proof. By induction on the complexity of χ. Let $\vec{w}\in M$.

Let χ be an atomic formula of the form $u\in v$ or $u=v$. If t_0 and t_1 are variables there is nothing to show. The other cases correspond to the following equivalences:

$$\begin{aligned}
(y\in\{x\,|\,\varphi\})^M &\leftrightarrow (\varphi\frac{y}{x})^M\\
&\leftrightarrow \varphi^M\frac{y}{x}\\
&\leftrightarrow (x\in M\wedge\varphi^M)\frac{y}{x}\\
&\leftrightarrow y\in\{x\,|\,x\in M\wedge\varphi^M\}=\{x\in M\,|\,\varphi^M\}\\
&\leftrightarrow y^M\in\{x\,|\,\varphi\}^M.
\end{aligned}$$

This equivalence is already used in:

$$\begin{aligned}
(\{x\,|\,\varphi\}=\{y\,|\,\psi\})^M &\leftrightarrow (\forall z(z\in\{x\,|\,\varphi\}\leftrightarrow z\in\{y\,|\,\psi\}))^M\\
&\leftrightarrow \forall z\in M(z\in\{x\,|\,\varphi\}^M\leftrightarrow z\in\{y\,|\,\psi\}^M)\\
&\leftrightarrow \forall z(z\in\{x\,|\,\varphi\}^M\leftrightarrow z\in\{y\,|\,\psi\}^M),\text{ since }\{x|\varphi\}^M\subseteq M,\\
&\leftrightarrow \{x\,|\,\varphi\}^M=\{y\,|\,\psi\}^M.
\end{aligned}$$

Note, that $x\subseteq M$ by the transitivity of M:

$$\begin{aligned}
(x=\{y\,|\,\psi\})^M &\leftrightarrow (\forall z(z\in x\leftrightarrow z\in\{y\,|\,\psi\}))^M\\
&\leftrightarrow \forall z\in M(z\in x\leftrightarrow z\in\{y\,|\,\psi\}^M)\\
&\leftrightarrow \forall z(z\in x\leftrightarrow z\in\{y\,|\,\psi\}^M),\text{ since }x\subseteq M,\\
&\leftrightarrow x^M=\{y\,|\,\psi\}^M.
\end{aligned}$$

$$\begin{aligned}
(\{x\,|\,\varphi\}\in\{y\,|\,\psi\})^M &\leftrightarrow (\exists z(\psi\frac{z}{y}\wedge z=\{x\,|\,\varphi\}))^M\\
&\leftrightarrow \exists z\in M(\psi^M\frac{z}{y}\wedge z=\{x\,|\,\varphi\}^M)\\
&\leftrightarrow \exists z(z\in M\wedge\psi^M\frac{z}{y}\wedge z=\{x\,|\,\varphi\}^M)\\
&\leftrightarrow \{x\,|\,\varphi\}^M\in\{y\,|\,y\in M\wedge\psi^M\}=\{y|\psi\}^M.
\end{aligned}$$

$$\begin{aligned}(\{x\,|\,\varphi\}\in y)^M &\leftrightarrow (\exists z\,(z\in y\wedge z=\{x\,|\,\varphi\}))^M\\ &\leftrightarrow \exists z\in M\,(z\in y\wedge z=\{x\,|\,\varphi\}^M)\\ &\leftrightarrow \exists z(z\in y\wedge z=\{x\,|\,\varphi\}^M),\text{ since } y\subseteq M,\\ &\leftrightarrow \{x\,|\,\varphi\}^M\in y=y^M.\end{aligned}$$

Now assume that χ is a complex formula and the theorem holds for all proper subformulas. If $\chi=\neg\psi$ and $\vec{w}\in M$ then

$$(\chi(t_0,\ldots,t_{n-1}))^M\leftrightarrow\neg(\psi(t_0,\ldots,t_{n-1}))^M\leftrightarrow\neg\psi^M(t_0^M,\ldots,t_{n-1}^M)\leftrightarrow\chi^M(t_0^M,\ldots,t_{n-1}^M).$$

If $\chi=\varphi\vee\psi$ and $\vec{w}\in M$ then

$$\begin{aligned}(\chi(t_0,\ldots,t_{n-1}))^M &\leftrightarrow (\varphi(t_0,\ldots,t_{n-1}))^M\vee(\psi(t_0,\ldots,t_{n-1}))^M\\ &\leftrightarrow \varphi^M(t_0^M,\ldots,t_{n-1}^M)\vee\psi^M(t_0^M,\ldots,t_{n-1}^M)\\ &\leftrightarrow \chi^M(t_0^M,\ldots,t_{n-1}^M).\end{aligned}$$

If $\chi=\exists x\varphi$ and $\vec{w}\in M$ then

$$\begin{aligned}(\chi(t_0,\ldots,t_{n-1}))^M &\leftrightarrow \exists x\in M\,(\varphi(x,t_0,\ldots,t_{n-1}))^M\\ &\leftrightarrow \exists x\in M\varphi^M(x,t_0^M,\ldots,t_{n-1}^M)\\ &\leftrightarrow \chi^M(t_0^M,\ldots,t_{n-1}^M).\end{aligned}$$

□

e

Theorem 5.7. *Let M be a non-empty transitive term. Assume that M satisfies the following closure properties:*

a) $\forall x,y\in M\ \{x,y\}\in M$;

b) $\forall x\in M\ \bigcup x\in M$;

c) $\omega\in M$;

d) for all terms A: $\forall x\in M\,x\cap A^M\in M$;

e) for all terms F: if F^M is a function then $\forall x\,F^M[x]\in M$.

Then ZF$^-$ *holds in M.*

Proof. (1) The axiom of extensionality holds in M.
Proof. Consider $x,y\in M$. By the axiom of extensionality in V

$$x\subseteq y\wedge y\subseteq x\rightarrow x=y.$$

Since M is transitive, $x\cap M=x$, $y\cap M=y$ and

$$x\cap M\subseteq y\wedge y\cap M\subseteq x\rightarrow x=y.$$

This is equivalent to

$$(\forall z\in M\,(z\in x\rightarrow z\in y)\wedge\forall z\in M\,(z\in y\rightarrow z\in x))\rightarrow x=y$$

and

$$(x \subseteq y \wedge y \subseteq x \rightarrow x = y)^M.$$

Thus

$$(\forall x, y\, (x \subseteq y \wedge y \subseteq x \rightarrow x = y))^M.$$

(2) The pairing axiom holds in M.
Proof. Observe that for $x, y \in M$

$$\{x, y\}^M = \{z \in M \mid z = x \vee z = y\} = \{x, y\}.$$

Moreover $V^M = \{x \in M \mid x = x\} = M$. By assumption a),

$$\begin{array}{l} \forall x, y \in M\ \{x, y\} \in M \\ \forall x, y \in M\ \{x, y\}^M \in V^M \\ (\forall x, y\ \{x, y\} \in V)^M, \end{array}$$

i.e., the pairing axiom holds in M.
(3) The union axiom holds in M.
Proof. Observe that for $x \in M$,

$$\begin{aligned} (\bigcup x)^M &= \{z \in M \mid \exists y \in M\, (y \in x \wedge z \in y)\} \\ &= \{z \in M \mid \exists y\, (y \in x \wedge z \in y)\}, \text{ since } x \subseteq M, \\ &= \{z \mid \exists y\, (y \in x \wedge z \in y)\}, \text{ since } \forall y \in x \forall z \in y\, z \in M, \\ &= \bigcup x \end{aligned}$$

By assumption b),

$$\begin{array}{l} \forall x \in M\ \bigcup x \in M \\ \forall x \in M\ (\bigcup x)^M \in V^M \\ (\forall x\ \bigcup x \in V)^M, \end{array}$$

i.e., the union axiom holds in M.
(4) The axiom of infinity holds in M.
Proof. Let $x = \omega \in M$. Then

$$\emptyset \in x \wedge \forall y\, (y \in x \rightarrow y \cup \{y\} \in x).$$

The universal quantifier may be restricted to M:

$$\emptyset \in x \wedge \forall y \in M (y \in x \rightarrow y \cup \{y\} \in x).$$

Since $(y \cup \{y\})^M = y \cup \{y\}$ this formula is equivalent to

$$(\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))^M.$$

Then

$$\exists x \in M\, (\emptyset \in x \wedge \forall y (y \in x \rightarrow y \cup \{y\} \in x))^M,$$

i.e., the axiom of infinity holds in M.
(5) The axiom schema of subsets holds in M.
Proof. Let $A(\vec{y}\,)$ be a term and $x, \vec{y} \in M$. By assumption,

$$x \cap A^M(\vec{y}\,) \in M.$$

Note that

$$\begin{aligned}
x\cap A^M(\vec{y}) &= \{v\,|\,v\in x\wedge v\in A^M(\vec{y})\}\\
&= \{v\,|\,(v\in x\wedge v\in A(\vec{y}))^M\}\\
&= \{v\in M\,|\,(v\in x\wedge v\in A(\vec{y}))^M\}, \text{ since } x\subseteq M,\\
&= \{v\,|\,v\in x\wedge v\in A(\vec{y})\}^M\\
&= (x\cap A)^M.
\end{aligned}$$

So

$$(x\cap A)^M\in M=V^M.$$

This proves

$$\forall x,\vec{y}\in M\,(x\cap A\in V)^M,$$

i.e., the axiom scheme of subsets holds relativized to M.
(6) The axiom scheme of replacement holds relativized to M.
Proof. Let $F(\vec{y})$ be a term, and let $x,\vec{y}\in M$ such that (F is a function)^M. Note that

$$\begin{aligned}
F^M[x] &= \{v\,|\,\exists u\in x\,(u,v)\in F^M\}\\
&= \{v\in M\,|\,\exists u\,(u\in x\wedge(u,v)\in F^M)\}, \text{ since } F^M\subseteq M \text{ and } M \text{ is transitive},\\
&= \{v\in M\,|\,\exists u\in M\,(u\in x\wedge(u,v)\in F)^M\\
&= \{v\,|\,\exists u\,(u\in x\wedge(u,v)\in F)\}^M\\
&= (F[x])^M.
\end{aligned}$$

The assumption implies

$$\begin{gathered}
F^M[x]\in M\\
F^M[x]\in M=V^M\\
(F[x]\in V)^M
\end{gathered}$$

Thus

$$\begin{gathered}
\forall x,\vec{y}\in M\,(F \text{ is a function } \to F[x]\in V)^M, \text{ and}\\
(\forall x,\vec{y}\,(F \text{ is a function } \to F[x]\in V))^M,
\end{gathered}$$

as required.
(7) The axiom schema of foundation holds in M.
Proof. Let $A(\vec{y})$ be a term and let $\vec{y}\in M$ such that $(A\neq\emptyset)^M$. Then $A^M\neq\emptyset$. By the replacement schema in V, take $x\in A^M$ such that $x\cap A^M=\emptyset$. We have seen before that $x\cap A^M=(x\cap A)^M$. So $(x\cap A)^M=\emptyset$ and $(x\cap A=\emptyset)^M$. Hence

$$\begin{gathered}
\exists x\in M\,(x\in A^M\wedge(x\cap A=\emptyset)^M)\\
(\exists x\,(x\in A\wedge x\cap A=\emptyset)^M
\end{gathered}$$

Thus

$$(A\neq\emptyset\to\exists x\,(x\in A\wedge x\cap A=\emptyset)^M,$$

i.e., the foundation schema holds in M. $\square$

The converse to this theorem will be shown later.

Theorem 5.8. *Let M be a non-empty transitive term such that*

$$\forall x \in M \, \mathcal{P}(x) \cap M \in M.$$

Then the power set axiom holds in M.

Proof. Note that for $x \in M$

$$\begin{aligned}
(\mathcal{P}(x))^M &= \{y \mid y \subseteq x\}^M \\
&= \{y \in M \mid (y \subseteq x)^M\} \\
&= \{y \in M \mid (y \subseteq x)^M\} \\
&= \{y \in M \mid (\forall z \, (z \in y \rightarrow z \in x))^M\} \\
&= \{y \in M \mid \forall z \in M \, (z \in y \rightarrow z \in x)\} \\
&= \{y \in M \mid \forall z (z \in y \rightarrow z \in x)\}, \text{ da } y \subseteq M, \\
&= \{y \in M \mid y \subseteq x\} = \mathcal{P}(x) \cap M.
\end{aligned}$$

The assumption yields

$$\begin{gathered}
\forall x \in M \, \mathcal{P}(x) \cap M \in M \\
\forall x \in M \, \mathcal{P}(x)^M \in V^M \\
(\forall x \, \mathcal{P}(x) \in V)^M,
\end{gathered}$$

i.e., the power set axiom holds in M. □

Chapter 6
Definite formulas and terms

6.1 Definiteness

In the set existence axioms of the theory ZF^- every element of a term whose existence is postulated is determined by some parameters of the axiom. In the replacement scheme, e.g., every element

$$v \in \{F(x) \,|\, x \in z\}$$

is of the form $v = F(x)$ and is thus definable from the "simpler" parameter x by the term F. In contrast, there is no way to define an arbitrary element of an infinite power set from simple parameters; this impression can be made more formal by using CANTOR's diagonal argument. The axiom of choice also is a pure existence statement. There exists a choice functions, but it is in general not definable from the parameters of the situation at hand.

The notion of *defineteness* aims to capture the concrete nature of ZF^- as compared to full ZFC. It will be seen that most basic notions of set theory are definite and that these notions can be decided in ZF^- independently of the specific transitive model of ZF^-. The definition of *definite term* tries to capture the "absolute" part of the theory ZF^-.

Definition 6.1. *Define the collections of* ***definite formulas*** *and* ***definite terms*** *by a common recursion on syntactic complexities:*

a) *the atomic formulas* $x \in y$ *and* $x = y$ *are definite;*

b) *if* φ *and* ψ *are definite formulas then* $\varphi \vee \psi$ *and* $\neg\varphi$ *are definite;*

c) *if* φ *is a definite formula then* $\forall x \in y\, \varphi$ *and* $\exists x \in y\, \varphi$ *are definite formulas;*

d) x, $\{x, y\}$, $\bigcup x$ *and* ω *are definite terms;*

e) *if* $s(x_0, \ldots, x_{n-1})$ *and* $t_0, \ldots, t_{n-1}$ *are definite terms then* $s(t_0, \ldots, t_{n-1})$ *is a definite term;*

f) *if* $\varphi(x_0, \ldots, x_{n-1})$ *is a definite formula and* $t_0, \ldots, t_{n-1}$ *are definite terms then* $\varphi(t_0, \ldots, t_{n-1})$ *is a definite formula;*

g) *if* φ *is a definite formula then* $\{x \in y \,|\, \varphi(x, \vec{z})\}$ *is a definite term;*

h) *if* $t(x, \vec{z})$ *is a definite term then* $\{t(x, \vec{z}) \,|\, x \in y\}$ *is a definite term;*

i) *if* G *is a definite term then the canonical term* F *defined by* $\in$*-recursion with* $F(x) = G(F \upharpoonright x)$ *is definite.*

The majority of basic notions of set theory (and of mathematics) are definite. The following theorems list some representative examples.

Theorem 6.2. *The following terms are definite:*

a) $x\setminus y$

b) (x,y)

c) $x\times y$

d)

$$\begin{cases} x, \text{ if } \varphi \\ y, \text{ if } \neg\varphi \end{cases},$$

where φ is a definite formula ("definition by cases")

Proof. a) $x\setminus y=\{z\in x\,|\,z\notin y\}$.
b) $(x,y)=\{\{x\},\{x,y\}\}$.
c) $x\times y=\bigcup\,\{x\times\{v\}\,|\,v\in y\}=\bigcup\,\{\{(u,v)\,|\,u\in x\}\,|\,v\in y\}$.
d) $\left\{\begin{smallmatrix} x, \text{ if } \varphi \\ y, \text{ if } \neg\varphi \end{smallmatrix}\right.$ can be defined definitely by

$$\{u\in x\,|\,\varphi\}\cup\{u\in y\,|\,\neg\varphi\}.$$

□

Theorem 6.3. *The following formulas are definite:*

a) x *is transitive*

b) x *is an ordinal*

c) x *is a successor ordinal*

d) x *is a limit ordinal*

e) x *is a natural number*

Proof. All these formulas are equivalent to Σ_0-formulas. □

Recursion on the ordinals is a special case of $\in$-recursion which also leads to definite terms.

Theorem 6.4. *Let G_0, G_{succ} and G_{limit} be definite terms defining a term F:* $\text{Ord}\to V$ *by the following recursion:*

– $F(0)=G_0$ *;*

– $F(\alpha+1)=G_{\text{succ}}(F\restriction(\alpha+1))$ *;*

– $F(\lambda)=G_{\text{limit}}(F\restriction\lambda)$ *for limit ordinals λ.*

Then the term $F(\alpha)$ is definite.

Proof. Let F' be the canonical term defined by the $\in$-recursion

$$F'(x)=\begin{cases} 0, \text{ if } x=0, \\ G_{\text{succ}}(F'\restriction x), \text{ if } x \text{ is a successor ordinal,} \\ G_{\text{limit}}(F'\restriction x), \text{ if } x \text{ is a limit ordinal,} \\ 0, \text{ if } x\notin\text{Ord}. \end{cases}$$

By (an extension of) Theorem 6.2 d) on definition by cases, the recursion condition is definite and so is $F'(x)$. Then $F = F' \restriction \mathrm{Ord}$. $\square$

Exercise 6.1. Show that the terms ${}^n x$, V_n, V_ω are definite.

6.2 Absoluteness

Definition 6.5. *Let W be a transitive non-empty class. Let $\varphi(\vec{x})$ be an $\in$-formula and $t(\vec{x})$ be a term. Then*

a) *φ is W-**absolute** iff $\forall \vec{x} \in W\, (\varphi^W(\vec{x}) \leftrightarrow \varphi(\vec{x}))$;*

b) *t is W-**absolute** iff $\forall \vec{x} \in W\, (t^W(\vec{x}) \in W \leftrightarrow t(\vec{x}) \in V)$ and $\forall \vec{x} \in W\, t^W(\vec{x}) = t(\vec{x})$.*

Theorem 6.6. *Let W be a transitive model of* ZF^-. *Then*

a) *if $t(\vec{x})$ is a definite term then $\forall \vec{x}\, t(\vec{x}) \in V$;*

b) *every definite formula is W-absolute;*

c) *every definite term is W-absolute.*

Proof. a) may be proved by induction on the complexity of the definite term t. Most cases are immediate from the ZF^--axioms; if t is a canonical term defined by recursion with a definite recursion rule then the existence of $t(\vec{x})$ follows from the recursion principle.

The properties b) and c) are proved by a common induction along the generation rules of Definition 6.1 for definite formulas and terms. If $t(\vec{x})$ is a definite term, then by a)

$$(\forall \vec{x}\, t(\vec{x}) \in V)^W \rightarrow \forall \vec{x} \in W\, t^W(\vec{x}) \in W$$

so that always

$$\forall \vec{x} \in W\, (t^W(\vec{x}) \in W \leftrightarrow t(\vec{x}) \in V).$$

Thus for the W-absoluteness of t one only has to check

$$\forall \vec{x} \in W\, t^W(\vec{x}) = t(\vec{x}).$$

We now begin the induction. The cases 6.1 a) and b) are trivial.

6.1 c): Let $\varphi(x, \vec{z})$ be definite and assume that $\varphi(x, \vec{z})$ is W-absolute. Let y, $\vec{z} \in W$. Then $y \subseteq W$ and $y \cap W = y$, since W is transitive.

$$\begin{aligned}
(\forall x \in y\, \varphi(x, \vec{z}))^W &\leftrightarrow (\forall x\, (x \in y \rightarrow \varphi(x, \vec{z})))^W \\
&\leftrightarrow \forall x \in W\, (x \in y \rightarrow \varphi^W(x, \vec{z})) \\
&\leftrightarrow \forall x\, (x \in y \cap W \rightarrow \varphi^W(x, \vec{z})) \\
&\leftrightarrow \forall x\, (x \in y \rightarrow \varphi(x, \vec{z})), \text{ since } \varphi \text{ is } W\text{-absolute,} \\
&\leftrightarrow \forall x \in y\, \varphi(x, \vec{z}).
\end{aligned}$$

Thus $\forall x \in y\, \varphi(x, \vec{z})$ is W-absolute. Similarly, $\exists x \in y\, \varphi(x, \vec{z})$ is W-absolute.

Let us remark that cases 6.1 a) to c) imply that every $\in$-formula in which every quantifier is bounded is W-absolute. Such formulas are called Σ_0-formulas.
6.1 d): The only non-trivial case is the term

$$\omega = \{\alpha \in \mathrm{Ord} \mid \forall \beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal})\}.$$

(1) The formula $\alpha \in \omega$ is W-absolute.
Proof. By the remark above it suffices to see that the formula $\alpha \in \omega$ is equivalent to a Σ_0-formula.

$$\begin{aligned} \alpha \in \omega &\leftrightarrow \alpha \in \mathrm{Ord} \wedge \forall \beta \in \alpha + 1\, (\beta = 0 \vee \beta \text{ is a successor ordinal}) \\ &\leftrightarrow \mathrm{Trans}(\alpha) \wedge \forall y \in \alpha\, \mathrm{Trans}(y) \wedge \forall \beta \in \alpha\, (\forall x \in \beta\, x \neq x \vee \\ &\quad\ \exists \gamma \in \beta\ \beta = \gamma + 1) \wedge (\forall x \in \alpha\ x \neq x \vee \exists \gamma \in \alpha\ \alpha = \gamma + 1) \\ &\leftrightarrow \forall u \in \alpha \forall v \in u\, v \in \alpha \wedge \forall y \in \alpha \forall u \in y \forall v \in u\, v \in y \wedge \\ &\quad\ \forall \beta \in \alpha\, (\forall x \in \beta\, x \neq x \vee \exists \gamma \in \beta\, (\forall u \in \beta\, (u \in \gamma \vee u = \gamma) \wedge \\ &\quad\ \forall u \in \gamma\, u \in \beta \wedge \gamma \in \beta)) \wedge (\forall x \in \alpha\, x \neq x \vee \\ &\quad\ \exists \gamma \in \alpha\, (\forall u \in \alpha\, (u \in \gamma \vee u = \gamma) \wedge \forall u \in \gamma\, u \in \alpha \wedge \gamma \in \alpha)) \end{aligned}$$

qed(1)
(2) $\omega \subseteq W$.
Proof. By complete induction. $0 \in W$ since W is a non-empty transitive term. Assume that $n \in \omega$ and $n \in W$. Then, since $(\mathrm{ZF}^-)^W$, $(n \cup \{n\})^W \in W$.

$$\begin{aligned} (n \cup \{n\})^W &= \{x \in W \mid (x \in n \vee x \in \{n\})^W\} \\ &= \{x \in W \mid (x \in n \vee x = n)^W\} \\ &= \{x \in W \mid x \in n \vee x = n\} \\ &= \{x \mid x \in n \vee x = n\}, \text{ since } n \cup \{n\} \subseteq W, \\ &= n \cup \{n\}. \end{aligned}$$

Hence $n + 1 \in W$. *qed*(2)
(3) $\omega^M = \omega$.
Proof.

$$\begin{aligned} \omega^M &= \{x \in M \mid (x \in \omega)^M\} \\ &= \{x \in M \mid x \in \omega\}, \text{ since } x \in \omega \text{ is } W\text{-absolute}, \\ &= \{x \mid x \in \omega\}, \text{ since } \omega \subseteq M, \\ &= \omega. \end{aligned}$$

qed(3)

By our previous remarks this concludes case 6.1 d).
6.1 e): Let $\vec{y}$ be the free variables of the terms $t_0, \ldots, t_{n-1}$ and let $\vec{y} \in W$. Then by the inductive assumption

$$\begin{aligned} (s(t_0, \ldots, t_{n-1}))^W(\vec{y}) &= s^W(t_0^W(\vec{y}), \ldots, t_{n-1}^W(\vec{y})) \\ &= s^W(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\ &= s(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y})) \\ &= s(t_0, \ldots, t_{n-1})(\vec{y}). \end{aligned}$$

6.1 f): Let $\vec{y}$ be the free variables of the terms $t_0, \ldots, t_{n-1}$ and let $\vec{y} \in W$. Then by the inductive assumption

$$\begin{aligned}(\varphi(t_0, \ldots, t_{n-1}))^W(\vec{y}) &\leftrightarrow \varphi^W(t_0^W(\vec{y}), \ldots, t_{n-1}^W(\vec{y}))\\ &\leftrightarrow \varphi^W(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y}))\\ &\leftrightarrow \varphi(t_0(\vec{y}), \ldots, t_{n-1}(\vec{y}))\\ &\leftrightarrow \varphi(t_0, \ldots, t_{n-1})(\vec{y}).\end{aligned}$$

6.1 g): Let $y, \vec{z} \in W$. Then $y \subseteq W$ since W is transitive. By the inductive assumption

$$\begin{aligned}\{x \in y \mid \varphi(x, \vec{z})\}^W &= \{x \mid x \in y \wedge \varphi(x, \vec{z})\}^W\\ &= \{x \in W \mid x \in y \wedge \varphi^W(x, \vec{z})\}\\ &= \{x \in W \mid x \in y \wedge \varphi(x, \vec{z})\}\\ &= \{x \mid x \in y \wedge \varphi(x, \vec{z})\}, \text{ since } y \subseteq W,\\ &= \{x \in y \mid \varphi(x, \vec{z})\}.\end{aligned}$$

6.1 h): Let $y, \vec{z} \in W$. Then $y \subseteq W$ since W is transitive, and

$$\begin{aligned}\{t(x, \vec{z}) \mid x \in y\}^W &= \{z \mid \exists x \in y\ z = t(x, \vec{z})\}^W\\ &= \{z \mid \exists x (x \in y \wedge z = t(x, \vec{z})\}^W\\ &= \{z \in W \mid \exists x \in W (x \in y \wedge z = t^W(x, \vec{z})\}\\ &= \{z \mid \exists x \in W (x \in y \wedge z = t^W(x, \vec{z})\}, \text{ since } \forall x \in W t^W(x, \vec{z}) \in W,\\ &= \{z \mid \exists x \in W (x \in y \wedge z = t(x, \vec{z})\}, \text{ by inductive assumption,}\\ &= \{z \mid \exists x (x \in y \wedge z = t(x, \vec{z})\}, \text{ since } y \subseteq W,\\ &= \{t(x, \vec{z}) \mid x \in y\}.\end{aligned}$$

6.1 i): Let $G = G(z, \vec{y})$ with all free variables displayed and let F be the canonical term with

$$F(x, \vec{y}) = G(F \upharpoonright x, \vec{y}).$$

Let $\vec{y} \in W$. We show that $\forall x \in W F^W(x, \vec{y}) = F(x, \vec{y})$. Assume the contrary and let $x \in W$ be $\in$-minimal such that $F^W(x, \vec{y}) \neq F(x, \vec{y})$. Then by the recursion theorem in W,

$$\begin{aligned}F^W(x, \vec{y}) &= G^W(F^W \upharpoonright x, \vec{y})\\ &= G(F^W \upharpoonright x, \vec{y}), \text{ since } F^W \upharpoonright x \in W \text{ and } G \text{ is definite,}\\ &= G(F \upharpoonright x, \vec{y}), \text{ by the minimality of } x,\\ &= F(x, \vec{y}), \text{ contradiction.}\end{aligned}$$

□

Recursion can be used to show that certain terms involving *finiteness* are definite.

Definition 6.7. *Define* $\mathcal{P}_n(x) = \{y \subseteq x \mid \mathrm{card}(y) < n\}$ *for* $n \leqslant \omega$ *recursively by induction on* n*:*

- $\mathcal{P}_0(x) = \emptyset$;

– $\mathcal{P}_1(x) = \{\emptyset\}$;
– $\mathcal{P}_{n+1}(x) = \{y \cup \{z\} \mid y \in \mathcal{P}_n(x) \wedge z \in x\}$;
– $\mathcal{P}_\omega(x) = \bigcup_{n<\omega} \mathcal{P}_n(x)$.

Since this is an $\in$ *-recursion with a definite recursion rule the terms* $\mathcal{P}_n(x)$ *and* $\mathcal{P}_n(x)$ *are definite.*

We define a finitary version of the VON NEUMANN-hierarchy which agrees with the usual V_α-hierarchy for $\alpha \leqslant \omega$.

Definition 6.8. *Define* V_α^{fin} *for* $\alpha \in \text{Ord}$ *recursively:*

– $V_0^{\text{fin}} = \emptyset$,
– $V_{\alpha+1}^{\text{fin}} = \mathcal{P}_\omega(V_\alpha^{\text{fin}})$,
– $V_\lambda^{\text{fin}} = \bigcup_{\alpha<\lambda} V_\alpha^{\text{fin}}$ *for limit ordinals* λ.

Note that $V_\omega^{\text{fin}} = V_\omega$ *and that the term* V_α^{fin} *is definite. Hence* V_ω *is a definite term.*

Definition 6.9. *Define a well-order* $<_n$ *of* V_n *for* $n \leqslant \omega$ *recursively by induction on* n*:*

– $<_0 = \emptyset$;
– $<_{n+1} = <_n \cup (V_n \times (V_{n+1} \setminus V_n)) \cup$
$\cup \{(x, y) \in V_{n+1} \times V_{n+1} \mid \exists v \in y \setminus x \forall u \in V_n (u >_n v \rightarrow (u \in x \leftrightarrow u \in y))\}$;
– $<_\omega = \bigcup_{n<\omega} <_n$.

The terms $<_n$ *for* $n \leqslant \omega$ *are definite.*

We shall next give a definite definition of the set of finite sequences from a given set x which will later be used as the set of assignments in x.

Definition 6.10. *Define* ${}^n x = \{f \mid f \colon n \rightarrow x\}$ *for* $n \in \omega$ *by recursion on* n*:*

– ${}^0 x = \{\emptyset\}$;
– ${}^{n+1} x = \{f \cup \{(n, u)\} \mid f \in {}^n x \wedge u \in x\}$;
– ${}^{<\omega} x = \bigcup_{n<\omega} {}^n x$.

Call ${}^{<\omega} x$ *the set of* ***assigments*** *in* x.

There are natural operations on assignments:

Definition 6.11. *For* $f \in {}^{<\omega} x$, $a \in x$ *and* $k \in \text{dom}(f)$ *let*

$$f\frac{a}{k} = (f \setminus \{(k, f(k))\}) \cup \{(k, a)\}$$

be the ***substitution*** *of* a *into* f *at* k.

Chapter 7
Formalizing the logic of set theory

7.1 First-order logic

The theory ZF^- is able to formalize most basic mathematical notions. This general *formalization principle* also applies to first-order logic. For the definition of the constructible universe we shall be particularly interested in formalizing the logic of set theory within ZF^-, i.e., the logic of syntax and semantics of the language $\{\in\}$. Given some experience with definite formalizations the definite formalizability of first-order logic is quite obvious. For the sake of completeness we shall employ a concrete formalization as described in the monograph *Set Theory* by Frank Drake.

Standard first-order logic can be embedded into its formalized counterpart. So for every formula φ of the language of set theory we shall have a term $\ulcorner\varphi\urcorner$ which is a formalization of φ. Let us motivate the intended formalization by defining $\ulcorner\varphi\urcorner$ inductively over the complexity of φ.

Definition 7.1. *For each concrete $\in$-formula φ define its* **Goedel set** $\ulcorner\varphi\urcorner$ *by induction on the complexity of φ:*

- $\ulcorner v_i = v_j\urcorner = (0, i, j)$;
- $\ulcorner v_i \in v_j\urcorner = (1, i, j)$;
- $\ulcorner \varphi \wedge \psi\urcorner = (2, \ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$;
- $\ulcorner \neg\varphi\urcorner = (3, \ulcorner\varphi\urcorner)$;
- $\ulcorner \exists v_i\, \varphi\urcorner = (4, i, \ulcorner\varphi\urcorner)$.

Definition 7.2. *The formula* $\mathrm{Fm}(u, s, n)$ *describes that a formula u is constructed along a finite sequence s of length $n+1$ according to the construction principles of the previous definition:*

$$\begin{aligned}\mathrm{Fm}(u,s,n) \;\leftrightarrow\; & n \in \omega \wedge s \in {}^{n+1}V_\omega \wedge u = s(n) \wedge \\ & \wedge \forall k < n+1 \\ & (\exists i, j < \omega\ s(k) = (0, i, j) \vee \\ & \vee \exists i, j < \omega\ s(k) = (1, i, j) \vee \\ & \vee \exists l, m < k\ s(k) = (2, s(l), s(m)) \vee \\ & \vee \exists l < k\ s(k) = (3, s(l)) \vee \\ & \vee \exists l < k\ \exists i < \omega\ s(k) = (4, i, s(l))).\end{aligned}$$

Inspection of this definition shows that $\mathrm{Fm}(u,s,n)$ is definite.

Definition 7.3. *The formula* $\mathrm{Fmla}(u)$ *describes that* u *is a formalized* $\in$ *-formula:*

$$\mathrm{Fmla}(u) \leftrightarrow \exists n < \omega \exists s \in V_\omega \ \mathrm{Fm}(u,s,n).$$

The formula Fmla is also definite.

We formalize the TARSKIan satisfaction relation for the formulas u defined by Fmla. For each member of a construction sequence leading to u we consider the set of assignments in an $\in$-structure $(a, \in)$ which make the formula true.

Definition 7.4. *The formula* $S(s,a,r,t)$ *describes that* s *builds an* $\in$ *-formula as in Definition 7.2, and that* t *is a sequence of assignments of the variables* v_0, ..., v_{r-1} *in the* $\in$ *-structure* $(a, \in)$ *which make the corresponding* $\in$ *-formula of the sequence* s *true:*

$$\begin{aligned} S(s,a,r,t) \ \leftrightarrow \ & \exists u,n \in V_\omega \ \mathrm{Fm}(u,s,n) \wedge a \neq \emptyset \wedge r < \omega \wedge t\colon \mathrm{dom}(s) \to V_\omega \ \wedge \\ & \wedge \forall k \in \mathrm{dom}(s) \\ & \quad ((\exists i,j < \omega \ s(k) = (0,i,j) \wedge t(k) = \{b \in^r a \mid b(i) = b(j)\}) \vee \\ & \quad \vee (\exists i,j < \omega \ s(k) = (1,i,j) \wedge t(k) = \{b \in^r a \mid b(i) \in b(j)\}) \vee \\ & \quad \vee (\exists l,m < k \ s(k) = (2, s(l), s(m)) \wedge t(k) = t(l) \cap t(m)) \vee \\ & \quad \vee (\exists l < k \ s(k) = (3, s(l)) \wedge t(k) = {}^r a \setminus t(l)) \vee \\ & \quad \vee (\exists l < k \ \exists i < \omega \ s(k) = (4, i, s(l))) \ \wedge \\ & \qquad \wedge t(k) = \{b \in^r a \mid \exists x \in a \, (b \setminus \{(i, b(i))\}) \cup \{(i,x)\} \in t(l)\})). \end{aligned}$$

Then define the ***satisfaction relation*** $a \vDash u[b]$ *by* b *belonging to the assignments satisfying* u*:*

$$\begin{aligned} a \vDash u[b] \ \leftrightarrow \ & a \neq \emptyset \wedge \mathrm{Fmla}(u) \wedge b \in^{<\omega} a \ \wedge \\ & \wedge \exists s,r,t \in V_\omega \, (S(s,a,r,t) \wedge r = \mathrm{rk}(u) \wedge u = s(\mathrm{dom}(s) - 1) \ \wedge \\ & \wedge b \in t(\mathrm{dom}(s) - 1)). \end{aligned}$$

Note that

Theorem 7.5. *For each* $\in$ *-formula* $\varphi(v_0, ..., v_{n-1})$*:*

$$\forall a \forall x_0, ..., x_{n-1} \in a \, (\varphi^a(x_0, ..., x_{n-1}) \leftrightarrow a \vDash \ulcorner \varphi \urcorner [(x_0, ..., x_{n-1})].$$

On the right-hand side, $(x_0, ..., x_{n-1})$ *is the term*

$$\{(0, x_0), ..., (n-1, x_{n-1})\}.$$

Proof. By induction on the formula complexity of φ. □

7.2 Definable power sets

With these notions we can define a notion of definable power set crucial for the constructible hierarchy.

Definition 7.6. *a) For* $x \in V$*,* $\varphi \in \mathrm{Fml}$*, and* $\vec{a} \in^{<\omega} x$ *define the* ***interpretation*** *of* $(x, \varphi, \vec{a})$ *by*

$$I(x, \varphi, \vec{a}) = \{v \in x | x \vDash \varphi[\vec{a} \frac{v}{0}]\}$$

b) $\mathrm{Def}(x) = \{I(x, \varphi, \vec{p}) | \varphi \in \mathrm{Fml}, \vec{p} \in x\}$ *is the* ***definable power set*** *of* x*.*

The terms $I(x, \varphi, \vec{a})$ *and* $\mathrm{Def}(x)$ *are definite.*

Chapter 8
The constructible hierarchy

The constructible hierarchy is obtained by iterating the Def-operation along the ordinals.

Definition 8.1. *Define the* ***constructible hierarchy*** L_α , $\alpha \in \mathrm{Ord}$ *by recursion on* α*:*

$$\begin{aligned} L_0 &= \emptyset \\ L_{\alpha+1} &= \mathrm{Def}(L_\alpha) \\ L_\lambda &= \bigcup_{\alpha<\lambda} L_\alpha \text{, for } \lambda \text{ a limit ordinal.} \end{aligned}$$

The ***constructible universe*** L *is the union of that hierarchy:*

$$L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha .$$

The hierarchy satisfies natural hierarchical laws.

Theorem 8.2. a) $\alpha \leqslant \beta$ *implies* $L_\alpha \subseteq L_\beta$

b) L_β *is transitive*

c) $L_\beta \subseteq V_\beta$

d) $\alpha < \beta$ *implies* $L_\alpha \in L_\beta$

e) $L_\beta \cap \mathrm{Ord} = \beta$

f) $\beta \leqslant \omega$ *implies* $L_\beta = V_\beta$

g) $\beta \geqslant \omega$ *implies* $\mathrm{card}(L_\beta) = \mathrm{card}(\beta)$

Proof. By induction on $\beta \in \mathrm{Ord}$. The cases $\beta = 0$ and β a limit ordinal are easy and do not depend on the specific definition of the L_β-hierarchy.

Let $\beta = \gamma + 1$ where the claims hold for γ.

a) It suffices to show that $L_\gamma \subseteq L_\beta$. Let $x \in L_\gamma$. By b), L_γ is transitive and $x \subseteq L_\gamma$. Hence

$$x = \{v \in L_\gamma \,|\, v \in x\} = \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash (v \in w)\frac{x}{w}\} = I(L_\gamma, v \in w, x) \in L_{\gamma+1} = L_\beta .$$

b) Let $x \in L_\beta$. Let $x = I(L_\gamma, \varphi, \vec{p})$. Then by a) $x \subseteq L_\gamma \subseteq L_\beta$.

c) By induction hypothesis,

$$L_\beta = \mathrm{Def}(L_\gamma) \subseteq \mathcal{P}(L_\gamma) \subseteq \mathcal{P}(V_\gamma) = V_{\gamma+1} = V_\beta .$$

d) It suffices to show that $L_\gamma \in L_\beta$.

$$L_\gamma = \{v \in L_\gamma | v = v\} = \{v \in L_\gamma | (L_\gamma, \in) \vDash v = v\} = I(L_\gamma, v = v, \emptyset) \in L_{\gamma+1} = L_\beta .$$

e) $L_\beta \cap \mathrm{Ord} \subseteq V_\beta \cap \mathrm{Ord} = \beta$. For the converse, let $\delta < \beta$. If $\delta < \gamma$ the inductive hypothesis yields that $\delta \in L_\gamma \cap \mathrm{Ord} \subseteq L_\beta \cap \mathrm{Ord}$. Consider the case $\delta = \gamma$. We have to show that $\gamma \in L_\beta$. There is a formula $\varphi(v)$ which is Σ_0 and formalizes being an ordinal. This means that all quantifiers in φ are bounded and if z is transitive then

$$\forall v \in z \, (v \in \mathrm{Ord} \leftrightarrow (z, \in) \vDash \varphi(v)).$$

By induction hypothesis

$$\begin{aligned} \gamma &= \{v \in L_\gamma | v \in \mathrm{Ord}\} \\ &= \{v \in L_\gamma \,|\, (L_\gamma, \in) \vDash \varphi(v)\} \\ &= I(L_\gamma, \varphi, \emptyset) \\ &\in L_{\gamma+1} = L_\beta . \end{aligned}$$

f) Let $\beta < \omega$. By c) it suffices to see that $V_\beta \subseteq L_\beta$. Let $x \in V_\beta$. By induction hypothesis, $L_\gamma = V_\gamma$. $x \subseteq V_\gamma = L_\gamma$. Let $x = \{x_0, ..., x_{n-1}\}$. Then

$$\begin{aligned} x &= \{v \in L_\gamma \mid v = x_0 \vee v = x_1 \vee ... \vee v = x_{n-1}\} \\ &= \{v \in L_\gamma \mid (L_\gamma, \in) \vDash (v = v_0 \vee v = v_1 \vee ... \vee v = v_{n-1}) \frac{x_0\, x_1 \ldots x_{n-1}}{v_0\, v_1 \ldots v_{n-1}}\} \\ &= I(L_\gamma, (v = v_0 \vee v = v_1 \vee ... \vee v = v_{n-1}), x_0, x_1, ..., x_{n-1}) \\ &\in L_{\gamma+1} = L_\beta . \end{aligned}$$

g) Let $\beta > \omega$. By induction hypothesis $\mathrm{card}(L_\gamma) = \mathrm{card}(\gamma)$. Then

$$\begin{aligned} \mathrm{card}(\beta) &\leqslant \mathrm{card}(L_\beta) \\ &\leqslant \mathrm{card}(\{I(L_\gamma, \varphi, \vec{p}) | \varphi \in \mathrm{Fml}, \vec{p} \in L_\gamma\}) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}({}^{<\omega}L_\gamma) \\ &\leqslant \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}(L_\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma)^{<\omega} \\ &= \aleph_0 \cdot \mathrm{card}(\gamma), \text{ since } \gamma \text{ is infinite,} \\ &= \mathrm{card}(\gamma) \\ &= \mathrm{card}(\beta). \end{aligned}$$

□

The properties of the constructible hierarchy immediately imply the following for the constructible universe.

Theorem 8.3. *a) L is transitive.*

b) $\mathrm{Ord} \subseteq L$.

Theorem 8.4. $(L, \in)$ *is a model of* ZF.

Proof. By a previous theorem it suffices that L is transitive, *almost universal* and *closed under definitions*.

(1) L is almost universal, i.e., $\forall x\subseteq L\exists y\in L\,x\subseteq y$.

Proof. Let $x\subseteq L$. For each $u\in L$ let $\mathrm{rk}(u)=\min\{\alpha\,|\,u\in L_\alpha\}$ be its *constructible rank*. By replacement in V let $\beta=\bigcup\{\mathrm{rk}(u)\,|\,u\in x\}\in\mathrm{Ord}$. Then

$$x\subseteq L_\beta\in L.$$

(2) L is closed under definition, i.e., for every $\in$-formula $\varphi(x,\vec{y})$ holds

$$\forall a,\vec{y}\in L\,\{x\in a\,|\,\varphi^L(x,\vec{y})\}\in L.$$

Proof. Let $\varphi(x,\vec{y})$ be an $\in$-formula and $a,\vec{y}\in L$. Let $a,\vec{y}\in L_{\theta_0}$. By the Levy reflection theorem there is some $\theta\geqslant\theta_0$ such that φ is L_θ-L-absolute, i.e.,

$$\forall u,\vec{v}\in L_\theta\,(\varphi^{L_\theta}(u,\vec{v})\leftrightarrow\varphi^L(u,\vec{v})).$$

Then

$$\begin{aligned}\{x\in a\,|\,\varphi^L(x,\vec{y})\} &= \{x\in L_\theta\,|\,x\in a\wedge\varphi^L(x,\vec{y})\}\\ &= \{x\in L_\theta\,|\,x\in a\wedge\varphi^{L_\theta}(x,\vec{y})\}\\ &= \{x\in L_\theta\,|\,(x\in a\wedge\varphi(x,\vec{y}))^{L_\theta}\}\\ &= I(L_\theta,(x\in z\wedge\varphi(x,\vec{v})),\frac{a\,\vec{y}}{z\,\vec{v}})\in L_{\theta+1}\subseteq L.\end{aligned}$$

□

The recursive and definite definition of the L_α-hierarchy implies immediately:

Theorem 8.5. *The term L_α is definite.*

8.1 Wellordering L

We shall now prove an *external* choice principle and also an *external* continuum hypothesis for the constructible sets. These will later be internalized through the *axiom of constructibility*. Every constructible set x is of the form

$$x=I(L_\alpha,\varphi,\vec{p});$$

$(L_\alpha,\varphi,\vec{p})$ is a *name* for x.

Definition 8.6. *Define the class of **(constructible) names** or **locations** as*

$$\tilde{L}=\{(L_\alpha,\varphi,\vec{p})|\alpha\in\mathrm{Ord},\varphi(v,\vec{v})\in\mathrm{Fml},\vec{p}\in L_\alpha,\mathrm{length}(\vec{p})=\mathrm{length}(\vec{v})\}.$$

This class has a natural stratification

$$\tilde{L}_\alpha=\{(L_\beta,\varphi,\vec{p})\in\tilde{L}|\beta<\alpha\}\text{ for }\alpha\in\mathrm{Ord}.$$

*A location of the form $(L_\alpha,\varphi,\vec{p})$ is called an α-**location**.*

Definition 8.7. *Define wellorders* $<_\alpha$ *of* L_α *and* $\tilde{<}_\alpha$ *of* $\tilde{L}_\alpha$ *by recursion on* α.

- $<_0 = \tilde{<}_0 = \emptyset$ *is the vacuous ordering on* $L_0 = \tilde{L}_0 = \emptyset$;
- *if* $<_\alpha$ *is a wellordering of* L_α *then define* $\tilde{<}_{\alpha+1}$ *on* $\tilde{L}_{\alpha+1}$ *by:*
  $(L_\beta, \varphi, \vec{x}) \tilde{<}_{\alpha+1} (L_\gamma, \psi, \vec{y})$ *iff*
  $(\beta < \gamma)$ *or* $(\beta = \gamma \wedge \varphi < \psi)$ *or*
  $(\beta = \gamma \wedge \varphi = \psi \wedge \vec{x}$ *is lexicographically less than* $\vec{y}$ *with respect to* $<_\alpha$);
- *if* $\tilde{<}_{\alpha+1}$ *is a wellordering on* $\tilde{L}_{\alpha+1}$ *then define* $<_{\alpha+1}$ *on* $L_{\alpha+1}$ *by:*
  $y <_{\alpha+1} z$ *iff there is a name for* y *which is* $\tilde{<}_{\alpha+1}$*-smaller then every name for* z.
- *for limit* λ, *let* $<_\lambda = \bigcup_{\alpha<\lambda} <_\alpha$ *and* $\tilde{<}_\lambda = \bigcup_{\alpha<\lambda} \tilde{<}_\alpha$. ~

This defines two hierarchies of wellorderings linked by the interpretation function I.

Theorem 8.8. *a)* $<_\alpha$ *and* $\tilde{<}_\alpha$ *are well-defined*

b) $\tilde{<}_\alpha$ *is a wellordering of* $\tilde{L}_\alpha$

c) $<_\alpha$ *is a wellordering of* L_α

d) $\beta < \alpha$ *implies that* $\tilde{<}_\beta$ *is an initial segment of* $\tilde{<}_\alpha$

e) $\beta < \alpha$ *implies that* $<_\beta$ *is an initial segment of* $<_\alpha$

Proof. By induction on $\alpha \in \mathrm{Ord}$. □

We can thus define wellorders $<_L$ and $\tilde{<}$ of L and $\tilde{L}$ respectively:

$$<_L = \bigcup_{\alpha \in \mathrm{Ord}} <_\alpha \quad \text{and} \quad \tilde{<} = \bigcup_{\alpha \in \mathrm{Ord}} \tilde{<}_\alpha$$

Theorem 8.9. $<_L$ *is a wellordering of* L.

The above recursions are definite and yield:

Theorem 8.10. *The terms* $<_\alpha$ *and* $\tilde{<}_\alpha$ *are definite.*

8.2 An external continuum hypothesis

Theorem 8.11. $\mathcal{P}(\omega) \cap L \subseteq L_{\aleph_1}$.

"**Proof**". Let $m \in \mathcal{P}(\omega) \cap L$. By the downward LÖWENHEIM SKOLEM theorem let $K \prec L$ be a "sufficiently elementary" substructure such that

$$m \in K \text{ and } \mathrm{card}(K) = \aleph_0.$$

Let $\pi: (K, \in) \cong (K', \in)$ be the MOSTOWSKI transitivisation of K defined by

$$\pi(u) = \{\pi(v) | v \in u \wedge v \in K\}.$$

$\pi \restriction \omega = \mathrm{id} \restriction \omega$ and

$$\pi(m) = \{\pi(i) | i \in m \wedge i \in X\} = \{\pi(i) | i \in m\} = \{i | i \in m\} = m.$$

A *condensation argument* will show that there is $\eta \in \mathrm{Ord}$ with
$K' = L_\eta$. $\mathrm{card}(\eta) \leqslant \mathrm{card}(L_\eta) = \mathrm{card}(K) = \aleph_0$ and $\eta < \aleph_1$. Hence

$$m \in K' = L_\eta \subseteq L_{\aleph_1}.$$

Chapter 9
The Axiom of Constructibility

If $V = L$ holds then every set is constructible, and the above external arguments become internal. We shall show that $(V = L)^L$.

Definition 9.1. *The* ***axiom of constructibility*** *is the property* $V = L$.

Theorem 9.2. (ZF$^-$) *The axiom of constructibility holds in* L. *This can be also written as* $(V = L)^L$ *or* $L = L^L$.

Proof. By Theorem 8.5, the term L_α is definite. Thus the formula $x \in L_\alpha$ is absolute for the transitive ZF$^-$-model L. Since $L = \bigcup_{\alpha \in \mathrm{Ord}} L_\alpha$ we have

$\forall x \in L \exists \alpha \in \mathrm{Ord}\, x \in L_\alpha$
$\forall x \in L \exists \alpha \in L (\alpha \in \mathrm{Ord} \wedge x \in L_\alpha)$
$\forall x \in L \exists \alpha \in L ((\alpha \in \mathrm{Ord})^L \wedge (x \in L_\alpha)^L)$
$\forall x \in L \exists \alpha \in L ((\alpha \in \mathrm{Ord})^L \wedge (x \in L_\alpha)^L)$
$(\forall x \exists \alpha\, x \in L_\alpha)^L$
$(\forall x\, x \in L)^L$
$(V = L)^L$. □

Theorem 9.3. (ZF$^-$) *The axiom of choice holds in* L*:* AC^L.

Proof. Work in L. By the previous theorem, $V = L$. The relation $<_L$ is a wellordering of L, hence it is a wellordering of V. □

By previous discussions of relative consistency this yields:

Theorem 9.4. (Gödel) *If the theory* ZF *is consistent then the theory* ZFC *is also consistent.*

Theorem 9.5. L *is the* $\subseteq$-minimal *inner model of* ZF$^-$*, i.e., if* M *is a transitive model of* ZF$^-$ *which contains all the ordinals then* $L \subseteq M$.

Proof. Consider $x \in L_\alpha$. Since $\alpha \in M$ and since L_α is definite, $L_\alpha \in M$. By the transitivity of M, $x \in M$. □

Chapter 10

Constructible Operations and condensation

There are various ways of ensuring the condensation property for the structure K as used in the above argument for the continuum hypothesis. We shall only require closure under some basic operations of constructibility theory, in particular the interpretation operator I. An early predecessor for this approach to condensation and to hyperfine structure theory can be found in GÖDEL's 1939 paper [7]:

> Proof: Define a set K of constructible sets, a set O of ordinals and a set F of Skolem functions by the following postulates I-VII:
>
> I. $M_{\omega_\mu} \subseteq K$ and $m \in K$.
>
> II. If $x \in K$, the order of x belongs to O.
>
> III. If $x \in K$, all constants occuring in the definition of x belong to K.
>
> IV. If $\alpha \in O$ and $\phi_\alpha(x)$ is a propositional function over M_α all of whose constants belong to K, then:
>
> 1. The subset of M_α defined by ϕ_α belongs to K.
> 2. For any $y \in K \cdot M_\alpha$ the designated Skolem functions for ϕ_α and y or $\sim \phi_\alpha$ and y (according as $\phi_\alpha(y)$ or $\sim \phi_\alpha(y)$) belong to F.
>
> V. If $f \in F$, $x_1, ..., x_n \in K$ and $(x_1, ..., x_n)$ belongs to the domain of definition of f, then $f(x_1, ..., x_n) \in K$.
>
> VI. If $x, y \in K$ and $x - y \neq \Lambda$ the first element of $x - y$ belongs to K.
>
> VII. No proper subsets of K, O, F satisfy I–VI.
>
> ....
>
> ....

> Theorem 5. There exists a one-to-one mapping x' of K on M_η such that $x \in y \equiv x' \in y'$ for $x, y \in K$ and $x' = x$ for $x \in M_{\omega_\mu}$.
>
> Proof: The mapping x' (....) is defined by transfinite induction on the order, ...

10.1 Constructible operations

A substructure of the kind considered by GÖDEL may be obtained as a closure with respect to certain *constructible operations*.

Definition 10.1. *Define the constructible operations I, N, S by:*

a) ***Interpretation**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$I(L_\alpha, \varphi, \vec{x}) = \{y \in L_\alpha \mid (L_\alpha, \in) \vDash \varphi(y, \vec{x})\}$;

b) ***Naming**: for $y \in L$ let*
$N(y) =$ the $\tilde{<}$*-least name* $(L_\alpha, \varphi, \vec{x})$ *such that* $I(L_\alpha, \varphi, \vec{x}) = y$.

c) ***Skolem function**: for a name $(L_\alpha, \varphi, \vec{x})$ let*
$S(L_\alpha, \varphi, \vec{x}) =$ *the* $<_L$ *- least* $y \in L_\alpha$ *such that* $L_\alpha \vDash \varphi(y, \vec{x})$ *if such a* y *exists; set* $S(L_\alpha, \varphi, \vec{x}) = 0$ *if such a* y *does not exist.*

As we do not assume that α is a limit ordinal and therefore do not have pairing, we make the following convention.

For $X \subseteq L$, $(L_\alpha, \varphi, \vec{x})$ a name we write $(L_\alpha, \varphi, \vec{x}) \in X$ to mean that L_α and each component of $\vec{x}$ is an element of X.

Definition 10.2. *$X \subseteq L$ is **constructibly closed**, $X \triangleleft L$, iff X is closed under I, N, S:*

$$(L_\alpha, \varphi, \vec{x}) \in X \longrightarrow I(L_\alpha, \varphi, \vec{x}) \in X \text{ and } S(L_\alpha, \varphi, \vec{x}) \in X,$$
$$y \in X \longrightarrow N(y) \in X.$$

*For $X \subseteq L$, $L\{X\} =$ the $\subseteq$-smallest $Y \supseteq X$ such that $Y \triangleleft L$ is called the **constructible hull** of X.*

The constructible hull $L\{X\}$ of X can be obtained by closing X under the functions I, N, S in the obvious way. Hulls of this kind satisfy certain "algebraic" laws which will be stated later in the context of *fine* hulls. Clearly each L_α is constructibly closed.

Theorem 10.3. (Condensation Theorem) *Let X be constructibly closed and let π: $X \cong M$ be the* MOSTOWSKI *collapse of X onto the transitive set M. Then there is an ordinal α such that $M = L_\alpha$, and π preserves I, N, S and $<_L$:*

$$\pi: (X, \in, <_L, I, N, S) \cong (L_\alpha, \in, <_L, I, N, S).$$

Proof. We first show the legitimacy of performing a Mostowski collapse.
(1) $(X, \in)$ is extensional.
Proof. Let $x, y \in X$, $x \neq y$. Let $N(x) = (L_\alpha, \varphi, \vec{p}) \in X$ and $N(y) = (L_\beta, \psi, \vec{q}) \in X$.
Case 1. $\alpha < \beta$. Then $x \in L_\beta$ and $(L_\beta, \in) \vDash \exists v (v \in x \nleftrightarrow \psi(v, \vec{q}))$. Let

$$z = S(L_\beta, (v \in u \nleftrightarrow \psi(v, \vec{w})), \frac{x\,\vec{q}}{u\,\vec{w}}) \in X$$

Then $z \in x \nleftrightarrow z \in y$. *qed*(1)

We prove the theorem for $X \subseteq L_\gamma$, by induction on γ. There is nothing to show in case $\gamma = 0$. For γ a limit ordinal observe that

$$\pi = \bigcup_{\alpha < \gamma} \pi \restriction (X \cap L_\gamma)$$

where each $\pi \restriction (X \cap L_\gamma)$ is the Mostowski collapse of the constructibly closed set $X \cap L_\gamma$ which by induction already satisfies the theorem.

So let $\gamma = \beta + 1$, $X \subseteq L_{\beta+1}$, $X \nsubseteq L_\beta$, and the theorem holds for β. Let

$$\pi: (X, \in) \cong (\bar{X}, \in)$$

be the Mostowski collapse of X. $X \cap L_\beta$ is an $\in$-initial segment of X, hence $\pi \restriction X \cap L_\beta$ is the Mostowski collapse of $X \cap L_\beta$. $X \cap L_\beta$ is constructibly closed and so by the inductive assumption there is some ordinal $\bar{\beta}$ such that

$$\pi \restriction X \cap L_\beta: (X \cap L_\beta, \in, <_L, I, N, S) \cong (L_{\bar{\beta}}, \in, <_L, I, N, S).$$

Note that the inverse map π^{-1}: $L_{\bar{\beta}} \to L_\beta$ is elementary since $X \cap L_\beta$ is closed under Skolem functions for L_β.
(2) $L_\beta \in X$.
Proof. Take $x \in X \setminus L_\beta$. Let $N(x) = (L_\gamma, \varphi, \vec{p})$. Then $L_\gamma \in X$ and $L_\gamma = L_\beta$ since $x \notin L_\beta$. *qed*(2)
(3) $\pi(L_\beta) = L_{\bar{\beta}}$.
Proof. $\pi(L_\beta) = \{\pi(x) | x \in L_\beta \wedge x \in X\} = \{\pi(x) | x \in X \cap L_\beta\} = L_{\bar{\beta}}$.
(4) $X = \{I(L_\beta, \varphi, \vec{p}) | \vec{p} \in X \cap L_\beta\}$.
Proof. $\supseteq$ is clear. For the converse let $x \in X$.
Case 1. $x \in L_\beta$. Then $x = I(L_\beta, v \in v_1, \frac{x}{v_1})$ is of the required form.
Case 2. $x \in L \setminus L_\beta$. Let $N(x) = (L_\beta, \varphi, \vec{p})$, noting that the first component cannot be smaller than L_β. $\vec{p} \in X$ and $x = I(N(x)) = I(L_\beta, \varphi, \vec{p})$ is of the required form. *qed*(4)
(5) Let $\vec{x} \in X$. Then $\pi(I(L_\beta, \varphi, \vec{x})) = I(L_{\bar{\beta}}, \varphi, \pi(\vec{x}))$.
Proof.

$$\begin{aligned}
\pi(I(L_\beta, \varphi, \vec{x})) &= \{\pi(y) | y \in \pi(I(L_\beta, \varphi, \vec{x})) \wedge y \in X\} \\
&= \{\pi(y) | (L_\beta, \in) \vDash \varphi(y, \vec{x}) \wedge y \in X\} \\
&= \{\pi(y) | (L_{\bar{\beta}}, \in) \vDash \varphi(\pi(y), \pi(\vec{x})) \wedge y \in X\} \\
&= \{z \in L_{\bar{\beta}} | (L_{\bar{\beta}}, \in) \vDash \varphi(z, \pi(\vec{x}))\} \\
&= I(L_{\bar{\beta}}, \varphi, \pi(\vec{x})).
\end{aligned}$$

qed(5)
(6) $\bar{X} = L_{\bar{\beta}+1}$.
Proof. By (4,5),

$$\begin{aligned} L_{\bar{\beta}+1} &= \{I(L_{\bar{\beta}}, \varphi, \vec{x}) \mid \vec{x} \in L_{\bar{\beta}}\} \\ &= \{I(L_{\bar{\beta}}, \varphi, \pi(\vec{p})) \mid \vec{p} \in X \cap L_\beta\}, \text{ since } \pi \restriction X \cap L_\beta \colon X \cap L_\beta \cong L_{\bar{\beta}}\ , \\ &= \{\pi(I(L_\beta, \varphi, \vec{p})) \mid \vec{p} \in X \cap L_\beta\} \\ &= \pi''\{I(L_\beta, \varphi, \vec{p}) \mid \vec{p} \in X \cap L_\beta\} \\ &= \pi''X = \bar{X}. \end{aligned}$$

qed(6)
(7) Let $y \in X$. Then $\pi(N(y)) = N(\pi(y))$. This means: if $N(y) = (L_\delta, \varphi, \vec{x})$ then $N(\pi(y)) = (\pi(L_\delta), \varphi, \pi(\vec{x})) = (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. Let $N(y) = (L_\delta, \varphi, \vec{x})$. Then $y = I(L_\delta, \varphi, \vec{x})$ and by (5) we have $\pi(y) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq N(\pi(y))$. Let $N(\pi(y)) = (L_\eta, \psi, \vec{y})$. By the minimality of names we have $(L_\eta, \psi, \vec{y}) \tilde{<} (L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Then $(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \tilde{<} (L_\delta, \varphi, \vec{x})$. By the minimality of $(L_\delta, \varphi, \vec{x}) = N(y)$, $I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y})) \neq I(L_\delta, \varphi, \vec{x}) = y$. Since π is injective and by (5),

$$\begin{aligned} \pi(y) &\neq \pi(I(L_{\pi^{-1}(\eta)}, \psi, \pi^{-1}(\vec{y}))) \\ &= I(L_\eta, \psi, \vec{y}) \\ &= I(N(y)) = y. \end{aligned}$$

Contradiction. *qed*(7)
(8) Let $x, y \in X$. Then $x <_L y$ iff $\pi(x) <_L \pi(y)$.
Proof. $x <_L y$ iff $N(x) \tilde{<} N(y)$ iff $\pi(N(x)) \tilde{<} \pi(N(y))$ (since inductively π preserves $<_L$ on $X \cap L_\beta$ and $\tilde{<}$ is canonically defined from $<_L$) iff $N(\pi(x)) \tilde{<} N(\pi(y))$ iff $\pi(x) <_L \pi(y)$. *qed*(8)
(9) Let $(L_\delta, \varphi, \vec{x}) \in X$. Then $\pi(S(L_\delta, \varphi, \vec{x})) = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$.
Proof. We distinguish cases according to the definition of $S(L_\delta, \varphi, \vec{x})$.
Case 1. There is no $v \in I(L_\delta, \varphi, \vec{x})$, i.e., $I(L_\delta, \varphi, \vec{x}) = \emptyset$ and $S(L_\delta, \varphi, \vec{x}) = \emptyset$. Then by (5),

$$I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \pi(I(L_\delta, \varphi, \vec{x})) = \pi(\emptyset) = \emptyset$$

and $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) = \emptyset$. So the claim holds in this case.
Case 2. There is $v \in I(L_\delta, \varphi, \vec{x})$, and then $S(L_\delta, \varphi, \vec{x})$ is the $<_L$-smallest element of $I(L_\delta, \varphi, \vec{x})$. Let $y = S(L_\delta, \varphi, \vec{x})$. By (5),

$$\pi(y) \in \pi(I(L_\delta, \varphi, \vec{x})) = I(L_{\pi(\delta)}, \varphi, \pi(\vec{x})).$$

So $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$ is well-defined as the $<_L$-minimal element of $I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. Assume for a contradiction that $S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \neq \pi(y)$. Let $z = S(L_{\pi(\delta)}, \varphi, \pi(\vec{x})) \in I(L_{\pi(\delta)}, \varphi, \pi(\vec{x}))$. By the minimality of SKOLEM values, $z <_L \pi(y)$. By (8), $\pi^{-1}(z) <_L y$. Since π is $\in$-preserving, $\pi^{-1}(z) \in I(L_\delta, \varphi, \vec{x})$. But this contradicts the $<_L$-minimality of $y = S(L_\delta, \varphi, \vec{x})$ □

Chapter 11
GCH in L

Theorem 11.1. $(L,\in)\vDash\mathrm{GCH}$.

Proof. $(L,\in)\vDash V=L$. It suffices to show that

$$\mathrm{ZFC}+V=L\vdash\mathrm{GCH}.$$

Let $\omega_\mu\geqslant\aleph_0$ be an infinite cardinal.
(1) $\mathcal{P}(\omega_\mu)\subseteq L_{\omega_\mu^+}$.
Proof. Let $m\in\mathcal{P}(\omega_\mu)$. Let $K=L\{L_{\omega_\mu}\cup\{m\}\}$ be the constructible hull of $L_{\omega_\mu}\cup\{m\}$. By the Condensation Theorem take an ordinal η and and the MOSTOWSKI isomorphism

$$\pi\colon(K,\in)\cong(L_\eta,\in).$$

Since $L_{\omega_\mu}\subseteq K$ we have $\pi(m)=m$.

$$\eta<\mathrm{card}(\eta)^+=\mathrm{card}(L_\eta)^+=\mathrm{card}(K)^+=\mathrm{card}(L_{\omega_\mu})^+=\omega_\mu^+.$$

Hence $m\in L_\eta\subseteq L_{\omega_\mu^+}$. *qed*(1)

Thus $\omega_\mu^+\leqslant\mathrm{card}(\mathcal{P}(\omega_\mu))\leqslant\mathrm{card}(L_{\omega_\mu^+})=\omega_\mu^+$. □

Chapter 12
Trees

Throughout these lectures we shall prove combinatorial principles in L and apply them to construct specific structures that cannot be proved to exist in ZFC alone. We concentrate on the construction of infinite *trees* since they are purely combinatorial objects which are still quite close to ordinals and cardinals. One could extend these considerations and also construct unusual topological spaces or uncountable groups.

Definition 12.1. *A* ***tree*** *is a strict partial order $T = (T, <_T)$, such that $\forall t \in T \{s \in T \mid s <_T t\}$ is well-ordered by $<_T$. For $t \in T$ let $\mathrm{ht}_T(t) = \mathrm{otp}(\{s \in T \mid s <_T t\})$ be the* ***height*** *of t in T. For $X \subseteq \mathrm{Ord}$ let T_X be the set of points in the tree whose heights lie in X:*

$$T_X = \{t \in T \mid \mathrm{ht}_T(t) \in X\}.$$

In particular, $T_{\{\alpha\}}$ is the α-th ***level*** *of the tree and T_α is the initial segment of T below α. We let*

$$\mathrm{ht}(T) = \min \{\alpha \mid T = T_\alpha\}$$

be the ***height*** *of the tree T.*

A ***chain*** *in T is a linearly ordered subset of T. An $\subseteq$-maximal chain is called a* ***branch****.*

Definition 12.2. *A tree T of cardinality λ all of whose levels and branches are of cardinality $< \lambda$ is called a λ-**Aronszajn tree**. If $\lambda = \omega_1$, T is called an* ***Aronszajn tree****.*

Theorem 12.3. *Let κ be regular and $\forall \lambda < \kappa\, 2^\lambda \leqslant \kappa$. Then there is a κ^+-Aronszajn tree.*

Hence in ZFC one can show the existence of an (ω_1-)Aronszajn tree. The generalized continuum hypothesis implies the assumption $\forall \lambda < \kappa\, 2^\lambda \leqslant \kappa$, so in L there are κ^+-Aronszajn trees for every regular κ.

Theorem 12.4. *Let κ be an infinite cardinal. Then there is a linear order $(Q, \prec)$ such that $\mathrm{card}(Q) = \kappa$ and every $\alpha < \kappa^+$ can be order-embedded into every proper interval of Q.*

Proof. Let $Q = \{a \in {}^\omega \kappa \mid \exists m \in \omega \forall n \in \omega\, (n > m \rightarrow a(n) = 0\}$ be the set of ω-sequences from κ which are eventually zero and define the lexicographic linear order $\prec$ on Q by:

$$a \prec b \leftrightarrow \exists n \in \omega (a \restriction n = b \restriction n \wedge a(n) < b(n)).$$

We first prove the embedding property for $\alpha = \kappa$:
(1) If $a \prec b$ then there is an order-preserving embedding

$$f: (\kappa, <) \to ((a, b), \prec)$$

into the interval $(a, b) = \{c \mid a \prec c \prec b\}$.
Proof. Take $n \in \omega$ such that

$$a \upharpoonright n = b \upharpoonright n \wedge a(n) < b(n).$$

Then define $f: (\kappa, <) \to ((a, b), \prec)$ by

$$f(i) = (a \upharpoonright n+1) \cup \{(n+1, a(n+1)+1+i)\} \cup (0 \mid n+2 \leqslant l < \omega).$$

qed(1)

We prove the full theorem by induction on $\alpha < \kappa^+$. Let $\alpha < \kappa^+$ and assume that the theorem holds for all $\beta < \alpha$. Let (a, b) be a proper interval of Q, $a \prec b$.
Case 1: $\alpha = \beta + 1$ is a successor ordinal. By (1) take $b' \in (a, b)$. By the inductive assumption take an order-preserving map $f': (\beta, <) \to (a, b')$. Extend f' to an order-preserving map $f: (\alpha, <) \to (a, b)$ by setting $f(\beta) = b'$.
Case 2: α is a limit ordinal. Since $\alpha < \kappa^+$ let $\alpha = \bigcup_{i<\kappa} \alpha_i$ such that $\forall i < \kappa\, \alpha_i < \alpha$. By (1) let $f: (\kappa, <) \to ((a, b), \prec)$ order-preservingly. By the inductive assumption choose a sequence $(g_i \mid i < \kappa)$ of order-preserving embeddings

$$g_i: (\alpha_i, <) \to ((f(i), f(i+1)), \prec).$$

Then define an order-preserving embedding

$$h: (\alpha, <) \to ((a, b), \prec)$$

by $h(\beta) = g_i(\beta)$, where $i < \kappa$ is minimal such that $\beta \in \alpha_i$. □

Proof of Theorem 12.3. Let $(Q, \prec)$ be a linear order as in Theorem 12.4. We define a tree

$$T \subseteq \{t \mid \exists \alpha < \kappa^+ \, t: (\alpha, <) \to (Q, \prec) \text{ is order-preserving}\}$$

with strict inclusion $\subset$ as the tree order such that:

a) T is closed under initial segments, i.e., $\forall t \in T \forall \xi \in \mathrm{Ord}\; t \upharpoonright \xi \in T$;

b) for all $\alpha < \kappa^+$, $T_{\{\alpha\}} = \{t \in T \mid \mathrm{dom}(t) = \alpha\}$ has cardinality $\leqslant \kappa$;

c) for all limit ordinals $\alpha < \kappa^+$ with $\mathrm{cof}(\alpha) < \kappa$

$$T_{\{\alpha\}} = \{t \mid t: \alpha \to Q \wedge \forall \beta < \alpha \;\; t \upharpoonright \beta \in T_{\{\beta\}}\}.$$

d) for all $\alpha < \beta < \kappa^+$, $t \in T_{\{\alpha\}}$, $a \prec b \in Q$ such that $\forall \xi \in \alpha\; t(\xi) \prec a$ there exists $t' \in T_{\{\beta\}}$ such that $t \subset t'$ and $\forall \xi \in \beta\; t'(\xi) \prec b$.

We define the levels $T_{\{\alpha\}}$ by recursion on $\alpha < \kappa^+$.
Let $T_{\{0\}} = \{\emptyset\}$.
Let $\alpha = \beta + 1$ and assume that $T_{\{\beta\}}$ is defined according to a) - d). For any $t \in T_{\{b\}}$ and $a \prec b \in Q$ such that $\forall \xi \in \beta\, t(\xi) \prec a$ choose an extension $t'_{a,b}$ such that

– $t'_{a,b}: (\alpha, <) \to (Q, \prec)$ is order-preserving;

- $t'_{a,b} \supset t$;
- $\forall \xi \in \alpha \; t'_{a,b}(\xi) \prec b$.

One could for example set $t'_{a,b}(\beta) = a$. Then set

$$T_{\{\alpha\}} = \{t'_{a,b} \,|\, t \in T_{\{\beta\}}, a \prec b, \forall \xi \in \beta \, t(\xi) \prec a\}.$$

Obviously, conditions a) - d) are satisfied.
Let $\alpha < \kappa^+$ be a limit ordinal so that for all $\beta < \alpha$ $T_{\{\beta\}}$ is defined according to a) - d). These are the levels of the tree T_α.
Case 1: $\text{cof}(\alpha) < \kappa$. Let the sequence $(\alpha_i \,|\, i < \text{cof}(\alpha))$ be continuous and cofinal in α with $\text{cof}(\alpha) < \alpha_0$. By c) we must set

$$T_{\{\alpha\}} = \{t \,|\, t: \alpha \to Q \wedge \forall \beta < \alpha \;\, t \upharpoonright \beta \in T_{\{\beta\}}\}.$$

Let us check that properties a) - d) hold for this definition. a) is immediate. For b), note that every $t \in T_{\{\alpha\}}$ is determined by $(t \upharpoonright \beta \,|\, \beta \in C)$:

$$\begin{aligned} \text{card}(T_{\{\alpha\}}) &\leqslant \text{card}({}^{\text{cof}(\alpha)}(T_\alpha)) \\ &\leqslant \text{card}({}^{\text{cof}(\alpha)} \bigcup_{\beta<\alpha} T_{\{\beta\}}) \\ &\leqslant \text{card}({}^{\text{cof}(\alpha)}\kappa \cdot \kappa) \\ &= \kappa^{\text{cof}(\alpha)} \\ &\leqslant \sum_{\nu<\kappa} \nu^{\text{cof}(\alpha)} \\ &\leqslant \sum_{\nu<\kappa} 2^{\nu \cdot \text{cof}(\alpha)} \\ &\leqslant \sum_{\nu<\kappa} \kappa \text{ , by the assumption } \forall \lambda < \kappa \, 2^\lambda \leqslant \kappa, \\ &= \kappa. \end{aligned}$$

For d), let $t \in T_\alpha$ and $a \prec b \in Q$ such that $\forall \xi \in \text{dom}(s) \; t(\xi) \prec a$. By Theorem 12.4 there is an order-preserving embedding f: $(\text{cof}(\alpha), <) \to ((a, b), \prec)$. We may assume that $\text{ht}(t) < \alpha_0$. We may recursively choose sequences $t_i \in T_{\{\alpha_i\}}$ such that

- $\forall i < j < \text{cof}(\alpha) \; t \subset t_i \subset t_j$;
- $\forall i < \text{cof}(\alpha) \forall \xi \in \alpha_i \; t_i(\xi) \prec f(i)$.

For non-limit ordinals $i < \text{cof}(\alpha)$ use the extension property d). For limit ordinals $i < \text{cof}(\alpha)$ note that α_i is the limit of $(\alpha_j \,|\, j < i)$ and is thus singular with $\text{cof}(\alpha_i) \leqslant i < \text{cof}(\alpha) < \kappa$. We can then take $t_i = \bigcup_{j<i} t_j$ which is an element of $T_{\{\alpha_i\}}$ by c).

Then take $t' = \bigcup_{i<\text{cof}(\alpha)} t_i$. $t' \in T_{\{\alpha\}}$ by the definition of $T_{\{\alpha\}}$. t' is an extension of t and $\forall \xi \in \alpha \, t'(\xi) \prec b$ as required.
Case 2: $\text{cof}(\alpha) = \kappa$. Let the sequence $(\alpha_i \,|\, i < \kappa)$ be continuous and cofinal in α. For each $t \in T_\alpha$ and $a \prec b \in Q$ with $\forall \xi \in \text{dom}(t) \, t(\xi) \prec a$ we shall construct an extension $t'_{a,b}$ in T appropriate for the extension property d): By Theorem 12.4 there is an order-preserving embedding f: $(\kappa, <) \to ((a, b), \prec)$. We may assume that $\text{ht}(t) < \alpha_0$. Recursively choose sequences $t_i \in T_{\{\alpha_i\}}$ such that

- $\forall i < j < \kappa \;\, t \subset t_i \subset t_j$;

– $\forall i<\kappa \forall \xi \in \alpha_i\, t_i(\xi) \prec f(i)$.

For non-limit ordinals $i<\mathrm{cof}(\alpha)$ use the extension property d). For limit ordinals $i<\mathrm{cof}(\alpha)$ note that α_i is the limit of $(\alpha_j\,|\,j<i)$ and is thus singular with $\mathrm{cof}(\alpha_i)\leqslant i<\kappa$. We can then take $t_i=\bigcup_{j<i} t_j$ which is an element of $T_{\{\alpha_i\}}$ by c).

Then set $t'_{a,b}=\bigcup_{i<\kappa} t_i$. $t'_{a,b}$ is an extension of t and $\forall \xi \in \alpha\, t'_{a,b}(\xi) \prec b$ as required in c).

Now define

$$T_{\{\alpha\}}=\{t'_{a,b}\,|\,t\in T_\alpha, a\prec b, \forall \xi \in \mathrm{dom}(t)\ t(\xi)\prec a\}.$$

The properties a) - d) are easily checked. a) follows by construction. For b) note that

$$\begin{aligned}\mathrm{card}(T_{\{\alpha\}}) &\leqslant \mathrm{card}(T_\alpha)\cdot \mathrm{card}(Q)\cdot \mathrm{card}(Q)\\ &\leqslant (\mathrm{card}(\alpha)\cdot\kappa)\cdot\kappa\cdot\kappa\\ &\leqslant \kappa\cdot\kappa\cdot\kappa\cdot\kappa\leqslant\kappa.\end{aligned}$$

c) does not apply for $T_{\{\alpha\}}$ and d) holds by construction.

This defines the tree $T=\bigcup_{\alpha<\kappa^+} T_{\{\alpha\}}$. We show that T is a κ^+-Aronszajn tree.

(1) $\mathrm{ht}(T)=\kappa^+$.

Proof. Property d) ensures that $\forall \alpha<\kappa^+\ T_{\{\alpha\}}\neq\emptyset$. By construction, $T_{\{\kappa^+\}}=\emptyset$, hence $\mathrm{ht}(T)=\kappa^+$. *qed*(1)

(2) $\mathrm{card}(T)=\kappa^+$, since by property b) $\kappa^+=\mathrm{ht}(T)\leqslant\mathrm{card}(T)\leqslant\kappa^+\cdot\kappa=\kappa^+$.

(3) $\forall\alpha<\mathrm{ht}(T)\,\mathrm{card}(T_{\{\alpha\}})\leqslant\kappa$, by property b).

(4) Every branch of T has cardinality $\leqslant\kappa$.

Proof. Let $B\subseteq T$ be a branch of T. Then $\bigcup B$: $(\theta,<)\rightarrow(Q,\prec)$ is an order-preserving embedding for some $\theta\in\mathrm{Ord}$. Since $\bigcup B$ is an injection from θ into Q, $\mathrm{card}(\theta)\leqslant\kappa$. Then $\mathrm{card}(B)\leqslant\theta\leqslant\kappa$. □

Chapter 13
The principle $\diamondsuit$

We shall study a principle which was introduced by Ronald Jensen and may be seen as a strong form of a continuum hypothesis. We shall use the principle to construct Aronszajn trees with stronger properties. The principle $\diamondsuit$ involves notions for "large" subsets of a regular uncountable cardinal: *closed unbounded* and *stationary* sets.

Definition 13.1. *Let κ be a regular uncountable cardinal.*

a) *$C \subseteq \kappa$ is **closed unbounded** in κ if C is cofinal in κ and*

$$\forall \alpha < \kappa (C \cap \alpha \text{ is cofinal in } \alpha \rightarrow \alpha \in C).$$

b) *$\mathcal{C}_\kappa = \{X \subseteq \kappa \mid \exists C \subseteq X\, C$ is closed unbounded in $\kappa\}$ is the **closed unbounded filter** on κ.*

c) *$S \subseteq \kappa$ is **stationary** in κ if $\forall C \in \mathcal{C}_\kappa\, S \cap C \neq \emptyset$.*

Theorem 13.2. *Let $\kappa > \omega$ be a regular cardinal. Then $\mathcal{C}_\kappa$ is a non-trivial filter on κ which is $<\kappa$-complete, i.e.,*

$$\forall \beta < \kappa \forall \{X_\xi \mid \xi < \beta\} \subseteq \mathcal{C}_\kappa \bigcap_{\xi < \beta} X_\xi \in \mathcal{C}_\kappa .$$

Proof. Exercise. □

Definition 13.3. *Let κ be a regular uncountable cardinal. Then $\diamondsuit_\kappa$ is the principle: there is a sequence $(S_\alpha \mid \alpha < \kappa)$ such that*

$$\forall S \subseteq \kappa \, \{\alpha < \kappa \mid S \cap \alpha = S_\alpha\} \text{ is stationary in } \kappa.$$

Theorem 13.4. *Assume $\diamondsuit_{\kappa^+}$. Then $2^\kappa = \kappa^+$.*

Proof. Let $(S_\alpha \mid \alpha < \kappa)$ be a sequence satisfying $\diamondsuit_{\kappa^+}$. Consider $x \subseteq \kappa$. By the $\diamondsuit_{\kappa^+}$-property there is $\alpha \in (\kappa, \kappa^+)$ such that $x = x \cap \alpha = S_\alpha$. Hence

$$\mathcal{P}(\kappa) \subseteq \{S_\alpha \mid \alpha < \kappa^+\}$$

and

$$2^\kappa = \mathrm{card}(\mathcal{P}(\kappa)) \leqslant \kappa^+.$$

□

Theorem 13.5. *Assume $V = L$. Then $\diamondsuit_\kappa$ holds for all regular uncountable cardinals κ.*

Proof. Define $(S_\alpha\,|\alpha<\kappa)$ by recursion on α. Consider $\beta<\kappa$ and let $(S_\alpha\,|\alpha<\beta)$ be appropriately defined. If β is not a limit ordinal, set $S_\beta=\emptyset$. If β is a limit ordinal, let (S_β, C_β) be the $<_L$-minimal pair such that C_β is closed unbounded in β, $S_\beta\subseteq\beta$ and $\forall\alpha\in C_\beta\; S_\beta\cap\alpha\neq S_\alpha$, if this exists; otherwise let $S_\beta=\emptyset$.

We show that $(S_\alpha\,|\alpha<\kappa)$ satisfies $\diamondsuit_\kappa$. Assume not. Then there is a set $S\subseteq\kappa$ such that $\{\alpha<\kappa\,|\,S\cap\alpha=S_\alpha\}$ is not stationary in κ. Hence there is a closed unbounded set $C\subseteq\kappa$ such that

$$\{\alpha<\kappa\,|\,S\cap\alpha=S_\alpha\}\cap C=\emptyset,$$

i.e.,

$$\forall\alpha\in C\; S\cap\alpha\neq S_\alpha.$$

We may assume that $(S,\ C)$ is the $<_L$-minimal pair such that C is closed unbounded in κ and $\forall\alpha\in C\; S\cap\alpha\neq S_\alpha$.

Take a level L_θ such that $(\mathrm{ZF}^-)^{L_\theta}$ and $\kappa,(S_\alpha\,|\alpha<\kappa),S,C\in L_\theta$.

(1) There is $X\lhd L$ such that L_θ, κ, $(S_\alpha\,|\alpha<\kappa)$, S, $C\in X$, and $\beta=X\cap\kappa$ is a limit ordinal $<\kappa$.

Proof. Define sequence $X_0\subseteq X_1\subseteq\ldots$ and $\beta_0<\beta_1<\ldots<\kappa$ by recursion so that $\mathrm{card}(X_n)<\kappa$ and $X_n\cap\kappa\subseteq\beta_n$. Let

$$X_0=L\{\{L_\theta,\kappa,(S_\alpha\,|\alpha<\kappa),S,C\}\}\lhd L.$$

X_0 is countable and so $\mathrm{card}(X_0)<\kappa$.

Let X_n be defined such that $\mathrm{card}(X_n)<\kappa$. Since κ is a regular cardinal, $X_n\cap\kappa$ is bounded below κ. Take $\beta_n<\kappa$ such that $X_n\cap\kappa\subseteq\beta_n$. Then let

$$X_{n+1}=L\{X_n\cup(\beta_n+1)\}.$$

$$\mathrm{card}(X_{n+1})\leqslant\mathrm{card}(X_n)+\mathrm{card}(\beta_n)+\aleph_0<\kappa.$$

Let $X=\bigcup_{n<\omega}X_n$ and $\beta=\bigcup_{n<\omega}\beta_n$. Since κ is regular uncountable, β is a limit ordinal and $\beta<\kappa$. By construction,

$$X=\bigcup_{n<\omega}X_n=\bigcup_{n<\omega}L\{X_n\cup(\beta_n+1)\}=L\{\bigcup_{n<\omega}(X_n\cup(\beta_n+1))\}\lhd L.$$

$$\beta=\bigcup_{n<\omega}\beta_n\subseteq(\bigcup_{n<\omega}X_{n+1})\cap\kappa\subseteq X\cap\kappa\subseteq\bigcup_{n<\omega}X_n\cap\kappa\subseteq\bigcup_{n<\omega}\beta_n=\beta.$$

qed(1)

By the condensation theorem let

$$\pi\colon(X,\in,<_L,I,N,S)\cong(L_\delta,\in,<_L,I,N,S)$$

for some $\delta\in\mathrm{Ord}$. We compute the images of various sets.

(2) $\pi\restriction\beta=\mathrm{id}\restriction\beta$, since $\beta=X\cap\kappa\subseteq X$ is transitive.

(3) $\pi(\kappa)=\beta$, since $\pi(\kappa)=\{\pi(\xi)\,|\,\xi\in\kappa\wedge\xi\in X\}=\{\pi(\xi)\,|\,\xi\in\beta\}=\{\xi\,|\,\xi\in\beta\}=\beta$.

(4) $\pi(S)=S\cap\beta$, since

$$\begin{aligned}\pi(S)&=\{\pi(\xi)\,|\,\xi\in S\wedge\xi\in X\}\\&=\{\pi(\xi)\,|\,\xi\in S\cap X\}\\&=\{\pi(\xi)\,|\,\xi\in S\cap\beta\}\\&=\{\xi\,|\,\xi\in S\cap\beta\}\\&=S\cap\beta.\end{aligned}$$

Similarly
(5) $\pi(C) = C \cap \beta$.
(6) $\pi((S_\alpha \,|\, \alpha < \kappa)) = (S_\alpha \,|\, \alpha < \beta)$.
Proof.

$$\begin{aligned}\pi((S_\alpha \,|\, \alpha < \kappa)) &= \pi(\{(\alpha, S_\alpha) | \alpha \in \kappa\}) \\ &= \{\pi((\alpha, S_\alpha)) | \alpha \in \beta\} \\ &= \{(\pi(\alpha), \pi(S_\alpha)) | \alpha \in \beta\} \\ &= \{(\alpha, S_\alpha) | \alpha \in \beta\} \\ &= (S_\alpha \,|\, \alpha < \beta).\end{aligned}$$

qed(6)
(7) $X \cap L_\theta$ is an elementary substructure of $(L_\theta, \in)$.
Proof. Since $L_\theta \in X$, the initial segment $X \cap L_\theta$ is closed with respect to the Skolem functions $S(L_\theta, \_, \_)$ for L_θ. *qed*(7)

Let $\bar{\theta} = \pi(\theta)$. Then
(8) $\pi^{-1} \upharpoonright L_{\bar{\theta}} : (L_{\bar{\theta}}, \in) \to (L_\theta, \in)$ is an elementary embedding.

Now we use elementarity and absoluteness to derive a contradiction.
(9) $C \cap \beta$ is closed unbounded in β, $S \cap \beta \subseteq \beta$ and $\forall \alpha \in C \cap \beta\; S \cap \alpha \neq S_\alpha$.
Proof. C is closed unbounded in κ. Since this is a definite property, $(L_\theta, \in) \vDash C$ is closed unbounded in κ. By elementarity, $(L_{\bar{\theta}}, \in) \vDash C \cap \beta$ is closed unbounded in β. By the absoluteness of being closed unbounded, $C \cap \beta$ is closed unbounded in β.

The other properties follow by the assumptions on C and S. *qed*(9)
(10) $(S \cap \beta, C \cap \beta) = (S_\beta, C_\beta)$.
Proof. Assume not. By the minimality of (S_β, C_β) and (9), we get

$$(S_\beta, C_\beta) <_L (S \cap \beta, C \cap \beta).$$

Since $L_{\bar{\theta}}$ is an initial segment of $<_L$ we have $(S_\beta, C_\beta) \in L_{\bar{\theta}}$. The defining properties for (S_β, C_β) are absolute for $(L_{\bar{\theta}}, \in)$:

$$(L_{\bar{\theta}}, \in) \vDash C_\beta \text{ is closed unbounded in } \beta,\ S_\beta \subseteq \beta \text{ and } \forall \alpha \in C_\beta\; S_\beta \cap \alpha \neq S_\alpha.$$

By the elementarity of $\pi^{-1} \upharpoonright L_{\bar{\theta}}$:

$$(L_\theta, \in) \vDash \pi^{-1}(C_\beta) \text{ is cl. unb. in } \kappa,\ \pi^{-1}(S_\beta) \subseteq \kappa,\ \forall \alpha \in \pi^{-1}(C_\beta)\; \pi^{-1}(S_\beta) \cap \alpha \neq S_\alpha.$$

By the absoluteness of these properties for transitive ZF^--models,

$$\pi^{-1}(C_\beta) \text{ is cl. unb. in } \kappa,\ \pi^{-1}(S_\beta) \subseteq \kappa,\ \forall \alpha \in \pi^{-1}(C_\beta)\; \pi^{-1}(S_\beta) \cap \alpha \neq S_\alpha,$$

i.e., the pair $(\pi^{-1}(S_\beta), \pi^{-1}(C_\beta))$ satisfies the defining property for (S, C). Since π^{-1} preserves $<_L$,

$$(\pi^{-1}(S_\beta), \pi^{-1}(C_\beta)) <_L (\pi^{-1}(S \cap \beta), \pi^{-1}(C \cap \beta)) = (S, C).$$

This contradicts the $<_L$-minimal cloice of (S, C). *qed*(10)

By (9), β is a limit point of C and hence $\beta \in C$. By (10), $S \cap \beta = S_\beta$. This contradicts the choice of the pair (S, C), i.e., there is no counterexample against the $\diamondsuit_\kappa$-property of the sequence $(S_\alpha \,|\, \alpha < \kappa)$. □

Chapter 14

Combinatorial principles and Suslin trees

Definition 14.1. *Let $T=(T,<_T)$ be a tree.*

a) *A set $A\subseteq T$ is an* ***antichain*** *in T if $\forall s,t\in A\,(s\neq t\rightarrow(s\not<_T t\wedge t\not<_T s))$.*

b) *Let κ be a cardinal. T is called a κ-**Suslin tree** if $\operatorname{card}(T)=\kappa$ and every chain and antichain in T has cardinality $<\kappa$.*

Obviously every level of a tree is an antichain. Hence a κ-Suslin tree is also a κ-Aronszajn tree.

Theorem 14.2. *Let κ be an infinite cardinal. Let $T=(T,<_T)$ be a tree with $\operatorname{card}(T)=\kappa$ such that every antichain in T has cardinality $<\kappa$ and T is* ***branching****, i.e.*

$$\forall s\in T\,\exists t,t'\in T\,(s<_T t\wedge s<_T t'\wedge \operatorname{ht}_T(t)=\operatorname{ht}_T(t')=\operatorname{ht}_T(s)+1\wedge t\neq t').$$

Then T is a κ-Suslin tree.

Proof. It suffices to see that every chain in T has cardinality $<\kappa$. Let $C\subseteq T$ be a chain. For every $s\in C$ choose $t,t'\in T$ such that

$$s<_T t\wedge s<_T t'\wedge \operatorname{ht}_T(t)=\operatorname{ht}_T(t')=\operatorname{ht}_T(s)+1\wedge t\neq t'$$

Then at least one of t,t' is not an element of C. So for each $s\in C$ we can choose $s^*>_T s$ such that $s^*\notin C$ and $\operatorname{ht}_T(s^*)=\operatorname{ht}_T(s)+1$.
(1) If $s,t\in C$ and $s\neq t$ then $s^*\not<_T t^*\wedge t^*\not<_T s^*$.
Proof. Assume not. Without loss of generality assume $s^*<_T t^*$. Since t is the immediate $<_T$-predecessor of t^* we have $s^*\leqslant_T t$ and $s^*\in C$. Contradiction. *qed*(1)

Hence $\{s^*\mid s\in C\}$ is an antichain in T. By assumption $\operatorname{card}(\{s^*\mid s\in C\})<\kappa$. Since the assignment $s\mapsto s^*$ is injective, we have $\operatorname{card}(C)<\kappa$. □

Theorem 14.3. *Assume $\diamondsuit_{\omega_1}$. Then there exists an ω_1-Suslin tree.*

Proof. Let $(S_\alpha\mid\alpha<\omega_1)$ be a $\diamondsuit_{\omega_1}$-sequence. We construct a tree $T=(T,<_T)$ of the form $T=\bigcup_{\alpha<\omega_1}T_{\{\alpha\}}$ such that every level T_α is countable. We can arrange that

$$T_{\{0\}}=\{0\}\text{ and }\forall\alpha\in[1,\omega_1)\,T_{\{\alpha\}}=\omega\cdot(\alpha+1)\setminus\omega\cdot\alpha.$$

By recursion on $\alpha < \omega_1$ we shall determine the $<_T$-predecessors of $x \in T_{\{\alpha\}}$. We shall also ensure the following recursive condition which guarantees that the tree can always be continued:
(1) for all $\xi < \zeta \leqslant \alpha$ and $s \in T_{\{\xi\}}$ there exists $t \in T_{\{\zeta\}}$ such that $s <_T t$.

For $\alpha = 0$ there is nothing to determine.

For $\alpha = 1$, let every element of $T_{\{1\}}$ be a $<_T$-successor of $0 \in T_{\{0\}}$.

Let $\alpha = \beta + 1 > 1$ and let $<_T \restriction T_\alpha$ be determined so that (1) is satisfied. We let every $s \in T_{\{\beta\}}$ have two immediate successors in $T_{\{\alpha\}}$: if $s = \omega \cdot \beta + m \in T_{\{\beta\}}$ and $t = \omega \cdot \alpha + n \in T_{\{\alpha\}}$ then set

$$s <_T t \text{ iff } n = 2 \cdot m \text{ or } n = 2 \cdot m + 1.$$

Since $<_T$ has to be a transitive partial order, this determines all $<_T$-predecessors of $x \in T_{\{\alpha\}}$. Also (1) holds for $<_T \restriction T_{\alpha+1}$.

Let α be a limit ordinal and let $<_T \restriction T_\alpha$ be determined so that (1) is satisfied.
(2) For every $s_0 \in T_\alpha$ there is a branch B of the tree $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$ such that $s_0 \in B$ and $\text{otp}(B) = \alpha$.
Proof. Choose an ω-sequence

$$\text{ht}_T(s) = \alpha_0 < \alpha_1 < \ldots < \alpha_n < \ldots < \alpha$$

which is cofinal in α. Using (1) choose a sequence

$$s_0 <_T s_1 <_T \ldots <_T s_n <_T \ldots$$

such that $\forall n < \omega \, \text{ht}_T(s_n) = \alpha_n$. Then

$$B = \{t \in T_\alpha \mid \exists n < \omega \, t <_T s_n\}$$

satisfies the claim. *qed*(2)

Define a set $S'_\alpha \subseteq T_\alpha$ as follows: if S_α is a maximal antichain in the tree $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$ then set

$$S'_\alpha = \{r \in T_\alpha \mid \exists s \in S_\alpha \, s \leqslant_T r\};$$

otherwise set $S'_\alpha = T_\alpha$. The set S'_α is countable. Let $S'_\alpha = \{s_i \mid i < \omega\}$ be an enumeration of S'_α. For each $i < \omega$ use (2) to choose a branch B_i of T_α with $s_i \in B_i$ and $\text{otp}(B_i) = \alpha$. For $x = \omega \cdot \alpha + i \in T_{\{\alpha\}}$ and $s \in T_\alpha$ define

$$s <_T x \text{ iff } s \in B_i.$$

(3) Property (1) holds for $T_{\alpha+1}$.
Proof. Let $s \in T_\alpha$. It suffices to find $t \in T_{\{\alpha\}}$ such that $s <_T t$.
Case 1: $S'_\alpha = T_\alpha$. Then $s = s_i$ for some $i < \omega$, $s_i \in B_i$, and $s_i <_T \omega \cdot \alpha + i \in T_{\{\alpha\}}$.
Case 2: $S'_\alpha = \{r \in T_\alpha \mid \exists s \in S_\alpha \; s \leqslant_T r\}$, where S_α is a maximal antichain in $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$. By the maximality of S_α there is $s' \in S_\alpha$ which is comparable with s:

$$s \leqslant_T s' \text{ or } s' \leqslant_T s.$$

Case 2.1: $s \leqslant_T s'$. Then $s' \in S'_\alpha$, say $s' = s_i$, $s \in B_i$, and $s <_T \omega \cdot \alpha + i \in T_{\{\alpha\}}$.
Case 2.2: $s' \leqslant_T s$. Then $s \in S'_\alpha$, say $s = s_i$, $s \in B_i$, and $s <_T \omega \cdot \alpha + i \in T_{\{\alpha\}}$. *qed*(3)

This concludes the recursive definition of the tree $T=(T,<_T)$. It is straightforward to check, that the predetermined sets $T_{\{\alpha\}}$ are indeed the α-th levels of the tree. By the construction at successors, the tree is branching. By the previous theorem it suffices to show that every antichain in T has cardinality $<\omega_1$.

Let $A\subseteq T$ be an antichain in T. Using the lemma of ZORN we may assume that A is maximal with respect to $\subseteq$.
(4) The set $C=\{\alpha<\omega_1\,|A\cap\alpha$ is a maximal antichain in $T_\alpha\}$ is closed unbounded in ω_1.
Proof. Let us first show unboundedness. Let $\alpha_0<\omega_1$. Construct an ω-sequence

$$\alpha_0<\alpha_1<\ldots<\omega_1$$

as follows. Let $\alpha_n<\omega_1$ be defined. By the maximality of A every $s\in T_{\alpha_n}$ is $<_T$-comparable to some $t\in A$. By the regularity of ω_1 one can take $\alpha_{n+1}\in(\alpha_n,\omega_1)$ such that

$$\forall s\in T_{\alpha_n}\exists t\in A\cap\alpha_{n+1}\,(s\leqslant_T t\vee t\leqslant_T s).$$

Let $\alpha=\bigcup_{n<\omega}\alpha_n<\omega_1$. $A\cap\alpha$ is an antichain in T, since it consists of pairwise incomparable elements. So $A\cap\alpha$ is an antichain in T_α. For the maximality consider $s\in T_\alpha$. Let $s\in T_{\alpha_n}$. By construction there is $t\in A\cap\alpha_{n+1}$ such that $s\leqslant_T t\vee t\leqslant_T s$. So every element of T_α is comparable with some element of $A\cap\alpha$.

For the closure property consider some $\alpha<\omega_1$ such that $C\cap\alpha$ is cofinal in α. To show that $\alpha\in C$ it suffices to show that $A\cap\alpha$ is a *maximal* antichain in T_α. Consider $s\in T_\alpha$. Take $\beta\in C\cap\alpha$ such that $s\in T_\beta$. Then $A\cap\beta$ is a maximal antichain in T_β and there exists $t\in A\cap\beta\subseteq A\cap\alpha$ which is comparable with s. Thus for every $s\in T_a$ there exists $t\in A\cap\alpha$ which is comparable with s. Thus $\alpha\in C$. *qed*(4)

By the $\diamondsuit_{\omega_1}$-property, $\{\alpha<\omega_1\,|A\cap\alpha=S_\alpha\}$ is stationary in ω_1. Take $\alpha\in C$ such that $A\cap\alpha=S_\alpha$. Then $A\cap\alpha=S_\alpha$ is a maximal antichain in T_α.
(5) $A=A\cap\alpha$.
Proof. Let $t\in A$. We show that every $r\in T$ is comparable with some $s\in A\cap\alpha$. Since $A\cap\alpha$ is a maximal antichain in T_α this is clear for $r\in T_\alpha$ and we may assume that $r\in T\setminus T_\alpha$. Then $\mathrm{ht}_T(r)\geqslant\alpha$ and we can take the unique $\bar r\in T_{\{\alpha\}}$ such that $\bar r\leqslant_T r$. By construction of $T_{\{\alpha\}}$ there is some $s\in S_\alpha=A\cap\alpha$ such that

$$s<_T\bar r\leqslant_T r$$

qed(5)

By (5), $A=A\cap\alpha$ is countable. Since T is a branching tree all whose antichains are countable, T is a Suslin tree. □

We shall now study generalizations from ω_1-Suslin trees to κ^+-Suslin trees for $\kappa>\omega$. We first consider the case when κ is regular. There are now different kinds of limit cases α in the construction: $\mathrm{cof}(\alpha)<\kappa$ and $\mathrm{cof}(\alpha)=\kappa$. To ensure the analogue of property (1) of the previous proof, we

- extend all paths through T_α when $\mathrm{cof}(\alpha)<\kappa$;
- use the set S_α of the $\diamondsuit$-sequence as above when $\mathrm{cof}(\alpha)=\kappa$.

In the first case one assumes that $\kappa^{\mathrm{cof}(\alpha)} \leqslant \kappa^{<\kappa} = \kappa$ which is a consequence of GCH. For the second case to yield the desired result a $\diamondsuit$-principle for ordinals of cofinality κ is needed. Note that the set $\mathrm{Cof}_\kappa = \{\alpha < \kappa^+ \mid \mathrm{cof}(\alpha) = \kappa\}$ is stationary in κ^+.

Definition 14.4. *Let κ be a regular uncountable cardinal and let $D \subseteq \kappa$ be stationary in κ. Then $\diamondsuit_\kappa(D)$ is the principle: there is a sequence $(S_\alpha \mid \alpha < \kappa)$ such that*

$$\forall S \subseteq \kappa \, \{\alpha \in D \mid S \cap \alpha = S_\alpha\} \text{ is stationary in } \kappa.$$

Theorem 14.5. *Assume $V = L$. Let κ be a regular uncountable cardinal and $D \subseteq \kappa$ be stationary. Then $\diamondsuit_\kappa(D)$ holds.*

This is very much proved like $\diamondsuit_\kappa = \diamondsuit_\kappa(\kappa)$. We only indicate the necessary changes in the previous proof.

Proof. Let $\beta < \kappa$ and let $(S_\alpha \mid \alpha < \beta)$ be appropriately defined. If β is not a limit ordinal or $\beta \notin D$, set $S_\beta = \emptyset$. If β is a limit ordinal and $\beta \in D$, let (S_β, C_β) be the $<_L$-minimal pair such that C_β is closed unbounded in β, $S_\beta \subseteq \beta$ and $\forall \alpha \in D \cap C_\beta \; S_\beta \cap \alpha \neq S_\alpha$, if this exists; otherwise let $S_\beta = \emptyset$.

Assume that $(S_\alpha \mid \alpha < \kappa)$ does not satisfy $\diamondsuit_\kappa$. Then there is a set $S \subseteq \kappa$ such that $\{\alpha \in D \mid S \cap \alpha = S_\alpha\}$ is not stationary in κ. Let (S, C) be the $<_L$-minimal pair such that C is closed unbounded in κ and $\forall \alpha \in D \cap C \; S \cap \alpha \neq S_\alpha$.

Take a level L_θ such that $(\mathrm{ZF}^-)^{L_\theta}$ and $\kappa, D, (S_\alpha \mid \alpha < \kappa), S, C \in L_\theta$.

(1) There is $X \lhd L$ such that L_θ, κ, D, $(S_\alpha \mid \alpha < \kappa)$, S, $C \in X$, $\beta = X \cap \kappa$ is a limit ordinal, and $\beta \in D$.

Proof. We basically show that the set of $\beta < \kappa$ with the first two properties is closed unbounded in κ. Let

$$A = \{\beta < \kappa \mid \beta = L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa\}.$$

We first show the unboundedness of A. Let $\beta_0 < \kappa$ and define an ω-sequence $\beta_0 < \beta_1 < \ldots < \kappa$ by recursion: if $\beta_n < \kappa$ is defined, let $\beta_{n+1} < \kappa$ be minimal such that $\beta_{n+1} > \beta_n$ and

$$L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa < \beta_{n+1}.$$

β_{n+1} exists, since

$$\mathrm{card}(L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\}) \leqslant \mathrm{card}(\beta_n) + \aleph_0 < \kappa$$

and since κ is regular.

Let $\beta = \bigcup_{n<\omega} \beta_n$. Since κ is regular uncountable, β is a limit ordinal and $\beta < \kappa$. By construction,

$$\begin{aligned} \beta &\subseteq L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa \\ &= \bigcup_{n<\omega} (L\{\beta_n \cup \{L_\theta, \kappa, D, (S_\alpha \mid \alpha < \kappa), S, C\}\} \cap \kappa) \\ &\subseteq \bigcup_{n<\omega} \beta_{n+1} \\ &= \beta, \end{aligned}$$

hence $\beta \in A$.

A similar argument shows that A is closed in κ. Since D is stationary in κ take $\beta \in D \cap A$. Then

$$X = L\{\beta \cup \{L_\theta, \kappa, D, (S_\alpha \,|\, \alpha < \kappa), S, C\}\}$$

has the required properties. *qed*(1)

By the condensation theorem let

$$\pi\colon (X, \in, <_L, I, N, S) \cong (L_\delta, \in, <_L, I, N, S)$$

for some $\delta \in \mathrm{Ord}$. The proof then follows the previous proof of $\diamondsuit_\kappa$. □

Theorem 14.6. *Let κ be a regular cardinal such that $\kappa^{<\kappa} = \kappa$. Assume $\diamondsuit_{\kappa^+}(\{\alpha < \kappa^+ \,|\, \mathrm{cof}(\alpha) = \kappa\})$. Then there exists a κ^+-Suslin tree.*

Proof. Let $(S_\alpha \,|\, \alpha < \kappa^+)$ be a $\diamondsuit_{\kappa^+}(\{\alpha < \kappa^+ \,|\, \mathrm{cof}(\alpha) = \kappa\})$-sequence. We construct a tree $T = (T, <_T)$ of the form $T = \bigcup_{\alpha < \kappa^+} T_{\{\alpha\}}$ such that every level T_α has cardinality $\leqslant \kappa$. We can arrange that

$$T_{\{0\}} = \{0\} \text{ and } \forall \alpha \in [1, \kappa^+)\, T_{\{\alpha\}} = \kappa \cdot (\alpha + 1) \setminus \kappa \cdot \alpha.$$

By recursion on $\alpha < \kappa^+$ we shall determine the $<_T$-predecessors of $x \in T_{\{\alpha\}}$. We shall also ensure the following two recursive conditions which guarantee that the tree can always be continued:
(1) for all $\xi < \zeta \leqslant \alpha$ and $s \in T_{\{\xi\}}$ there exists $t \in T_{\{\zeta\}}$ such that $s <_T t$;
(2) if $\alpha' < \alpha$ is a limit ordinal with $\mathrm{cof}(\alpha') < \kappa$ and B is a branch through $T_{\alpha'}$ with $\mathrm{otp}(B) = \alpha'$ then there is $t \in T_{\{\alpha'\}}$ such that $\forall s \in B\, s <_T t$.

For $\alpha = 0$ there is nothing to determine.

For $\alpha = 1$, let every element of $T_{\{1\}}$ be a $<_T$-successor of $0 \in T_{\{0\}}$.

Let $\alpha = \beta + 1 > 1$ and let $<_T \restriction T_\alpha$ be determined so that (1), (2) are satisfied. We let every $s \in T_{\{\beta\}}$ have two immediate successors in $T_{\{\alpha\}}$: if $s = \kappa \cdot \beta + \mu + m \in T_{\{\beta\}}$ and $t = \omega \cdot \alpha + \nu + n \in T_{\{\alpha\}}$ with limit ordinals $\mu, \nu < \kappa$ and $m, n < \omega$ then set

$$s <_T t \text{ iff } \mu = \nu \text{ and } (n = 2 \cdot m \text{ or } n = 2 \cdot m + 1).$$

Since $<_T$ has to be a transitive partial order, this determines all $<_T$-predecessors of $x \in T_{\{\alpha\}}$. Also (1) and (2) hold for $<_T \restriction T_{\alpha+1}$.

Let α be a limit ordinal and let $<_T \restriction T_\alpha$ be determined so that (1) is satisfied.
(2) For every $s_0 \in T_\alpha$ there is a branch B of the tree $T_\alpha = (T_\alpha, <_T \restriction T_\alpha)$ such that $s_0 \in B$ and $\mathrm{otp}(B) = \alpha$.
Proof. Let $\gamma = \mathrm{cof}(\alpha)$. Take a γ-sequence

$$\mathrm{ht}_T(s) = \alpha_0 < \alpha_1 < \ldots < \alpha_i < \ldots < \alpha, i < \gamma$$

which is cofinal in α and continuous, i.e., if $i < \gamma$ is a limit ordinal then

$$\alpha_i = \bigcup_{j<i} \alpha_j.$$

Recursively choose a γ-sequence

$$s_0 <_T s_1 <_T \ldots <_T s_i <_T \ldots, i < \gamma$$

such that $\forall i<\gamma\,\mathrm{ht}_T(s_i)=\alpha_i$. The recursive construction is possible at successor ordinals $i<\gamma$ by (1). If $i<\gamma$ is a limit ordinal then

$$\mathrm{cof}(\alpha_i)\leqslant i<\gamma=\mathrm{cof}(\alpha)\leqslant\kappa.$$

Let $B_i=\{t\in T_{\alpha_i}\,|\,\exists j<i\,t<_T s_j\}$ be the branch through T_{α_i} determined so far. Then $s_i\in T_{\{\alpha_i\}}$ can be found by property (2). Then

$$B=\{t\in T_\alpha\,|\,\exists i<\gamma\,t<_T s_i\}$$

satisfies the claim. *qed*(2)
Case 1: $\mathrm{cof}(\alpha)<\kappa$. Then
(3) $\mathrm{card}(\{B\,|\,B$ is a branch through T_α of ordertype $\alpha\})=\kappa$.
Proof. Let $\gamma=\mathrm{cof}(\alpha)$. Take a γ-sequence

$$\mathrm{ht}_T(s)=\alpha_0<\alpha_1<\ldots<\alpha_i<\ldots<\alpha,\,i<\gamma$$

which is cofinal in α. A branch B through T_α of ordertype α is determined by the set $\{B\cap T_{\alpha_i}\,|\,i<\gamma\}$. The letter is basically a function from γ into κ. Hence

$$\kappa\leqslant\mathrm{card}(\{B\,|\,B\text{ is a branch through }T_\alpha\text{ of ordertype }\alpha\}\leqslant\mathrm{card}({}^\gamma\kappa)\leqslant\kappa^{<\kappa}\leqslant\kappa.$$

qed(3)

Let $(B_i\,|\,i<\kappa)$ be an injective enumeration of all branches through T_α of ordertype α. For $x=\kappa\cdot\alpha+i\in T_{\{\alpha\}}$, $i<\kappa$ and $s\in T_\alpha$ define

$$s<_T x\text{ iff }s\in B_i.$$

Obviously properties (1) and (2) hold for α.

Case 2: $\mathrm{cof}(\alpha)=\kappa$.

Define a set $S'_\alpha\subseteq T_\alpha$ as follows: if S_α is a maximal antichain in the tree $T_\alpha=(T_\alpha,<_T\restriction T_\alpha)$ then set

$$S'_\alpha=\{r\in T_\alpha\,|\,\exists s\in S_\alpha\,s\leqslant_T r\};$$

otherwise set $S'_\alpha=T_\alpha$. Obviously $\mathrm{card}(S'_\alpha)=\kappa$. Let $S'_\alpha=\{s_i\,|\,i<\kappa\}$ be an enumeration of S'_α. For each $i<\kappa$ use (2) to choose a branch B_i of T_α with $s_i\in B_i$ and $\mathrm{otp}(B_i)=\alpha$. For $x=\kappa\cdot\alpha+i\in T_{\{\alpha\}}$ and $s\in T_\alpha$ define

$$s<_T x\text{ iff }s\in B_i.$$

(3) Property (1) holds for $T_{\alpha+1}$.
Proof. Let $s\in T_\alpha$. It suffices to find $t\in T_{\{\alpha\}}$ such that $s<_T t$.
Case 1: $S'_\alpha=T_\alpha$. Then $s=s_i$ for some $i<\kappa$, $s_i\in B_i$, and $s_i<_T\kappa\cdot\alpha+i\in T_{\{\alpha\}}$.
Case 2: $S'_\alpha=\{r\in T_\alpha\,|\,\exists s\in S_\alpha\,s\leqslant_T r\}$, where S_α is a maximal antichain in $T_\alpha=(T_\alpha,<_T\restriction T_\alpha)$. By the maximality of S_α there is $s'\in S_\alpha$ which is comparable with s:

$$s\leqslant_T s'\text{ or }s'\leqslant_T s.$$

Case 2.1: $s\leqslant_T s'$. Then $s'\in S'_\alpha$, say $s'=s_i$, $s\in B_i$, and $s<_T\kappa\cdot\alpha+i\in T_{\{\alpha\}}$.
Case 2.2: $s'\leqslant_T s$. Then $s\in S'_\alpha$, say $s=s_i$, $s\in B_i$, and $s<_T\kappa\cdot\alpha+i\in T_{\{\alpha\}}$. *qed*(3)

This concludes the recursive definition of the tree $T = (T, <_T)$. It is straightforward to check, that the predetermined sets $T_{\{\alpha\}}$ are indeed the α-th levels of the tree. By the construction at successors, the tree is branching. By the previous theorem it suffices to show that every antichain in T has cardinality $\leqslant \kappa$.

Let $A \subseteq T$ be an antichain in T. Using the lemma of ZORN we may assume that A is maximal with respect to $\subseteq$. As before one can show
(4) The set $C = \{\alpha < \kappa^+ \mid A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$ is closed unbounded in κ^+.

By the $\diamondsuit_{\kappa^+}$-property, $\{\alpha < \kappa^+ \mid A \cap \alpha = S_\alpha\}$ is stationary in κ^+. Take $\alpha \in C$ such that $A \cap \alpha = S_\alpha$. Then $A \cap \alpha = S_\alpha$ is a maximal antichain in T_α.
(5) $A = A \cap \alpha$.
Proof. Let $t \in A$. We show that every $r \in T$ is comparable with some $s \in A \cap \alpha$. Since $A \cap \alpha$ is a maximal antichain in T_α this is clear for $r \in T_\alpha$ and we may assume that $r \in T \setminus T_\alpha$. Then $\text{ht}_T(r) \geqslant \alpha$ and we can take the unique $\bar{r} \in T_{\{\alpha\}}$ such that $\bar{r} \leqslant_T r$. By construction of $T_{\{\alpha\}}$ there is some $s \in S_\alpha = A \cap \alpha$ such that

$$s <_T \bar{r} \leqslant_T r$$

qed(5)

By (5), $A = A \cap \alpha$ has cardinality $\leqslant \kappa$. Since T is a branching tree all whose antichains have cardinality $\leqslant \kappa$, T is a κ^+-Suslin tree. □

Motivation der Suslin Kostruktion, 14.7. Definition von $\square_\kappa$ mit stationärer Menge. 14.8. Satz: Wenn $\square_\kappa$ und das ensprechende $\diamondsuit_{\kappa^+}(S)$ gelten, dann existiert ein Suslin-Baum. Konstruktion wie vorher, die induktive Bedingung für Limites ist so, das linkst-mögliche Pfade Forsetzungen haben, wenn das Niveau nicht in S ist.

Chapter 15
$\square$-principles and trees

The above construction of a κ^+-Suslin tree used strongly that κ was a *regular* cardinal. The discussion of the case when κ is *singular* will lead to the formulation of a combinatorial principle $\square$ which we shall later prove in the constructible universe.

In the recursive construction of a κ^+-Suslin tree a balance has to be found between

a) extending enough branches so that the recursion may be continued to all levels $<\kappa^+$, and

b) keeping the cardinalities of levels and antichains below κ^+.

In the recursion, these requirements are distributed between limit stages of various cofinalities. When κ was regular and $\mathrm{cof}(\beta) < \kappa$ we continued every branch through T_β at level $T_{\{\beta\}}$. This was fine in the recursion, because by assumption

$$\mathrm{card}([T_\beta]) \leqslant \kappa^{<\kappa} = \kappa .$$

If κ is singular, this leads to problems since $\kappa^{<\kappa} > \kappa$. In general, one cannot continue every branch through T_β . Instead we shall only continue "left-most" branches. Consider, e.g., an β with $\mathrm{cof}(\beta) = \omega$. Let $(\beta_n | n < \omega)$ be a strictly increasing ω-sequence which is cofinal in β. For a branch $B \subseteq T_\beta$ of ordertype β define the elements b_n to be the unique element of $B \cap T_{\{\beta_n\}}$. Then B is left-most if, at least for a final segment of the $n < \omega$, b_{n+1} is the least $<_T$-successor of b_n on level $T_{\{\beta_{n+1}\}}$. The choice of left-most branches depends on the choice of singularizing sequences for ordinals.

For cofinalities $\delta > \omega$ the following *coherency* problem arises. Let $(\beta_i | i < \delta)$ be strictly increasing, continuous and cofinal in β and assume one is defining a sequence $(b_i | i < \delta)$ of elements $b_i \in T_{\{b_i\}}$ recursively. Then b_ω should be a left-most extension of the sequence $(b_n | n < \omega)$ so far which means that the branch through T_{β_ω} determined by $(b_n | n < \omega)$ should have been extended in the previous stages of the recursion. So $(\beta_n | n < \omega)$ should have been the singularizing sequence for β_ω.

These considerations lead to the following combinatorial principle, which provides singularizing sequences for subtle uncountable constructions.

Definition 15.1. *Let $\kappa \geqslant \omega$ be a cardinal and let $E \subseteq \kappa^+$. Then the principle $\square_\kappa(E)$ postulates that there exists a system $(C_\beta \,|\, \mathrm{Lim}(\beta), \beta < \kappa^+)$ such that for all β*

a) C_β is closed unbounded in β;

b) if $\mathrm{cof}(\beta) < \kappa$ then $\mathrm{otp}(C_\beta) < \kappa$;

c) (coherency) if $\bar{\beta}$ is a limit point of C_β the $\bar{\beta} \notin E$ and $C_{\bar{\beta}} = C_\beta \cap \bar{\beta}$.

Write $\square_{\kappa^+}$ for $\square_{\kappa^+}(\emptyset)$.

In applications, the set E will be stationary in κ^+ and on E we shall apply the oracle principle $\diamondsuit_{\kappa^+}(E)$. It is important that limit points of C_β do not get into conflict with such points, i.e., $\bar{\beta} \notin E$.

Theorem 15.2. *Let κ be an infinite cardinal. Let $E \subseteq \kappa^+$ be stationary such that $\square_\kappa(E)$ and $\diamondsuit_{\kappa^+}(E)$ hold. Then there is a κ^+-Suslin tree.*

Proof. Let $(C_\alpha | \alpha < \kappa^+ \wedge \lim(\alpha))$ and $(S_\alpha | \alpha < \kappa^+)$ be sequences satisfying $\square_\kappa(E)$ and $\diamondsuit_{\kappa^+}(E)$ resp. Again we construct a tree $T = (T, <_T)$ of the form $T = \bigcup_{\alpha < \kappa^+} T_{\{\alpha\}}$ where

$$T_{\{0\}} = \{0\} \text{ and } \forall \alpha \in [1, \kappa^+)\, T_{\{\alpha\}} = \kappa \cdot (\alpha + 1) \setminus \kappa \cdot \alpha.$$

By recursion on $\alpha < \kappa^+$ we shall determine the $<_T$-predecessors of $x \in T_{\{\alpha\}}$, i.e., $<_T \cap (T_\alpha \times T_{\{\alpha\}})$. We shall ensure the following two recursive conditions which guarantee that the tree can always be continued:
(1) for all $\xi < \zeta \leqslant \alpha$ and $s \in T_{\{\xi\}}$ there exists $t \in T_{\{\zeta\}}$ such that $s <_T t$;
(2) if $\alpha' < \alpha$ is a limit ordinal with $\mathrm{cof}(\alpha') < \kappa$ and B is a branch through $T_{\alpha'}$ with $\mathrm{otp}(B) = \alpha'$ then there is $t \in T_{\{\alpha'\}}$ such that $\forall s \in B\; s <_T t$.

For $\alpha = 0$ there is nothing to determine.

For $\alpha = 1$, let every element of $T_{\{1\}}$ be a $<_T$-successor of $0 \in T_{\{0\}}$.

Let $\alpha = \beta + 1 > 1$ and let $<_T \restriction T_\alpha$ be determined such that (1) and (2) hold. We let every $s \in T_{\{\beta\}}$ have two immediate successors in $T_{\{\alpha\}}$: if $s = \kappa \cdot \beta + \mu + m \in T_{\{\beta\}}$ and $t = \omega \cdot \alpha + \nu + n \in T_{\{\alpha\}}$ with limit ordinals $\mu, \nu < \kappa$ and $m, n < \omega$ then set

$$s <_T t \text{ iff } \mu = \nu \text{ and } (n = 2 \cdot m \text{ or } n = 2 \cdot m + 1).$$

Since $<_T$ has to be a transitive partial order, this determines all $<_T$-predecessors of $x \in T_{\{\alpha\}}$ and (1) and (2) are preserved.

Let α be a limit ordinal and let $<_T \restriction T_\alpha$ be already determined so that (1) and (2) are satisfied.

For any $x \in T \restriction \alpha$ define a *canonical leftmost* branch b_α^x through $T \restriction \alpha$ such that $x \in b_\alpha^x$ and $\mathrm{opt}(b_\alpha^x) = \alpha$ as follows: let $(\gamma_\alpha(\nu) | \nu < \lambda$ ...

We have to argue that the construction of the tree does not break down. The construction cannot break down for $\alpha = 0$ or α a successor. At limit α the construction can only break down if the *canonical leftmost* branches b_α^x cannot be constructed. Note that

$$b_\alpha^x = \{y \in T \restriction \alpha \,|\, \exists \nu < \lambda_\alpha\, y \leqslant_T p_\alpha^x(\nu)\}.$$

So we have to argue that the recursive construction of $p_\alpha^x(\nu)$ for $\nu < \lambda_\alpha$ does not break down. This construction cannot break down for $\nu = 0$ and ν a successor ordinal by property (1). For limit ν, $\bar{\alpha} = \gamma_\alpha(\nu) < \alpha$ is a limit point of C_α. Then $C_{\bar{\alpha}} = \bar{\alpha} \cap C_\alpha$ and $\bar{\alpha} \notin E$. By construction, $p_\alpha^x(i) = p_{\bar{\alpha}}^x(i)$ for $i < \nu$. Since $\bar{\alpha} \notin E$ the branch $b_{\bar{\alpha}}^x$ which contains all the $p_{\bar{\alpha}}^x(i)$ for $i < \nu$ was extended at level $\bar{\alpha}$ of the tree. This means that $p_\alpha^x(\nu)$ is welldefined. Thus the construction of the tree does not break down.

To show that T is a Suslin tree it suffices to see that every antichain in T has cardinality $\leqslant \kappa$. Let $A \subseteq T$ be an $\subseteq$-maximal antichain in T. The set

$$C = \{\alpha < \kappa^+ \mid A \cap \alpha \text{ is a maximal antichain in } T_\alpha\}$$

is closed unbounded in κ^+. By the $\diamondsuit_{\kappa^+}$-property, $\{\alpha < \kappa^+ \mid A \cap \alpha = S_\alpha\}$ is stationary in κ^+. Take $\alpha \in C$ such that $A \cap \alpha = S_\alpha$. Then $A \cap \alpha = S_\alpha$ is a maximal antichain in T_α.
(5) $A = A \cap \alpha$.
Proof. Let $t \in A$. We show that every $r \in T$ is comparable with some $s \in A \cap \alpha$. Since $A \cap \alpha$ is a maximal antichain in T_α this is clear for $r \in T_\alpha$ and we may assume that $r \in T \setminus T_\alpha$. Then $\mathrm{ht}_T(r) \geqslant \alpha$ and we can take the unique $\bar{r} \in T_{\{\alpha\}}$ such that $\bar{r} \leqslant_T r$. By construction of $T_{\{\alpha\}}$ there is some $s \in S_\alpha = A \cap \alpha$ such that

$$s <_T \bar{r} \leqslant_T r$$

qed(5)
By (5), $A = A \cap \alpha$ has cardinality $\leqslant \kappa$. Since T is a branching tree all whose antichains have cardinality $\leqslant \kappa$, T is a κ^+-Suslin tree. $\square$

To complete the picture we show

Theorem 15.3. *Assume $\square_\kappa$. Then there exists a stationary $E \subseteq \kappa^+$ such that $\square_\kappa(E)$.*

Before the proof we state two properties of stationary sets.

Lemma 15.4. *Let $\theta > \omega_1$ be a regular cardinal. Then*

a) If $\mu < \theta$ is regular then $\{\alpha \in \theta \mid \mathrm{cof}(\alpha) = \mu\}$ is stationary in θ.

b) If $W \subseteq \theta$ is stationary and $f: W \to \eta$ for some $\eta < \theta$ then there is some $\nu < \eta$ such that $\{\alpha \in W \mid f(\alpha) = \nu\}$ is stationary in θ.

Proof. Exercise. $\square$

We can now prove the theorem. The proof is an example for a combinatorial construction.

Proof. Let $(A_\lambda \mid \lambda < \kappa^+ \wedge \lim(\lambda))$ be a $\square_\kappa$-sequence. For $\lambda < \kappa^+ \wedge \lim(\lambda)$ let $B_\lambda = \{\gamma \in A_\lambda \mid \gamma \text{ is a limit point of } A_\lambda\}$. Then
(1) B_λ is a closed subset of λ;
(2) if $\mathrm{cof}(\lambda) > \omega$ then B_λ is unbounded in λ;
(3) $\gamma \in B_\lambda \to B_\gamma = \gamma \cap B_\lambda$;
(4) $\mathrm{cof}(\lambda) < \kappa \to \mathrm{otp}(B_\lambda) < \kappa$.
Let $W = \{\alpha \in \kappa^+ \mid \mathrm{cof}(\alpha) = \omega\}$. W is stationary in κ^+. Define $f: W \to \kappa + 1$ by $f(\alpha) = \mathrm{otp}(B_\alpha)$. Take $\nu \leqslant \kappa$ such that $E = \{\alpha \in W \mid f(\alpha) = \nu\}$ is stationary in κ^+. We shall show $\square_\kappa(E)$.
For $\lambda < \kappa^+ \wedge \lim(\lambda)$ define

$$D_\lambda = \begin{cases} B_\lambda, \text{ if } \mathrm{otp}(B_\lambda) \leqslant \nu \\ B_\lambda \setminus \{\alpha \in B_\lambda \mid \mathrm{otp}(B_\alpha) \leqslant \nu\}, \text{ otherwise.} \end{cases}$$

The sequence $(D_\lambda|\lambda<\kappa^+\wedge\lim(\lambda))$ satisfies properties (1)-(4) above. Also
(5) $D_\lambda\cap E)=\emptyset$.

For $\lambda<\kappa^+\wedge\lim(\lambda)$ define recursively

$$C_\lambda=\begin{cases}\bigcup\ \{C_\gamma|\gamma\in D_\lambda\},\text{ if }\sup(D_\lambda)=\lambda\\ \bigcup\ \{C_\gamma|\gamma\in D_\lambda\}\cup\{\theta_n^\lambda|n<\omega\},\text{ otherwise,}\end{cases}$$

where $(\theta_n^\lambda|n<\omega)$ is strictly increasing cofinal in λ with $\theta_0^\lambda=\sup(D_\lambda)$. We claim that $(C_\lambda|\lambda<\kappa^+\wedge\lim(\lambda))$ is a $\Box_\kappa(E)$-sequence.
(6) Each C_λ is unbounded in λ.
Proof. Induction on λ. $\lambda=\omega$. Then $D_\omega=B_\omega=\emptyset$ is bounded below ω and C_ω is unbounded in ω by construction. If $\lambda>\omega$ and (6) holds for $\lambda'<\lambda$ then (6) holds at λ by construction. *qed*
(7) Let $\gamma\in D_\lambda$. Then $C_\lambda=\gamma\cap C_\lambda$.
Proof. Assume that (7) holds below λ. By definition of C_λ, $C_\gamma\subseteq C_\lambda$ and $C_\gamma\subseteq\gamma\cap C_\lambda$. For the converse let $\xi\in\gamma\cap C_\lambda$. By the definition of C_λ take $\delta\in D_\delta$ such that $\xi\in\gamma\cap C_\delta$. If $\delta=\gamma$ then $\xi\in C_\gamma$ and we are done.

So suppose that $\delta<\gamma$. Then γ ...

$\Box$

15.1 Global square

Theorem 15.5. (Jensen) *Assume $V=L$. Then there exists a system $(C_\beta\ |\beta$ singular) such that*

a) C_β is closed unbounded in β;

b) C_β has ordertype less than β;

c) (coherency) if $\bar\beta$ is a limit point of C_β the $\bar\beta$ is singular and $C_{\bar\beta}=C_\beta\cap\bar\beta$.

The existence of such a system is usually described as global square *or $\Box$ without a cardinal index.*

Theorem 15.6. *Assume $\Box$ and let κ be an infinite cardinal. Then there is a stationary set $E\subseteq\kappa^+$ such that $\Box_\kappa(E)$ holds.*

Theorem 15.7. *Assume $V=L$. Then for every infinite cardinal κ there exists a κ^+-Suslin tree.*

Chapter 16
Hyperfine structure

The presentation of fine structure theory is guided by ideas for proving the combinatorial principle $\square$. One would like to choose a *square sequence* C_β for a given β in a very canonical way, say of minimal complexity or at a minimal place in L. The coherency property is difficult to arrange, it will come out of an involved condensation argument with a structure *in* which β is still regular but *over* which the singularity of β becomes apparent.

Let us consider the process of singularisation of β in L in detail. Let $L \vDash \beta$ is singular. Let γ be minimal such that over L_γ we can define a cofinal subset C of β of smaller ordertype; we can assume that C takes the form

$$C = \{z \in \beta \mid \exists x < \alpha \colon z \text{ is } <_L\text{-minimal such that } L_\gamma \vDash \varphi(z, \vec{p}, x)\}$$

where $\alpha < \beta$, φ is a first order formula, and $\vec{p}$ is a parameter sequence from L_γ. Using the Skolem function S we can write this as

$$C = \{S(L_\gamma, \varphi, \vec{p}^\frown x) \mid x < \alpha\}.$$

Here the locations $(L_\gamma, \varphi, \vec{p}^\frown x)$ are $\tilde{<}$-cofinal in the location $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$. The singularization of β may thus be carried out with the Skolem function S *restricted to arguments smaller* than $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$. This suggests to say that β is singularised *at the location* $(L_\gamma, \varphi, \vec{p}^\frown \alpha)$ and that the adequate singularizing structure for β is of the form

$$L_{(L_\gamma, \varphi, \vec{p}^\frown \alpha)} = (L_\gamma, \in, <_L, I, N, S \upharpoonright (L_\gamma, \varphi, \vec{p}^\frown \alpha));$$

where $S \upharpoonright (L_\gamma, \varphi, \vec{p}^\frown \alpha)$ means that we have the function $S \upharpoonright L_\gamma$ available as well as the Skolem assignments $S(L_\gamma, \psi, \vec{q})$ for $(L_\gamma, \psi, \vec{q}) \tilde{<} (L_\gamma, \varphi, \vec{p}^\frown \alpha)$.

These structures are indexed by locations and provide us with a fine interpolation between successive L_γ-levels:

$$L_\gamma, \ldots, L_{(L_\gamma, \varphi, \vec{p}\, \alpha)}, \ldots, L_{\gamma+1}, \ldots.$$

The interpolated *fine* hierarchy is very slow-growing but satisfies condensation and natural hulling properties which will allow the construction of a $\square$-system. The theory of the fine hierarchy is called *hyperfine structure theory* and was developed by Sy Friedman and the present author [4].

16.1 The fine hierarchy

Definition 16.1. *For a location $s=(L_\alpha,\varphi,\vec{x})$ define the* ***restricted*** SKOLEM ***function***

$$S\restriction s=S\restriction\{t\in\tilde{L}|t\tilde{<}s\}.$$

Define the ***fine level***

$$L_s=(L_\alpha,\in,<_L,I,N,S\restriction s).$$

Then $(L_s)_{s\in\tilde{L}}$ is the ***fine hierarchy****, it is indexed along the wellorder $\tilde{<}$.*

This hierarchy is equipped with algebraic hulling operations. To employ the restricted SKOLEM function $S\restriction s$ at "top locations" $t\tilde{<}s$ of the form $t=(L_\alpha,\varphi,\vec{x})$ we pretend that L_α itself is a constant of the structure $L_s=(L_\alpha,\in,<_L,I,N,S\restriction s)$, i.e. considering L_s and some $Y\subseteq L_\alpha$ we write $(L_\alpha,\varphi,\vec{x})\in Y$ iff $\vec{x}\in Y$.

Definition 16.2. *Let $s=(L_\alpha,\varphi,\vec{x})$ be a location. $Y\subseteq L_\alpha$ is* ***closed*** *in L_s, $Y\lhd L_s$, if Y is an algebraic substructure of L_s, i.e., Y is closed under I, N, and $S\restriction s$. For $X\subseteq L_\alpha$ let $L_s\{X\}$ be the $\subseteq$-smallest $Y\lhd L_s$ such that $Y\supseteq X$; $L_s\{X\}$ is called the L_s-****hull*** *of X.*

By our convention, $Y\lhd L_s$ means:

$$\begin{aligned}L_\beta,\vec{x}\in Y &\longrightarrow I(L_\beta,\varphi,\vec{x})\in Y \text{ and } S(L_\beta,\varphi,\vec{x})\in Y,\\ \vec{x}\in Y\wedge(L_\alpha,\varphi,\vec{x})\tilde{<}s &\longrightarrow S(L_\alpha,\varphi,\vec{x})\in Y\\ y\in Y\wedge N(y)=(L_\beta,\varphi,\vec{x}) &\longrightarrow L_\beta,\vec{x}\in Y.\end{aligned}$$

The fine hierarchy with its associated hull operators again satisfies condensation:

Theorem 16.3. (*Condensation*) *Let $s=(L_\alpha,\varphi,\vec{x})$ be a location and suppose that $X\lhd L_s$. Then there is a minimal location $\bar{s}$ so that there is an isomorphism*

$$\pi\colon(X,\in,<_L,I,N,S\restriction s)\cong L_{\bar{s}}=(L_{\bar{\alpha}},\in,<_L,I,N,S\restriction\bar{s});$$

concerning locations $t\in X$ of the form $t=(L_\alpha,\psi,\vec{y})$ this means

a) $\pi(t)=(L_{\bar{\alpha}},\psi,\pi(\vec{y}))$;

b) $t\tilde{<}s$ *iff* $\pi(t)\tilde{<}\bar{s}$ *and then* $S(\pi(t))=\pi(S(t))$.

Since π is the MOSTOWSKI *collapse of X the isomorphism π is uniquely determined.*

Proof. Let

$$\pi\colon(X,\in,<_L,I,N,S)\cong(L_{\bar{\alpha}},\in,<_L,I,N,S)$$

be the unique isomorphism given by the coarse Condensation Theorem 16.3. Let $\bar{S}=\{\pi(t)|t\in X\wedge t\tilde{<}s\}$.
(1) $\bar{S}$ is an initial segment of $(\tilde{L},\tilde{<})$.
Proof. Let $\pi(t)\in\bar{S}$, $t\in X$, $t\tilde{<}s$ and $r\tilde{<}\pi(t)$. Let $r=(L_\delta,\psi,\vec{y})$. Since π is surjective there is a location $r'\in X$ such that $r=\pi(r')$. $\pi(r')\tilde{<}\pi(t)$. Since π preserves $<_L$ we have $r'\tilde{<}t\tilde{<}s$. Thus $r\in\bar{S}$. *qed*(1)

Take $\bar{s}$ $\tilde{<}$-minimal such that $\bar{s} \notin \bar{S}$. Then $\bar{S} = \{r \in \tilde{L} | r \tilde{<} \bar{s}\}$. We now have to prove property b) of the theorem. Let $t = (L_\alpha, \psi, \vec{y}) \in X$ be a top location. Then (1) and the definition of $\bar{s}$ imply

(2) $t \tilde{<} s$ iff $\pi(t) \tilde{<} \bar{s}$.

Assume that $t \tilde{<} s$.

(3) $S(\pi(t)) = \pi(S(t))$.

Proof. Let $x = S(t)$, i.e., x is the $<_L$-smallest element of L_α such that

$$(L_\alpha, \in) \vDash \psi(x, \vec{y}).$$

Since $X \lhd L_s$ we have $x \in X$. One can be show by induction on the subformulas of ψ that the map π^{-1}: $(L_{\bar{\alpha}}, \in) \to (L_\alpha, \in)$ is elementary for every subformula. This is clear for atomic formulas and for propositional connectives; if the subformula is of the form $\exists v \chi$ then $\chi < \psi \leqslant \varphi$ in Fml and X is closed under the Skolem function $S(L_\alpha, \chi, .)$ for the formula $\exists v \chi$; hence π^{-1} is elementary for $\exists v \chi$.

Therefore,

$$(L_{\bar{\alpha}}, \in) \vDash \psi(\pi(x), \pi(\vec{y})),$$

and $S(\pi(t)) = S(L_{\bar{\alpha}}, \psi, \pi(\vec{y}))$ is defined as the $<_L$-minimal $z \in L_{\bar{\alpha}}$ such that

$$(L_{\bar{\alpha}}, \in) \vDash \psi(z, \pi(\vec{y})).$$

Assume for a contradiction that $z = S(\pi(t)) \neq \pi(x)$. By minimality, $z <_L \pi(x)$. Then $\pi^{-1}(z) <_L x$ and again by the elementarity of ψ with respect to π^{-1}:

$$(L_\alpha, \in) \vDash \psi(\pi^{-1}(z), \vec{y}).$$

But this contradicts the minimal definition of $x = S(t)$. □

16.2 Fine hulls

We prove a couple of further laws about the hulling operation $L_s\{.\}$ which can be seen as *fundamental laws of fine structure theory*. It is conveivable that these laws can be strengthened so that they alone capture the combinatorial content of L and might allow abstract proofs of combinatorial principles. Some of our laws are well-known for any kind of hull by generating functions. A specific and crucial law of hyperfine structure theory is the *finiteness property* (Theorem 16.8). It corresponds to a similar property in the theory of Silver machines ([13], see also [12]) which was an older attempt to simplify fine structure theory and which is also characterized by hulls and condensations.

Theorem 16.4. (Monotonicity) *Consider locations* $s = (L_\alpha, \varphi, \vec{x}) \tilde{\leqslant} t = (L_\beta, \psi, \vec{y})$ *and a set* $X \subseteq L_\alpha$.

a) If $\alpha = \beta$ *then* $L_s\{X\} \subseteq L_t\{X\}$.

b) If $\alpha < \beta$ *then* $L_s\{X\} \subseteq L_t\{X \cup \{\alpha\}\}$.

Proof. a) holds, since all hulling functions of L_s are available in L_t.
b) Note that $L_\alpha \in L_t\{X \cup \{\alpha\}\}$, since $N(\alpha) = (L_\alpha, ., .)$. Then the hulling function of L_s of the form $I(L_\alpha, ., .)$ and $S(L_\alpha, ., .)$ are also available in $L_t\{X \cup \{\alpha\}\}$. □

The next two theorems are obvious for hulls generated with finitary functions.

Theorem 16.5. (Compactness) *Let* $s=(L_\alpha, \varphi, \vec{x}) \in \tilde{L}$ *and* $X \subseteq L_\alpha$. *Then*

$$L_s\{X\} = \bigcup \{L_s\{X_0\} \,|\, X_0 \text{ is a finite subset of } X\}.$$

Theorem 16.6. (Continuity in the generators) *Let* $s = (L_\alpha, \varphi, \vec{x}) \in \tilde{L}$ *and let* $(X_i)_{i<\lambda}$ *be a* $\subseteq$*-increasing sequence of subsets of* L_α. *Then*

$$L_s\{\bigcup_{i<\lambda} X_i\} = \bigcup_{i<\lambda} L_s\{X_i\}.$$

Since the fine hierarchy grows discontinuously at limit locations (i.e., limits in $\tilde{<}$) of the form $(L_{\alpha+1}, 0, \emptyset)$, where 0 is the smallest element of Fml, we have to distinguish several constellations for the continuity in the locations.

Theorem 16.7. (Continuity in the locations)

a) If $s=(L_\alpha, 0, \emptyset)$ *is a limit location with* α *a limit ordinal and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = L\{X\} = \bigcup_{\beta<\alpha} L_{(L_\beta,0,\emptyset)}\{X \cap L_\beta\}.$$

b) If $s=(L_{\alpha+1}, 0, \emptyset)$ *is a limit location and* $X \subseteq L_\alpha$ *then*

$$\begin{aligned} L_s\{X \cup \{\alpha\}\} \cap L_\alpha &= L\{X \cup \{\alpha\}\} \cap L_\alpha \\ &= \bigcup \{L_r\{X\} \,|\, r \text{ is an } \alpha\text{-location}\}. \end{aligned}$$

c) If $s=(L_\alpha, \varphi, \vec{x}) \neq (L_\alpha, 0, \emptyset)$ *is a limit location and* $X \subseteq L_\alpha$ *then*

$$L_s\{X\} = \bigcup \{L_r\{X\} \,|\, r \text{ is an } \alpha\text{-location}, r \tilde{<} s\}.$$

Proof. a) is clear from the definitions since the hull operators considered only use the functions I, N, S.
b) The first equality is clear. The other is proved via two inclusions.
($\supseteq$) If z is an element of the right hand side, z is obtained from elements of X by successive applications of I, N, S and $S(L_\alpha, ., .)$. Since $L_\alpha \in L_s\{X \cup \{\alpha\}\}$, z can be obtained from elements of $X \cup \{\alpha\}$ by applications of I, N, S. Hence z is an element of the left hand side.
($\subseteq$) Consider $z \in L\{X \cup \{\alpha\}\} \cap L_\alpha$. There is a finite sequence

$$y_0, y_1, \ldots, y_k = z$$

which "computes" z in $L\{X \cup \{\alpha\}\}$. In this sequence each y_j is an element of $X \cup \{\alpha\}$ or it is obtained from $\{y_i \,|\, i<j\}$ by using I, N, S:

$$y_j = I(L_\beta, \varphi, \vec{y}) \text{ or } y_j = S(L_\beta, \varphi, \vec{y}) \text{ or } y_j \text{ is a component of } N(y) \tag{16.1}$$

for some $L_\beta, \vec{y}, y \in \{y_i \,|\, i<j\}$. We show by induction on $j \leqslant k$:

$$\text{if } y_j \in L_\alpha \text{ then } y_j \in U := \bigcup \{L_r\{X\} \,|\, r \text{ is an } \alpha\text{-location}\}.$$

Soe assume the claim for $i<j$ and that $y_j \in L_\alpha$.
Case 1. $y_j \in X \cup \{\alpha\}$. Then the claim is obvious.
Case 2. $y_j = I(L_\beta, \varphi, \vec{y})$ as in property (16.1) above. If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by induction hypothesis and hence $y_j \in U$.

If $\beta = \alpha$ then $\vec{y} \in U$ by induction hypothesis. Setting

$$\psi(v, \vec{w}) = \forall u (u \in v \leftrightarrow \varphi(u, \vec{w}))$$

we obtain $y_j = S(L_\alpha, \psi, \frac{\vec{y}}{\vec{w}}) \in U$.
Case 3. $y_j = S(L_\beta, \varphi, \vec{y})$ as in property (16.1) above. If $\beta < \alpha$, then $\beta, \vec{y} \in U$ by induction hypothesis and hence $y_j \in U$. If $\beta = \alpha$ then $\vec{y} \in U$ by induction hypothesis and $y_j = S(L_\alpha, \varphi, \vec{y}) \in U$.
Case 4. y_j is a component of $N(y_i)$ for some $i<j$ as in property (16.1) above.
Case 4.1. $y_i \in L_\alpha$. Then $y_i \in U$ by induction hypothesis. As U is closed under N, we have $N(y_i) \in U$. So each component of $N(y_i)$ and in particular y_j is an element of U.
Case 4.2. $y_i \in L_{\alpha+1} \setminus L_\alpha$. Since the values of N and S are "smaller" then corresponding arguments, then $y_i = \alpha$ or it is generated by the I-function: $y_i = I(L_\alpha, \psi, \vec{z})$ where $\vec{z} \in \{y_h | h < i\}$, $\vec{z} \in L_\alpha$, and by inductive assumption $\vec{z} \in U$. Since $\alpha = I(L_\alpha, \text{“}v \text{ is an ordinal”}, \emptyset)$ we may uniformly assume the case $y_i = I(L_\alpha, \psi, \vec{z})$. The name $N(y_i)$ will be of the form $(L_\alpha, \chi, (c_0, \ldots, c_{m-1}))$.

We claim that $c_0 \in U$: if

$$\chi_0(v_0, \vec{w}) \equiv \exists v_1 \ldots \exists v_{m-1} \forall u (\chi(u, v_0, v_1, \ldots, v_{m-1}) \leftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_0 then $c_0 = S(L_\alpha, \chi_0, \frac{\vec{z}}{\vec{w}} \vec{z}) \in U$.

We then obtain c_1 in U: if

$$\chi_1(v_1, \vec{w}) \equiv \exists v_2 \ldots \exists v_{m-1} \forall u (\chi(u, v_0, v_1, \ldots, v_{m-1}) \leftrightarrow \psi(u, \vec{w}))$$

with distinguished variable v_1 then $c_1 = S(L_\alpha, \chi_1, \frac{c_0 \frown \vec{z}}{v_0 \frown \vec{w}}) \in U$.

Proceeding in this fashion we get that $y_j \in U$.
c) Note that any element of $L_s\{X\}$ is generated from X by finitely many applications of the functions of L_s and thus only requires finitely many values $S(r)$ with $r \tilde{<} s$. □

Our final hull property is crucial for fine structural considerations. It states that the fine hierarchy grows in a "finitary" way. By incorporating information into finite generators or parameters one can arrange that certain effects can only take place at limit locations which then allows continuous approximations to that situation.

Theorem 16.8. (Finiteness Property) *Let s be an α-location and let s^+ be its immediate $\tilde{<}$-successor. Then there exists a set $z \in L_\alpha$ such that for any $X \subseteq L_\alpha$:*

$$L_{s^+}\{X\} \subseteq L_s\{X \cup \{z\}\}.$$

Proof. The expansion from L_s to L_{s^+} means to expand the SKOLEM function $S \upharpoonright s$ to $S \upharpoonright s^+ = (S \upharpoonright s) \cup \{(s, S(s))\}$. So $S \upharpoonright s^+$ provides at most one more possible value, namely $S(s)$. Then $z = S(s)$ is as required. □

Chapter 17
A proof of $\square$

Proposition 17.1. (Jensen) *Assume $V = L$. Then the global square principle $\square$ holds.*

In the proof, we have to define a singularizing sequence C_β for every singular limit ordinal β. Fix such a β. We shall be particularly interested in *closed* ordinals $\bar{\beta} < \beta$ where $\bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}$. The transitivization of $L_s\{\bar{\beta} \cup p\}$ will be the identity on $\bar{\beta}$ and map β to $\bar{\beta}$ which means that $\bar{\beta}$ inherits some properties from β. The singularizing set C_β will be defined from such ordinals. We study the singularity of β in terms of closed ordinals.

(1) There is a location $s = (\gamma, \varphi, \vec{x})$, $\gamma \geq \beta$, and a finite set $p \subseteq L_\gamma$ such that

$$\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}\}$$

is bounded in β.
Proof. Let $f: \alpha \to \beta$ cofinally, $\alpha < \beta$. Let $f \in L_\gamma$ and set $p = \{f\}$, $s = (\gamma, \varphi_{n+1}, \vec{0})$ where $\varphi_n \equiv v_0 = v_1(v_2)$ defines the operation of functional application with distinguished variable v_0. If $\alpha \leq \bar{\beta} < \beta$ then

$$\beta \cap L_s\{\bar{\beta} \cup p\} \supseteq \beta \cap L_s\{\alpha \cup p\} \supseteq f''\alpha$$

is cofinal in β and hence $\beta \cap L_s\{\bar{\beta} \cup p\} \neq \bar{\beta}$. *qed*(1)

Let $s = s(\beta)$ be $\tilde{<}$–minimal satisfying this lemma with some finite set $p \subseteq L_\gamma$. We show that s is a $\tilde{<}$–limit which can be nicely approximated from below.

(2) s is a limit location.
Proof. Assume that $s = r^+$ is the location successor of r. By the finiteness property there exists a $z \in L_\gamma$ such that for $\bar{\beta} < \beta$

$$L_s\{\bar{\beta} \cup p\} \subseteq L_r\{\bar{\beta} \cup p \cup \{z\}\}.$$

Then

$$\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap L_r\{\bar{\beta} \cup p \cup \{z\}\} \subseteq \{\bar{\beta} < \beta | \bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}\}$$

is bounded in β, contradicting the minimality of s. *qed*(2)

(3) $s \neq (\beta, \varphi_0, \vec{0})$.
Proof. Assume not and let $\gamma_0 < \beta$, $p \subseteq L_{\gamma_0}$, $s_0 = (\gamma_0, \varphi_0, \vec{0})$. Then

$$\{\bar{\beta} < \beta | \bar{\beta} = \beta \cap L_{s_0}\{\bar{\beta} \cup p\}\} \subseteq \{\bar{\beta} < \beta | \bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}\}$$

is bounded below β, contradicting the minimality of s. *qed*(3)

(4) $s \neq (\gamma, \varphi_0, \vec{0})$ for limit γ.
Proof. Assume not and let $\gamma_0 < \gamma$, $p \subseteq L_{\gamma_0}$, $s_0 = (\gamma_0, \varphi_0, \vec{0})$. Then

$$\{\bar{\beta} < \beta \mid \bar{\beta} = \beta \cap L_{s_0}\{\bar{\beta} \cup p\}\} \subseteq \{\bar{\beta} < \beta \mid \bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}\}$$

is bounded below β, contradicting the minimality of s. *qed*(4)

In defining C_β we shall consider three special cases and a generic case. In the special cases, β will have cofinality ω and we can pick any ω–sequence cofinal in β as C_β. The first special case deals with another degenerate type of limit location.

Special case 1. $s = (\alpha + 1, \varphi_0, \vec{0})$ for some α.
Note that every element of $L_{\alpha+1}$ can be "named" by α and finitely many elements of L_α. So we may assume that p is of the form $p = q \cup \{\alpha\}$ with $q \subseteq L_\alpha$. Let

$$\beta_0 = \max\{\bar{\beta} < \beta \mid \bar{\beta} = \beta \cap L_s\{\bar{\beta} \cup p\}\} < \beta.$$

Define $\beta_0 < \beta_1 < \ldots$ recursively: Choose $\beta_{n+1} > \beta_n$ least such that

$$\beta_{n+1} = \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_{n+1} \cup q\}.$$

$(\alpha, \varphi_n, \vec{0}) \tilde{<} s = (\alpha + 1, \varphi_0, \vec{0})$ and by the minimality of s, $\beta_{n+1} < \beta$ exists. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$.

$$\begin{aligned} \beta \cap L_s\{\beta_\omega \cup p\} &= \beta \cap L_s\{\beta_\omega \cup q \cup \{\alpha\}\} \\ &= \beta \cap \bigcup \{L_r\{\beta_\omega \cup q \mid r \text{is an } \alpha\text{–location}\} \\ &= \bigcup_{n<\omega} \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_\omega \cup q\} \\ &= \bigcup_{n<\omega} \beta \cap L_{(\alpha, \varphi_n, \vec{0})}\{\beta_{n+1} \cup q\} \\ &= \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega; \end{aligned}$$

the second equality uses a limit property of hulls, and the third and fourth use the monotonicity property. Now by the definition of β_0 we must have $\beta_\omega = \beta$, and we may define the set

$$C_\beta = \{\beta_n \mid n < \omega\}$$

cofinal in β.

Now assume that $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$. Then

(5) There is a finite $\bar{p} \subseteq L_\gamma$ such that $L_s\{\beta \cup \bar{p}\} = L_\gamma$.
Proof. By the condensation theorem there is a unique π such that

$$\pi \colon L_s\{\beta \cup p\} \cong L_{\bar{s}}.$$

Let $\bar{p} = \pi[p]$. Then $L_{\bar{s}} = L_{\bar{s}}\{\beta \cup \bar{p}\}$, $\pi \restriction \beta = \mathrm{id}$, and so for $\bar{\beta} < \beta$

$$\beta \cap L_s\{\bar{\beta} \cup p\} = \beta \cap L_{\bar{s}}\{\bar{\beta} \cup \bar{p}\}.$$

Hence

$$\{\bar\beta < \beta | \bar\beta = \beta \cap L_{\bar s}\{\bar\beta \cup \bar p\}\} = \{\bar\beta < \beta | \bar\beta = \beta \cap L_s\{\bar\beta \cup p\}\}$$

is bounded below β. The $\tilde{<}$-minimality of s implies $\bar s = s$. Thus $L_s = L_s\{\beta \cup \bar p\}$.
qed(5)

Let $<^*$ be the canonical wellordering of finite subsets of L induced by $<_L$:

$p_0 <^* p_1$ iff $p_0 \neq p_1$ and the $<_L$-maximal element of $p_0 \triangle p_1$ belongs to p_1.

Let $p(\beta)$ be the $<^*$-minimal $\bar p$ satisfying the previous lemma. Since the original parameter p is generated by $\beta \cup p(\beta)$:

(6) $\{\bar\beta < \beta | \bar\beta = \beta \cap L_s\{\bar\beta \cup p(\beta)\}\}$ is bounded below β; let $\beta_0 < \beta$ be the maximum element of this set.

By the lemma, we can write p instead of $p(\beta)$ without danger of confusion. We examine which locations below s are computed in $L_s\{X\}$: we say that $Y \subseteq L$ is *bounded below* s, if there is $s_0 \tilde{<} s$ such that if $r = (\gamma, \psi, \vec y) \tilde{<} s$, $\vec y \in Y$, then $r \tilde{<} s_0$.

Special case 2. $L_s\{\alpha \cup p\}$ is bounded below s for every $\alpha < \beta$.
Then define $\beta_0 < \beta_1 < \ldots < \beta$ recursively by

$$\beta_{n+1} = \bigcup (\beta \cap L_s\{(\beta_n + 1) \cup p\}).$$

By case assumption,

$$L_s\{(\beta_n + 1) \cup p\} = L_r\{(\beta_n + 1) \cup p\}$$

for some $r \tilde{<} s$. By the minimality of s, $\beta \cap L_r\{(\beta_n + 1) \cup p\}$ cannot be cofinal in β, and so $\beta_{n+1} < \beta$ exists. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n > \beta_0$.

$$\beta_\omega \subseteq \beta \cap L_s\{\beta_\omega \cup p\} \subseteq \bigcup_{n<\omega} \beta \cap L_s\{(\beta_n + 1) \cup p\} \subseteq \bigcup_{n<\omega} \beta_{n+1} = \beta_\omega.$$

By the definition of β_0 we have $\beta_\omega = \beta$ and we may define

$$C_\beta = \{\beta_n | n < \omega\}$$

cofinal in β.

Now assume that $L_s\{\alpha_0 \cup p\}$ is cofinal in s for some least $\alpha_0 = \alpha_0(\beta) < \beta$. We would like to use α_0 to steer the singularisation of β and obtain ordertype $C_\beta \leq \max\{\alpha_0, \omega\} < \beta$. If α_0 is not a limit ordinal or 0 we have to look for another steering ordinal. If $\alpha_0 = \alpha_0' + 1$, then choose $\alpha_1 = \alpha_1(\beta) < \alpha_0$ least such that

$$L_s\{\alpha_1 \cup p \cup \{\alpha_0'\}\}$$

is cofinal in s. If $\alpha_1 = \alpha_1' + 1$, then choose $\alpha_2 = \alpha_2(\beta) < \alpha_1$ least such that

$$L_s\{\alpha_2 \cup p \cup \{\alpha_0', \alpha_1'\}\}$$

is cofinal in s. Continue this way until $\alpha = \alpha(\beta) = \alpha_k(\beta)$ is a limit ordinal or 0.

Special case 3. $\alpha = 0$.
Then $L_s\{p \cup \{\alpha'_0, ..., \alpha'_{k-1}\}\}$ is a countable set which is $\tilde{<}$–unbounded in s. So s has “cofinality ω” in the ordering of locations and we can find γ–locations $s_1 \tilde{<} s_2 \tilde{<} s_3 \tilde{<} ...$ converging towards s. Let β_0 be defined as before and define $\beta_0 < \beta_1 < ... < \beta$ recursively by: Let $\beta_{n+1} > \beta_n$ be minimal such that

$$\beta_{n+1} = \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\}.$$

β_{n+1} exists, since $s_{n+1} \tilde{<} s$. Let $\beta_\omega = \bigcup_{n<\omega} \beta_n$.

$$\beta_\omega = \bigcup_{n<\omega} \beta_n = \bigcup_{n<\omega} \beta \cap L_{s_{n+1}}\{\beta_{n+1} \cup p\} = \beta \cap L_s\{\beta_\omega \cup p\},$$

and by the definition of β_0 we have $\beta_\omega = \beta$. We may define

$$C_\beta = \{\beta_n | n < \omega\}$$

cofinal in β.

So finally we arrive at the

Generic case. $s = (\gamma, \varphi, \vec{x}) \neq (\gamma, \varphi_0, \vec{0})$, and $L_s\{\alpha \cup p \cup \{\alpha'_0, ..., \alpha'_{k-1}\}\}$ is unbounded in s with α a limit ordinal $< \beta$.

Let $\beta_0 < \beta$ be defined as in (6). For each $0 < i \leq \alpha$ let s_i be the $\tilde{<}$–least strict upper bound of $L_s\{i \cup p \cup \{\alpha'_0, ..., \alpha'_{k-1}\}\}$ and $\beta_i = \beta_i(\beta)$ the least ordinal $> \beta_0$ such that

$$\beta_i = \beta \cap L_{s_i}\{\beta_i \cup p\}.$$

For $i < \alpha$, $\beta_i < \beta$ exists since $s_i \tilde{<} s$; also $s_\alpha = s$ and
(7) $0 < i < j < \alpha$ implies $s_i \tilde{\leqslant} s_j$ and $\beta_i \leq \beta_j$.
(8) $\{\beta_i | i < \alpha\}$ is closed unbounded in β.
Proof. Let $j \leq \alpha$ be a limit ordinal. We only have to show that $\beta_j = \bigcup_{i<j} \beta_i$. Since $\beta_j \geq \beta_i$ for $i < j$ it suffices to see that

$$\bigcup_{i<j} \beta_i = \bigcup_{i<j} \beta \cap L_{s_i}\{\beta_i \cup p\} = \beta \cap L_s\{\bigcup_{i<j} \beta_i \cup p\},$$

so that $\bigcup_{i<j} \beta_i$ satisfies the defining property of β_j. *qed*(8)

C_β will now be defined as an endsegment of such β_i's so that important elements of the preceding construction are visible below β_i or s_i. Set

$$\begin{aligned} I(\beta) = \{i < \alpha \mid\ & i > 0,\ \beta_i > \max(\{i\} \cup \{\alpha_l \mid l < k\}),\\ & s_i \text{ is a } \gamma\text{–location},\\ & s_i \tilde{>} \text{ the } \tilde{<}\text{–supremum of } L_s\{\alpha'_l \cup p \cup \{\alpha'_0, ..., \alpha'_{l-1}\}\}\\ & \text{for all } l < k, \text{ and}\\ & \beta < \gamma \longrightarrow \beta \in L_{s_i}\{\beta_i \cup p\}\}. \end{aligned}$$

Observe that:

(9) $\beta_i > \alpha$ for sufficiently high $i < \alpha$,
(10) $L_s\{\alpha'_l \cup p \cup \{\alpha'_0, ..., \alpha'_{l-1}\}\}$ is bounded below s by the definition of α_l,
(11) $\beta < \gamma$ implies $\beta \in L_s\{\beta \cup p\} = L_\gamma$.

So the conditions on i are satisfied for a non-empty final segment of α. Now define

$$C_\beta = \{\beta_i | i \in I(\beta)\}.$$

(12) C_β is closed unbounded in β and ordertype $C_\beta \leq \alpha < \beta$.

This completes the definition of the system (C_β), and we are left with proving the coherence property. Fix $\bar\beta \in \mathrm{Lim}(C_\beta)$, $\bar\beta < \beta$. We have to show $C_{\bar\beta} = C_\beta \cap \bar\beta$. β falls under the *generic case*, as ordertype $C_\beta > \omega$. Let $\bar\beta = \beta_{\bar\alpha}$ with $\bar\alpha < \alpha$ minimal; $\bar\alpha$ is a limit ordinal. Since $\mathrm{cof}(\beta_{\bar\alpha}) \leq \bar\alpha < \beta_{\bar\alpha}$, $\bar\beta$ is singular. By condensation let

$$\pi: L_{s_{\bar\alpha}}\{\bar\beta \cup p\} \cong L_{\bar s} \text{ and } q = \pi'' p.$$

(13) $\pi \restriction \bar\beta = \mathrm{id}$. If s is a β-location then $\bar s$ is a $\bar\beta$-location; if s is a γ-location with $\gamma > \beta$ the $\pi(\beta) = \bar\beta$.
Proof. If $\gamma > \beta$ then $\beta \in L_{s_{\bar\alpha}}\{\bar\beta \cup p\}$ and $\bar\beta = \beta \cap L_{s_{\bar\alpha}}\{\bar\beta \cup p\}$. *qed*(13)

(14) $\bar s = s(\bar\beta)$.
Proof. For $\delta < \bar\beta$, the condensation property of π implies

$$\delta = \beta \cap L_s\{\delta \cup p\} \text{ iff } \delta = \bar\beta \cap L_{\bar s}\{\delta \cup q\}.$$

As $\beta_0 < \bar\beta$, where β_0 was defined in (6),

$$\max\left\{\delta < \bar\beta \,|\, \delta = \bar\beta \cap L_{\bar s}\{\delta \cup q\}\right\} = \beta_0 < \bar\beta,$$

and so $\bar s$, q satisfy (1) for $\bar\beta$. Suppose that $r \tilde{<} \bar s$ and $\bar p$ satisfy (1). We may assume that $\bar s$ is a $\bar\gamma$-location, and, since we are in the *generic case*, r is without loss of generality also a $\bar\gamma$-location, $\bar p \subseteq L_{\bar\gamma}$. We may also assume that $\bar p \subseteq L_r\{\bar\beta \cup q\}$ by choosing r sufficiently high. Then r and q satisfy (1). By the determination of $\bar s$ in the proof of condensation, $\pi^{-1}(r) \tilde{<} s$. Let

$$r^* = \tilde{<} - \sup\left\{\pi^{-1}(t) | t \tilde{<} r\right\} \tilde{\leq} \pi^{-1}(r) \tilde{<} s.$$

For $\delta < \bar\beta$, the condensation property of π implies

$$\delta = \beta \cap L_r\{\delta \cup q\} = \beta \cap L_{r^*}\{\delta \cup p\}.$$

Then

$$\{\delta < \beta | \delta = \beta \cap L_{r^*}\{\delta \cup p\}\} = \{\delta < \beta | \delta = \beta \cap L_r\{\delta \cup q\}\}$$

is bounded below β contradicting the minimal choice of s. *qed*(14)

(15) $\bar\beta$ does not fall under *special case 1*.
(16) $q = p(\bar\beta)$.
Proof. $L_{\bar s}\{\bar\beta \cup q\} = L_{\bar\gamma}$, so $q \geq^* p(\bar\beta)$. Assume $q >^* p(\bar\beta)$. By property (5) at $\bar\beta$, $q \subseteq L_{\bar s}\{\bar\beta \cup p(\bar\beta)\}$, $p = \pi^{-1}{}'' q \subseteq L_s\{\bar\beta \cup \pi^{-1}{}'' p(\bar\beta)\}$, and so $\pi^{-1}{}'' p(\bar\beta) <^* p = \pi^{-1}{}'' q$ satisfies (5), contrary to the minimal choice of $p = p(\beta)$. *qed*(16)

$L_{s_{\bar\alpha}}\{\bar\alpha \cup p\} = L_s\{\bar\alpha \cup p\}$ is cofinal in $s_{\bar\alpha}$. Hence $L_{\bar s}\{\bar\alpha \cup q\}$ is cofinal in $\bar s$, and $\bar\alpha < \alpha < \beta$. Hence
(17) $\bar\beta$ does not fall under *special case 2*.
(18) For $j < k$, $\alpha_j(\beta) = \alpha_j(\bar\beta)$.
Proof. By induction on $j < k$. Remember that $\alpha_j(k) =$ the smallest ν s.t.

$$L_s\{\nu \cup p \cup \{\alpha'_i \mid i < j\}\}$$

is cofinal in s. Now

$$L_s\{\bar\alpha \cup p \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is cofinal in $s_{\bar\alpha}$, hence

$$L_{\bar s}\{\bar\alpha \cup q \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is cofinal in $\bar s$ and

$$L_{\bar s}\{\alpha_j(\beta) \cup q \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is cofinal in $\bar s$.

Now let $\alpha_j(\beta) = \alpha'_j + 1$. Then

$$L_s\{\alpha'_j \cup p \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is bounded below s by some $s_i \tilde{<} s_{\bar\alpha}$. Hence

$$L_{s_{\bar\alpha}}\{\alpha'_j \cup p \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is bounded below $s_{\bar\alpha}$ by some $s_i \tilde{<} s_{\bar\alpha}$, and so by some location in $L_{s_{\bar\alpha}}\{\bar\beta \cup p\}$. Hence

$$L_{\bar s}\{\alpha'_j \cup q \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is bounded below $\bar s$ by some location $\tilde{<} \bar s$. So $\alpha_j(\beta) = \alpha_j(\bar\beta)$. *qed*(18)

(19) $\alpha_k(\bar\beta) = \bar\alpha$.
Proof. As above,

$$L_{\bar s}\{\bar\alpha \cup q \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is cofinal in $\bar s$. If we take $\alpha' < \bar\alpha$, then

$$L_{s_{\bar\alpha}}\{\alpha' \cup p \cup \{\alpha'_0, \dots, \alpha'_{k-1}\}\}$$

is bounded below $s_{\bar\alpha}$, by the minimality of $\bar\alpha$. So we have $\alpha_k(\bar\beta) = \bar\alpha$. *qed*(19)

(20) $\bar\beta$ does not fall under *special case 3*,
since $\bar\alpha \neq 0$. So we are again in the *generic case*.

(21) $\forall i < \bar\alpha \quad \beta_i(\beta) = \beta_i(\bar\beta)$.
Proof. $\beta_0 = \beta_0(\beta) =$ the largest s.t. $\beta_0 < \beta$ and $\beta_0 = \beta \cap L_s\{\beta_0 \cup p\}$. By definition of $\bar\beta = \beta_{\bar\alpha}$, $\beta_0 =$ the largest s.t. $\beta_0 < \bar\beta$ and $\beta_0 = \bar\beta \cap L_{s_{\bar\alpha}}\{\beta_0 \cup p\}$. As $L_{s_{\bar\alpha}}\{\beta_0 \cup p\} \cong L_{\bar s}\{\beta_0 \cup q\}$ by a map which is the identity on $\bar\beta$, $\beta_0 =$ the largest s.t. $\beta_0 < \bar\beta$ and $\beta_0 = \bar\beta \cap L_{\bar s}\{\beta_0 \cup q\}$, which is the definition of $\beta_0(\bar\beta)$.

Now consider $0 < i < \bar{\alpha}$.

$$\begin{aligned} s_i(\beta) &= \tilde{<}\text{-least upper bound of } L_s\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\} \text{ below } s \\ &= \tilde{<}\text{-least upper bound of } L_{s_{\bar{\alpha}}}\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\} \text{ below } s_{\bar{\alpha}} \\ &\quad \text{by the definition of } s_{\bar{\alpha}}\,; \\ s_i(\bar{\beta}) &= \tilde{<}\text{-least upper bound of } L_{\bar{s}}\{i \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\} \text{ below } \bar{s} \end{aligned}$$

For $\xi < \bar{\beta}$ we have

(*) $\pi: L_{s_i}\{\xi \cup p\} \cong L_{\bar{s}_i}\{\xi \cup q\}$, where we write $s_i = s_i(\beta)$ and $\bar{s}_i = s_i(\bar{\beta})$.
Proof. We have to show that the Skolem functions in L_s correspond to those in $L_{\bar{s}}$ via π. If $z = S(\delta, \psi, \vec{y})$, $(\delta, \psi, \vec{y}) \tilde{<} s_i$, $\vec{y} \in L_{s_{\bar{\alpha}}}\{\bar{\beta} \cup p\}$, then $(\delta, \psi, \vec{y}) \tilde{<} (\eta, \chi, \vec{z}) \tilde{<} s_i$ for some location $(\eta, \chi, \vec{z}) \in L_{s_{\bar{\alpha}}}\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$, i.e., $\vec{z} \in L_{s_{\bar{\alpha}}}\{i \cup p \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$. Now $\pi(\eta, \chi, \vec{z}) \in L_{\bar{s}}\{i \cup q \cup \{\alpha'_0, \ldots, \alpha'_{k-1}\}\}$ and

$$\pi(z) = S(\pi(\delta, \psi, \vec{y})),\ \pi(\delta, \psi, \vec{y}) \tilde{<} \pi(\eta, \chi, \vec{z}) \tilde{<} \bar{s}_i\,.$$

The converse proceeds analogously. *qed*(*)

Now $\beta_i(\beta)$ is the minimal $\beta_i(\beta) > \beta_0$ such that

$$\beta_i(\beta) = \beta \cap L_{s_i}\{\beta_i(\beta) \cup p\} = \bar{\beta} \cap L_{s_i}\{\beta_i(\beta) \cup p\}$$

and $\beta_i(\bar{\beta})$ is the minimal $\beta_i(\bar{\beta}) > \beta_0$ such that

$$\beta_i(\bar{\beta}) = \bar{\beta} \cap L_{\bar{s}_i}\{\beta_i(\bar{\beta}) \cup q\}.$$

By the isomorphism property (*) and that fact that $\pi \restriction \bar{\beta} = \mathrm{id}$ we have $\beta_i(\beta) = \beta_i(\bar{\beta})$ as required. *qed*(21)

Now to prove the coherence property it suffices to see that

$$\begin{aligned} I(\bar{\beta}) = \{i < \bar{\alpha} \ \mid \ & i > 0,\ \beta_i > \max(\{i\} \cup \{\alpha_l \mid l < k\}), \\ & s_i(\bar{\beta}) \text{ is a } \bar{\gamma}\text{–location}, \\ & s_i(\bar{\beta}) \tilde{>} \text{ the } \tilde{<}\text{–supremum of } L_{\bar{s}}\{\alpha'_l \cup q \cup \{\alpha'_0, \ldots, \alpha'_{l-1}\}\} \\ & \text{for all } l < k, \text{ and} \\ & \bar{\beta} < \bar{\gamma} \longrightarrow \bar{\beta} \in L_{s_i(\bar{\beta})}\{\beta_i \cup q\}\} \\ = \{i < \bar{\alpha} \ \mid \ & i > 0,\ \beta_i > \max(\{i\} \cup \{\alpha_l \mid l < k\}), \\ & s_i(\beta) \text{ is a } \gamma\text{–location}, \\ & s_i(\beta) \tilde{>} \text{ the } \tilde{<}\text{–supremum of } L_{s_{\bar{\alpha}}}\{\alpha'_l \cup q \cup \{\alpha'_0, \ldots, \alpha'_{l-1}\}\} \\ & \text{for all } l < k, \text{ and} \\ & \beta < \gamma \longrightarrow \beta \in L_{s_i(\beta)}\{\beta_i \cup p\}\} \\ = I(\beta) \cap \bar{\alpha}\,. \end{aligned}$$

Chapter 18
Embeddings of L

Transcending L

Theorem 18.1. *Elementary Embeddings: Definition, critical point, the critical point is regular, strong limit, there is a model of ZFC: if the critical point is a cardinal, then the powerset is an element of the other side, etc.*

Theorem 18.2. *The existence of such an elementary embedding can not be proven. No formula does this.*

Theorem 18.3. *If there is such an embedding then $V \neq L$. Assume $V = L$.*

$0^{\#}$ definieren!!!
Finestructural construction of such mappings.

Chapter 19
Morasses

19.1 Definition of a gap-1 morass

Combinatorial principles are general statements of infinitary combinatorics which yield construction principles for infinitary, mostly uncountable structures. The continuum hypothesis or the stronger principle $\diamondsuit$ are enumeration principles for subsets of ω or of ω_1 which can be used in recursive constructions. These principles are provable in the model L by non-finestructural methods.

Ronald Jensen has developed his fine structure theory with a view towards some stronger combinatorial principles. He could prove the full *gap*-1 *two-cardinal transfer property* in L using the combinatorial principle $\square$:

> if a countable first-order theory T has a model $\mathfrak{A} = (A, B, \ldots)$ with $\mathrm{card}(A) = \mathrm{card}(B)^+ \geqslant \aleph_1$ then for every infinite cardinal κ T has a model $\mathfrak{A}' = (A', B', \ldots)$ with $\mathrm{card}(A') = \kappa^+$ and $\mathrm{card}(B') = \kappa$.

Jensen could also prove the *gap*-2 *transfer* by defining and using *gap*-1 *morasses* in a similar way. We shall demonstrate that gap-1 morasses can be naturally constructed in hyperfine structure theory.

A morass is a commutative tree-like system of ordinals and embeddings. Let us first consider a trivial example of a system which may be used in constructions.

$$(\omega \cdot \alpha, <)_{\alpha \leqslant \omega_1}, (\mathrm{id} \upharpoonright \omega \cdot \alpha)_{\alpha \leqslant \beta \leqslant \omega_1}$$

is obviously a directed system whose final structure $(\omega_1, <)$ is determined by the previous structures, all of which are countable. If we have, e.g., a model-theoretic method which recursively constructs additional structure on the countable limit ordinals $(\omega \cdot \alpha, <)$ which is respected by the maps $\mathrm{id} \upharpoonright \omega \cdot \alpha$ then the system "automatically" yields a limit structure on $(\omega_1, <)$. Of course this is just the standard union-of-chains method, always available in ZFC, which is a main tool for many kinds of infinitary constructions.

A (gap-1) morass may be seen as a commutative system of directed systems. In an $(\omega_1, 1)$-morass a top directed system converges to a structure of size ω_2. That system consists of structures of size ω_1 and is itself the limit of a system of directed systems of size ω_0. In applications one has to determine the countable components of this system of systems. If the connecting maps between the countable components commute sufficiently then the morass "automatically" yields a "limit of limits" of size ω_2, whose properties can be steered by appropriate choices of the countable structures.

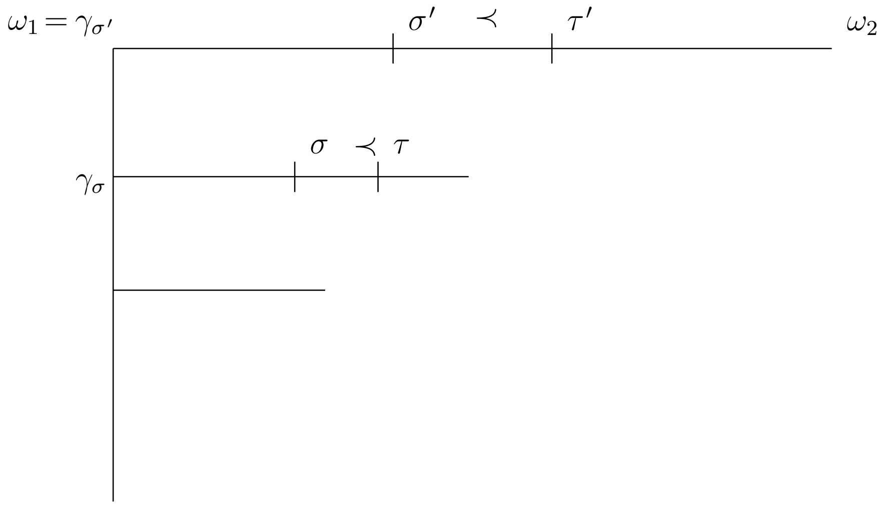
Hyperfine structure theory provides us with a host of structures and structure-preserving maps between them. Through hulls and condensation, one can approximate large structures L_s by countable structures $L_{\bar{s}}$. This motivates the following construction: carefully select a subsystem of the large hyperfine system or category and show that it satisfies JENSEN's structural axioms for an $(\omega_1, 1)$-morass. We could instead use arbitrary regular cardinals instead of ω_1. We assume $\mathrm{ZFC} + V = L$ for the rest of this paper.

The following construction is due to the present author and will be published in [5]. We approximate the structure L_{ω_2} by structures which look like L_{ω_2} . The heights of those structure will be *morass points*.

Definition 19.1. *A limit ordinal $\sigma < \omega_2$ is a **morass point** if*

- $L_\sigma = \bigcup \{L_\mu \mid \mu < \sigma \wedge L_\mu \models \mathrm{ZF}^-\}$ *and*
- $L_\sigma \models$ *"there is exactly one uncountable cardinal".*

For a morass point σ let γ_σ be the unique uncountable cardinal in L_σ . For morass points σ, τ define $\sigma \prec \tau$ iff $\sigma < \tau$ and $\gamma_\sigma = \gamma_\tau$. Let S^1 be the set of all morass points and $S^0 = \{\gamma_\sigma \mid \sigma \in S^1\}$.



The structures $L_{\sigma'} \subseteq L_{\tau'}$, $\sigma' \prec \tau'$ approximate L_{ω_2} ; the directed system $\sigma' \prec \tau'$ will be a limit of the countable directed systems $\sigma \prec \tau$ from below.

We shall assign levels of the fine hierarchy to morass points; the morass will consist of those levels and of suitable fine maps between them. Finite sets of *parameters* will be important in the sequel and they will often be chosen according to a canonical wellordering "by largest difference":

Definition 19.2. *Define a wellorder $<^*$ of the class $[V]^{<\omega}$ of finite sets: $p <^* q$ iff there exists $x \in q \setminus p$ such that for all $y >_L x$ holds $y \in p \leftrightarrow y \in q$.*

Lemma 19.3. *Let σ be a morass point. Then there is a $\tilde{<}$-least location $s(\sigma)$ such that there is a finite set $p \subseteq L_{s(\sigma)}$ with $L_{s(\sigma)}\{\gamma_\sigma \cup p\}$ being cofinal in σ. Let p_σ be the $<^*$-smallest such parameter. We call $M_\sigma = (L_{s(\sigma)}, p_\sigma)$ the* ***collapsing structure*** *of σ.*

Proof. Since $L_\sigma \vDash$ "there is exactly one uncountable cardinal" we have $\sigma \neq \omega_1$. Thus σ is not a cardinal in L. Let $f\colon \gamma_\sigma \to \sigma$ be surjective. Let $f \in L_\alpha$. Set $s = (L_{\alpha+1}, 0, \emptyset) \in \tilde{L}$ and $p = \{f, L_\alpha\}$.
(1) $\sigma \subseteq L_s\{\gamma_\sigma \cup p\}$.
Proof. Let $\zeta \in \sigma$. Let $\zeta = f(\xi)$, $\xi \in \gamma_\sigma$. Then

$$\begin{aligned}\zeta &= \text{the unique set such that } (\xi, \zeta) \in f\\ &= S(L_\alpha, \text{“}(v_1, v_0) \in v_2\text{”}, \frac{\xi\ f}{v_1\, v_2})\\ &\in L_s\{\gamma_\sigma \cup p\}\end{aligned}$$

□

Definition 19.4. *Define a strict partial order $\multimap$ on the set S^1 of morass points: $\sigma \multimap \tau$ if there exists a structure preserving map*

$$\pi\colon (L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \upharpoonright s(\tau))$$

such that

a) *π is elementary for existential statements of the form $\exists v_0 \ldots \exists v_{m-1}\, \psi$ where ψ is quantifier-free in the language for $(L_{s(\sigma)}, \in, <_L, I, N, S \upharpoonright s(\sigma))$;*

b) *$\pi \upharpoonright \gamma_\sigma = \mathrm{id} \upharpoonright \gamma_\sigma$, $\pi(\gamma_\sigma) = \gamma_\tau > \gamma_\sigma$, $\pi(\sigma) = \tau$, $\pi(p_\sigma) = p_\tau$;*

c) *if τ possesses an immediate $\prec$-predecessor τ' then $\tau' \in \mathrm{range}\, \pi$.*

We shall see that the system $(S^1, \multimap)$ with connecting maps as in this definition is a gap-1 morass. We first state some results about the "collapsing structures" $(L_{s(\sigma)}, p_\sigma)$.

Lemma 19.5. *Let $\sigma \in S^1$ be a morass point and $(L_{s(\sigma)}, p_\sigma)$ as defined above. Then*

a) *$s(\sigma)$ is a limit location.*

b) *$\sigma \subseteq L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$.*

c) *$L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} = L_{s(\sigma)}$.*

Proof. a) Assume for a contradiction that $s(\sigma)$ is a successor location of the form $s(\sigma) = s^+$. By the finiteness property (Theorem 16.8) there is a $z \in L_s$ such that

$$L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} = L_{s^+}\{\gamma_\sigma \cup p_\sigma\} \subseteq L_s\{\gamma_\sigma \cup p_\sigma \cup \{z\}\}.$$

But then $L_s\{\gamma_\sigma \cup p_\sigma \cup \{z\}\}$ is cofinal in σ, contradicting the minimality of $s(\sigma)$.
b) Let $\xi \in \sigma$. Since $L_\sigma \vDash$"γ_σ is the only uncountable cardinal" and $L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$ is cofinal in σ take $\zeta, L_\eta \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$ such that $\xi < \zeta \in L_\eta$, $\eta < \sigma$, and

$$L_\eta \vDash \exists f\, f\colon \omega_1 \twoheadrightarrow \zeta \text{ is surjective},$$

where "ω_1" is the ZF-term for the smallest uncountable cardinal. Then

$$g = S(L_\eta, v_0\colon \omega_1 \twoheadrightarrow v_1 \text{ is surjective}, \frac{\zeta}{v_1}) \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$$

is a surjective map $g\colon \gamma_\sigma \twoheadrightarrow \zeta$. Now

$$\xi \in \zeta = \text{range}\, g \subseteq L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}.$$

c) Let $X = L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\} \lhd L_{s(\sigma)}$. By the Condensation Theorem 16.3 there is a minimal location $\bar{s} \tilde{\preccurlyeq} s(\sigma)$ so that there is an isomorphism

$$\pi\colon (X, \in, <_L, I, N, S \restriction s(\sigma)) \cong L_{\bar{s}} = (L_{\bar{\alpha}}, \in, <_L, I, N, S \restriction \bar{s}).$$

Since $\sigma \subseteq X$ we have $\pi \restriction \sigma = \text{id} \restriction \sigma$. Let $\bar{p} = \pi(p_\sigma)$. Since π is a homomorphism, $L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$. Then $L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ is trivially cofinal in σ and by the minimal definition of $s(\sigma)$ and p_σ we get $\bar{s} = s(\sigma)$ and $\bar{p} = p_\sigma$. So

$$L_{s(\sigma)} = L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}. \qquad \square$$

Property a) of the preceding Lemma will be crucial; since $s(\sigma)$ is a limit it will be possible to continuous approximate the collapsing structure. The finiteness property of the fine hierarchy makes the hierarchy so slow that most interesting phenomena can be located at limit locations.

Lemma 19.6. *Let $\sigma \multimap \tau$ witnessed by a structure preserving map*

$$\pi\colon (L_{s(\sigma)}, \in, <_L, I, N, S \restriction s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \restriction s(\tau))$$

as in Definition 19.4. Then π is the unique map satisfying Definition 19.4.

Proof. Let $x \in L_{s(\sigma)}$. By Lemma 19.5c, $x = t^{L_{s(\sigma)}}(\vec{\xi}, \vec{p})$ for some $L_{s(\sigma)}$-term t, $\vec{\xi} < \gamma_\sigma$, where $\vec{p}$ is the $<_L$-increasing enumeration of p_σ. Since π preserves the constructible operations, and since $\pi(\vec{\xi}) = \vec{\xi}$ and $\pi(p_\sigma) = p_\tau$ we have

$$\pi(x) = t^{L_{s(\tau)}}(\vec{\xi}, \vec{q}),$$

where $\vec{q}$ is the $<_L$-increasing enumeration of p_τ. Hence $\pi(x)$ is uniquely determined by Definition 28. $\square$

This lemma is the basis for

Definition 19.7. *For $\sigma \multimap \tau$ let*

$$\pi_{\sigma\tau}\colon (L_{s(\sigma)}, \in, <_L, I, N, S \restriction s(\sigma)) \to (L_{s(\tau)}, \in, <_L, I, N, S \restriction s(\tau))$$

be the unique map satisfying Definition 19.4.

19.2 Proving the morass axioms

The main theorem states that we have defined a morass in the previous section.

Theorem 19.8. *The system*

$$(S^1, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau, (\pi_{\sigma\tau})_{\sigma \mathrel{-\!\!\!-\!\!\!3} \tau})$$

is an $(\omega_1, 1)$*-morass, i.e., it satisfies the following axioms:*

– (M0)

  a) *For all* $\gamma \in S^0$ *the set* $S_\gamma = \{\sigma \in S^1 | \gamma_\sigma = \gamma\}$ *is a set of ordinals which is closed in its supremum;*

  b) S_{ω_2} *is closed unbounded in* ω_2 *;*

  c) $S^0 \cap \omega_1$ *is cofinal in* ω_1 *;*

  d) $\mathrel{-\!\!\!-\!\!\!3}$ *is a tree-ordering on* S^1.

– (M1) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$. *Then:*

  a) *Let* $\nu < \sigma$. *Then* ν *is a morass point iff* $\pi_{\sigma\tau}(\nu)$ *is a morass point.*

  b) *For all* $\nu \preccurlyeq \sigma$ *holds:* ν *is* $\prec$*-minimal,* $\prec$*-successor,* $\prec$*-limit iff* $\pi_{\sigma\tau}(\nu)$ *is* $\prec$*-minimal,* $\prec$*-successor,* $\prec$*-limit, respectively.*

  c) *If* τ' *is the immediate* $\prec$*-predecessor of* τ *then* $\pi^{-1}(\tau')$ *is the immediate* $\prec$*-predecessor of* σ.

– (M2) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, $\bar{\sigma} \prec \sigma$. *Then* $\bar{\sigma} \mathrel{-\!\!\!-\!\!\!3} \pi_{\sigma\tau}(\bar{\sigma})$ *with correponding map* $\pi_{\bar{\sigma}\pi_{\sigma\tau}(\bar{\sigma})} = \pi_{\sigma\tau} \upharpoonright L_{s(\bar{\sigma})}$.

– (M3) *Let* $\tau \in S^1$. *Then* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *is closed in the ordinals* $< \gamma_\tau$.

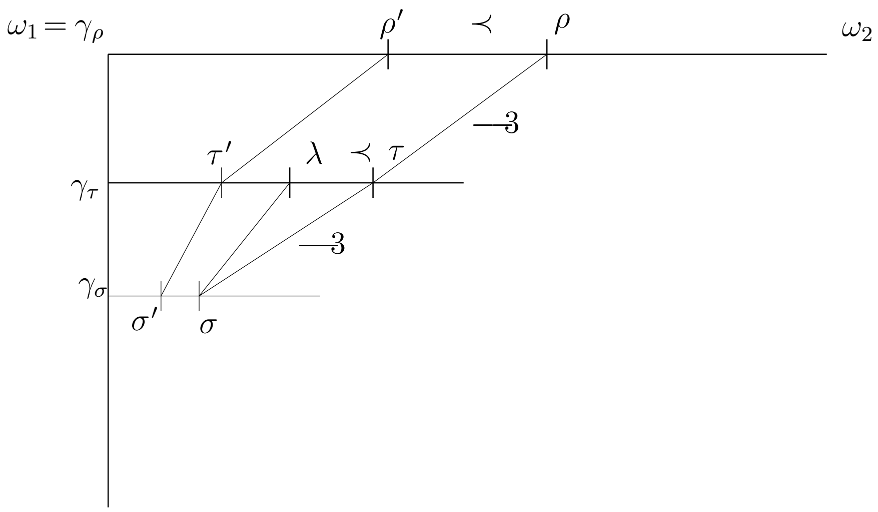
– (M4) *Let* $\tau \in S^1$ *and assume that* τ *is not* $\prec$*-maximal. Then* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *is cofinal in* γ_τ.

– (M5) *Let* $\{\gamma_\sigma \,|\, \sigma \mathrel{-\!\!\!-\!\!\!3} \tau\}$ *be cofinal in* γ_τ. *Then* $\tau = \bigcup_{\sigma \mathrel{-\!\!\!-\!\!\!3} \tau} \pi_{\sigma\tau}[\sigma]$.

– (M6) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, σ *a* $\prec$*-limit, and* $\lambda = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma < \tau$. *Then* $\sigma \mathrel{-\!\!\!-\!\!\!3} \lambda$ *with* $\pi_{\sigma\lambda} \upharpoonright \sigma = \pi_{\sigma\tau} \upharpoonright \sigma$.

– (M7) *Let* $\sigma \mathrel{-\!\!\!-\!\!\!3} \tau$, σ *a* $\prec$*-limit, and* $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma$. *Let* $\alpha \in S^0$ *such that* $\forall \bar{\sigma} \prec \sigma \exists \bar{\upsilon} \in S_\alpha \; \bar{\sigma} \mathrel{-\!\!\!-\!\!\!3} \bar{\upsilon} \mathrel{-\!\!\!-\!\!\!3} \pi_{\sigma\tau}(\bar{\sigma})$. *Then there exists* $\upsilon \in S_\alpha$ *such that* $\sigma \mathrel{-\!\!\!-\!\!\!3} \upsilon \mathrel{-\!\!\!-\!\!\!3} \tau$.

We shall show the morass axioms in a series of lemmas. The axioms can be motivated by the intended applications. Assume that one want to construct a structure of size ω_2 . Take ω_2 as the underlying set of the structure. We present ω_2 as the limit of a system of nicely cohering *countable* structures. The limit process has a two-dimensional nature: *inclusions* $\tau' \prec \tau$ (which implies $\tau' \subseteq \tau$) from left to right and morass maps $\pi_{\sigma\tau}$ going upwards. In the following picture the structure to be put on τ may be considered as enscribed on the vertical axis from 0 to γ_τ and on the horizontal level from γ_τ to τ. In a supposed construction, the horizontal levels are enscribed one after the other from bottom to top. To determine the enscriptions on a level S_α first map up all the enscriptions on levels S_β with $\beta < \alpha$ using the morass maps $\pi_{\sigma\tau}$ with $\sigma \in S_\beta$ and $\tau \in S_\alpha$. Often enough, this does not enscribe all of S_α so that on the non-enscribed parts the structure may be defined according to the specific aim of the construction.



The morass axioms will ensure that the general process above is possible: the morass maps are consistent with each other and with the inclusions from left to right, and the top level will be determined completely from the previous levels. Let us comment on some of the easier axioms. The intention of the complicated axioms M6 and M7 will only become apparent in actual constructions.

- M0 makes some general statements about the morass system: all of ω_2 is covered by morass points; the tree property ensures that a morass point can only be reached by one path from below.
- M1 and M3 give some further information along these lines.
- M2 is necessary for a consistent copying process from lower to higher levels.
- M4 says that a morass point τ which is not maximal on its level is a "limit" of the path leading to it. Together with M5 this completely determines the structure (the enscription) on τ. So the specific construction has to be performed for maximal points σ of levels which are not a limit of the path below.

Lemma 19.9. (M0) *holds.*

Proof. d) Let $\sigma, \sigma' \mathrel{-\!\!\!-\!\!3} \tau$, $\sigma \leqslant \sigma'$. Then the map $\pi_{\sigma'\tau}^{-1} \circ \pi_{\sigma\tau} \colon L_{s(\sigma)} \rightarrow L_{s(\sigma')}$ witnesses that $\sigma \mathrel{-\!\!\!-\!\!3} \sigma'$. So the $\mathrel{-\!\!\!-\!\!3}$-predecessors of any morass point are linearly ordered. Indeed they are wellordered since $\sigma \mathrel{-\!\!\!-\!\!3} \nu$ implies that $\sigma < \nu$. □

Lemma 19.10. (M1) *holds.*

Proof. (1) Let $\delta \in \mathrm{Ord} \cap L_{s(\sigma)}$. Then $\pi \restriction L_\delta \colon (L_\delta, \in) \to (L_{\pi(\delta)}, \in)$ is elementary.
Proof. For an $\in$-formula φ and $\vec{a} \in L_\delta$ note

$$\begin{aligned}(L_\delta, \in) \vDash \varphi(\vec{a}) \ &\text{iff} \ S(L_\delta, \varphi(\vec{w}) \wedge v_0 = 1, \frac{\vec{a}}{\vec{w}}) = 1 \\ &\text{iff} \ S(L_{\pi(\delta)}, \varphi(\vec{w}) \wedge v_0 = 1, \frac{\pi(\vec{a})}{\vec{w}}) = 1 \\ &\text{iff} \ (L_{\pi(\delta)}, \in) \vDash \varphi(\pi(\vec{a})) \ . \qquad qed(1)\end{aligned}$$

a) Being a morass point is absolute for transitive ZF^--models. σ is a morass point and so L_σ is a limit of ZF^--models. Take $\delta, \nu < \delta < \sigma$ so that L_δ is a ZF^--model. By (1), $L_{\pi(\delta)}$ is also a ZF^--model. Now ν is a morass point iff $(L_\delta, \in) \vDash \nu$ is a morass point iff $(L_{\pi(\delta)}, \in) \vDash \pi(\nu)$ is a morass point iff $\pi(\nu)$ is a morass point.
b) Also being a morass point which is $\prec$-minimal, $\prec$-successor, or $\prec$-limit can be expressed absolutely for ZF^--models and we can use the same technique as in a) to prove preservation.
c) $\pi^{-1}(\tau')$ is defined and it is a morass point by a). Assume for a contradiction that there is a morass point σ', $\pi^{-1}(\tau') \prec \sigma' \prec \sigma$. By a), $\pi(\sigma')$ is a morass point and $\tau' \prec \pi(\sigma') \prec \pi(\sigma) = \tau$, which contradicts the assumptions of c). □

Lemma 19.11. (M2) *holds*

Proof. Set $\bar{\tau} = \pi_{\sigma\tau}(\bar{\sigma})$. Take δ, $\bar{\sigma} < \delta < \sigma$ so that L_δ is a ZF^--model. The collapsing structure $(L_{s(\bar{\sigma})}, p(\bar{\sigma}))$ is definable in $(L_\delta, \in)$ from the parameter $\bar{\sigma}$. Then the same terms define $(L_{s(\bar{\tau})}, p(\bar{\tau}))$ in $(L_{\pi_{\sigma\tau}(\delta)}, \in)$ from the parameter $\bar{\tau}$, and the map $\pi_{\sigma\tau}$ restricted to the collapsing structure $L_{s(\bar{\sigma})}$ witnesses $\bar{\sigma} \longrightarrow\!\!\!\!\!3\ \pi_{\sigma\tau}(\bar{\sigma})$ by the lementarity of $\pi_{\sigma\tau} \restriction L_\delta \colon (L_\delta, \in) \to (L_{\pi_{\sigma\tau}(\delta)}, \in)$. □

Lemma 19.12. (M3) *holds*

Proof. Let $\bar{\alpha} < \gamma_\tau$ be a limit of $\{\gamma_\sigma \mid \sigma \longrightarrow\!\!\!\!\!3\ \tau\}$. Form the hull

$$L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\}$$

and by condensation obtain an isomorphism

$$\pi \colon L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\} \cong L_{\bar{s}} \text{ with } \bar{\tau} = \pi(\tau) \text{ and } \bar{p} = \pi(p_\tau).$$

(1) $\bar{\tau}$ is a morass point.
Proof. Let $\xi < \bar{\tau}$. Take $\sigma \longrightarrow\!\!\!\!\!3\ \tau$ such that $\gamma_\sigma < \bar{\alpha}$ and

$$\pi^{-1}(\xi) \in (L_{s(\tau)}\{\gamma_\sigma \cup \{p_\tau\}\} = \mathrm{range}\, \pi_{\sigma\tau} .$$

Let $\bar{\xi} < \sigma$ such that $\pi^{-1}(\xi) = \pi_{\sigma\tau}(\bar{\xi})$. Since σ is a morass point take an ordinal μ, $\bar{\xi} < \mu < \sigma$ such that $L_\mu \vDash \mathrm{ZF}^-$. Then $\pi_{\sigma\tau}(L_\mu) = L_{\pi_{\sigma\tau}(\mu)} \vDash \mathrm{ZF}^-$.

$$\pi_{\sigma\tau}(L_\mu) = L_{\pi_{\sigma\tau}(\mu)} \in \mathrm{range}\, \pi_{\sigma\tau} \subseteq L_{s(\tau)}\{\bar{\alpha} \cup \{p_\tau\}\} = \mathrm{range}\, \pi^{-1}.$$

Then $\pi(\pi_{\sigma\tau}(L_\mu)) = L_{\pi(\pi_{\sigma\tau}(\mu))}$ is a ZF^--model. Furthermore $\bar{\xi} < \mu < \sigma$ implies that $\pi_{\sigma\tau}(\bar{\xi}) = \pi^{-1}(\xi) < \pi_{\sigma\tau}(\mu) < \pi_{\sigma\tau}(\sigma) = \tau$ and $\xi = \pi(\pi^{-1}(\xi)) < \pi(\pi_{\sigma\tau}(\mu)) < \pi(\tau) = \bar{\tau}$. So $L_{\bar{\tau}}$ is a limit of ZF^--models.

Similarly one can show that $\bar{\alpha}$ is the only uncountable cardinal in $L_{\bar{\tau}}$.

Note that $L_{s(\tau)}\{\bar{\alpha} \cup p_\tau\} \cap \gamma_\tau = \bar{\alpha}$, since $\bar{\alpha}$ is the limit of $L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} \cap \gamma_\tau = \gamma_\sigma < \bar{\alpha}$. We show $\bar{s} = s(\bar{\tau})$: Clearly $s(\bar{\tau}) \tilde{<} \bar{s}$, since $L_{\bar{s}} = L_{\bar{s}}\{\bar{\alpha} \cup \bar{p}\}$ is cofinal in $\bar{\tau}$. Now assume for a contradiction that $s(\bar{\tau}) \tilde{<} \bar{s}$. Let $\pi_\sigma = \pi \circ \pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!\!3} \tau \,|\, \gamma_\sigma < \bar{\alpha}\}$. Choose σ large enough such that there exist $\tilde{s}, \tilde{p} \in L_{s(\sigma)}$ with $s(\bar{\tau}) = \pi_\sigma(\tilde{s})$ and $p_{\bar{\tau}} = \pi_\sigma(\tilde{p})$. By $s(\bar{\tau}) \tilde{<} \bar{s}$ we have $\tilde{s} \tilde{<} s(\sigma)$ and hence $L_{\tilde{s}}\{\gamma_\sigma \cup \tilde{p}\}$ bounded in σ, say by β. But this bound is preserved by $\pi_{\sigma\tau}$ and by π (hence by π_σ); therefore, we get that $L_{s(\bar{\tau})}\{\bar{\alpha} \cup p_{\bar{\tau}}\} \cap \bar{\tau}$ is bounded by $\pi_\sigma(\beta) < \bar{\tau}$ which contradicts the definition of $s(\bar{\tau})$ and $p_{\bar{\tau}}$.

To see that $\pi^{-1}\colon L_{s(\bar{\tau})} \to L_{s(\tau)}$ is a morass map and hence $\bar{\tau} \mathrel{-\!\!\!3} \tau$ with $\gamma_{\bar{\tau}} = \bar{\alpha}$, we need to show, that π^{-1} preserves Σ_1; the other properties follow by definition, for p_τ and the predecessor of τ (if any) note that dom π contains the ranges of morass maps as subsets.

As a collapsing map, π^{-1} is structure-preserving. Σ_1 is preserved upwards. Now assume, we have a Σ_1-formula in $L_{s(\tau)}$. It is preserved downwards by morass maps $\pi_{\sigma\tau}$ for $\sigma \in \{\sigma \mathrel{-\!\!\!3} \tau \,|\, \gamma_\sigma < \bar{\alpha}\}$ and hence has a witness in range $\pi_{\sigma\tau} \subset \operatorname{dom} \pi$. □

Lemma 19.13. (M4) *holds.*

Proof. Let $\upsilon \in S_{\gamma_\tau}$ with $\tau < \upsilon$. Let $\alpha < \gamma_\tau$ be arbitrary and η between τ and υ s.t. $L_{s(\tau)} \in L_\eta$ and $L_\eta \models ZF^-$. Let $X \prec L_\eta$ s.t. $L_{s(\tau)}\{\alpha \cup p_\tau\} \cup \{\tau\} \subset X$ and $\bar{\alpha} := X \cap \gamma_\tau \in \gamma_\tau$. Let $\pi\colon X \cong L_{\bar{\eta}}$, $\sigma = \pi(\tau)$, and $\bar{p} = \pi(p_\tau)$. So σ is a morass point and $\pi^{-1} \restriction L_{s(\sigma)}\colon L_{s(\sigma)} \to L_{s(\tau)}$ is elementary and therefore a morass map. Hence $\sigma \mathrel{-\!\!\!3} \tau$ and $\alpha \leq \gamma_\sigma = \bar{\alpha}$. □

Lemma 19.14. (M5) *holds.*

Proof. Let $\xi \in \tau \in S^1$. We have $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\}$ and by cofinality there exists a $\sigma \mathrel{-\!\!\!3} \tau$ with $\xi \in L_{s(\tau)}\{\gamma_\sigma \cup p_\tau\} = \operatorname{range} \pi_{\sigma\tau}$. □

Lemma 19.15. (M6) *holds.*

Proof. Let $\tilde{s} = \tilde{<}\text{-lub}\,\{\pi_{\sigma\tau}(t) | t \tilde{<} s(\sigma)\}$. We show that $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cap \tau = \lambda$: First assume $\lambda_0 \in \lambda$; then there is λ_1 with $\lambda_0 < \lambda_1 < \lambda$ and $\lambda_1 = \pi_{\sigma\tau}(\bar{\lambda}_1)$. Then $L_\sigma \models \operatorname{card}(\bar{\lambda}_1) \leq \gamma_\sigma$, hence there exists $\bar{f} \in L_\sigma$ s.t. $\bar{f} : \gamma_\sigma \to \bar{\lambda}_1$ is onto, in particular $\bar{f} \in L_{s(\sigma)}\{\gamma_\sigma \cup p_\sigma\}$. As $s(\sigma)$ is a limit location, we have $\bar{f} \in L_t\{\gamma_\sigma \cup p_\sigma\}$ for some $t \tilde{<} s(\sigma)$. Let $f = \pi_{\sigma\tau}(\bar{f}) \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$, then $f\colon \gamma_\tau \to \lambda_1$ is onto, so $\lambda_0 \in \operatorname{range} f$, hence $\lambda_0 \in L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$. On the other hand assume $\lambda_0 \in L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cap \tau$, then there is a $t \tilde{<} s(\sigma)$ s.t. $\lambda_0 \in L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\}$. But $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say), since $t \tilde{<} s(\sigma)$, hence also $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ, namely by $\pi_{\sigma\tau}(\beta) < \lambda$. So $\lambda_0 \in \lambda$ as required.

Let $\pi\colon L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\} \cong L_{s_0}$ and $p_0 = \pi(p_\tau)$ (then $\lambda = \pi(\tau)$). Note that $\lambda \in S_{\gamma_\tau}$. We show $L_{s_0}\{\gamma_\tau \cup p_0\} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$:

$s_0 = s(\lambda)$: First note that s_0 singularizes λ, so $s(\lambda) \mathrel{\tilde{<}} s_0$. Assume for contradiction that s_0 is strictly greater. As $p_\lambda \in L_{s_0}\{\gamma_\tau \cup p_0\}$, we have $p_\lambda \in L_{s_1}\{\gamma_\tau \cup p_0\}$ where $s(\lambda) \mathrel{\tilde{<}} s_1 \mathrel{\tilde{<}} s_0$ (and where $\alpha(s(\lambda))$ belongs to $L_{s_1}\{\gamma_\tau \cup p_0\}$ in case $\alpha(s(\lambda)) < \alpha(s_0)$; of course we are using the fact that s_0 is a limit location). Since $L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\} \subset L_{s_1}\{\gamma_\tau \cup p_0\}$, s_1 singularizes λ. By definition of s_0, $\pi^{-1}(s_1) \mathrel{\tilde{<}} \tilde{s}$. Further, by definition of $\tilde{s}$, there is a $t \mathrel{\tilde{<}} s(\sigma)$ s.t. $\pi^{-1}(s_1) \mathrel{\tilde{<}} \pi_{\sigma\tau}(t)$. By minimality of $s(\sigma)$, $L_t\{\gamma_\sigma \cup p_\sigma\} \cap \sigma$ is bounded below σ (by β say). Hence $L_{\pi_{\sigma\tau}(t)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (by $\pi_{\sigma\tau}(\beta)$). Since $\pi^{-1}(s_1) \mathrel{\tilde{<}} \pi_{\sigma\tau}(t)$, $L_{\pi^{-1}(s_1)}\{\gamma_\tau \cup p_\tau\} \cap \tau$ is bounded below τ (still by $\pi_{\sigma\tau}(\beta)$). Apply π: $L_{s_1}\{\gamma_\tau \cup p_0\} \cap \lambda$ is bounded below λ (by $\pi \circ \pi_{\sigma\tau}(\beta)$), contradiction.

$p_0 = p_\lambda$: $L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_0\}$ is cofinal in λ (as above using $s_0 = s(\lambda)$). Therefore, $p_\lambda \leq^* p_0$. Assume for contradiction that p_0 is strictly greater, then using $p_0 \in L_{s(\lambda)} = L_{s(\lambda)}\{\gamma_\tau \cup p_\lambda\}$ and applying π^{-1} we get $\pi^{-1}(p_\lambda) <^* p_\tau \in L_{\tilde{s}}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\} \subset L_{s(\tau)}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\}$. Therefore, $L_{s(\tau)} = L_{s(\tau)}\{\gamma_\tau \cup p_\tau\} = L_{s(\tau)}\{\gamma_\tau \cup \pi^{-1}(p_\lambda)\}$ contradicting the minimality of p_τ.

Let $\pi_0 = \pi \circ \pi_{\sigma\tau}$: $L_{s(\sigma)} \to L_{s(\lambda)}$. π_0 is well-defined as range $\pi_{\sigma\tau} = L_{\tilde{s}}\{\gamma_\sigma \cup p_\tau\} \subset \operatorname{dom} \pi$. Further, $\pi_0(\sigma) = \lambda$ and $\pi_0(p_\sigma) = p_\lambda$. Since λ is a $\prec$-limit, property 19.4c) of the definition of a morass map is vacuous. Finally, π_0 is Σ_1-preserving: First note that π_0 is structure-preserving. Σ_1 formulas are preserved by π_0 upwards, by π upwards (from $L_{s(\lambda)}$ to $L_{\tilde{s}}\{\gamma_\tau \cup p_\tau\}$), and by $\pi_{\sigma\tau}$ downwards, hence by π_0 both ways. Now $\pi_0 = \pi_{\sigma\lambda}$ is a morass map, hence $\sigma \mathrel{-\!\!\!-\!3} \lambda$ as required. □

Lemma 19.16. (M7) *holds.*

Proof. We first show that $L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau = \alpha$, clearly α is a subset of the left side. For the other direction note that since we assume $\tau = \sup \operatorname{range} \pi_{\sigma\tau} \upharpoonright \sigma$, the argument for (M6) shows that $s(\tau) = \mathrel{\tilde{<}}$-lub $\{\pi_{\sigma\tau}(t) | t \mathrel{\tilde{<}} s(\sigma)\}$. Let $\xi \in L_{s(\tau)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$, then there is $s_0 \mathrel{\tilde{<}} s(\sigma)$ s.t. $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Working downstairs we have that $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ does not collapse σ (by minimality of $s(\sigma) \mathrel{\tilde{>}} s_0$). Let π_0: $L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\} \cong L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$ where $\bar{p} = \pi_0^{-1}(p_\sigma)$. Then $\sigma' := \pi_0^{-1}(\sigma) < \sigma$. $L_{\bar{s}}$ cannot collapse σ', else there would be a map from γ_σ onto σ' and hence a map from γ_σ onto σ in $L_{s_0}\{\gamma_\sigma \cup p_\sigma\}$. Therefore, $L_{\bar{s}} \models \mathrm{Card}\sigma'$ and $L_\sigma \models \neg\mathrm{Card}\sigma'$, hence $L_{\bar{s}} \in L_\sigma$. Now, σ is a $\prec$-limit, so there is $\bar{\sigma} \prec \sigma$ s.t. $L_{\bar{s}}, \bar{p} \in L_{s(\bar{\sigma})} = L_{s(\bar{\sigma})}\{\gamma_\sigma \cup p_{\bar{\sigma}}\}$.

We shift the isomorphism π_0 to $L_{s(\tau)}$:

$$\text{“}\pi_{\sigma\tau}(\pi_0)\text{”: } L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\} \cong L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$$

We started with $\xi \in L_{\pi_{\sigma\tau}(s_0)}\{\alpha \cup p_\tau\} \cap \gamma_\tau$. Now we apply the isomorphism and infer $\xi \in L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau$ (since $\xi < \gamma_\tau$ it is not moved). Further, $L_{\pi_{\sigma\tau}(\bar{s})}\{\alpha \cup \pi_{\sigma\tau}(\bar{p})\} \cap \gamma_\tau \subset L_{s(\pi_{\sigma\tau}(\bar{\sigma}))}\{\alpha \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\} \cap \gamma_\tau = \alpha$, where the former holds since $\pi_{\sigma\tau}(\bar{p}) \in L_{\pi_{\sigma\tau}(\bar{\sigma})}\{\gamma_\sigma \cup p_{\pi_{\sigma\tau}(\bar{\sigma})}\}$ and $\pi_{\sigma\tau}(\bar{s}) \mathrel{\tilde{<}} s(\pi_{\sigma\tau}(\bar{\sigma}))$ and the latter holds by $\bar{\sigma} \mathrel{-\!\!\!-\!3} \bar{\upsilon} \mathrel{-\!\!\!-\!3} \pi_{\sigma\tau}(\bar{\sigma})$ for some $\bar{\upsilon} \in S_\alpha$. Hence $\xi \in \alpha$ as desired.

Now we define π: $L_{s(\tau)}\{\alpha \cup p_\tau\} \cong L_{s'}\{\alpha \cup p'\} = L_{s'}$ where $p' := \pi(p_\tau)$, $\upsilon := \pi(\tau)$. By the previous argument we have $\pi^{-1}(\alpha) = \gamma_\tau$. Using the system of morass maps we have $\upsilon \in S_\alpha$.

We have to show $s' = s(\upsilon)$: $L_{s'} = L_{s'}\{\alpha \cup p'\}$ collapses υ, hence $s(\upsilon) \tilde{<} s'$. Assume for a contradiction that $s(\upsilon) \tilde{<} s'$. Since $p_\upsilon \in L_{s'}$ we have that there is an s_0 s.t. $s(\upsilon) \tilde{<} s_0 \tilde{<} s'$ and $p_\upsilon \in L_{s_0}\{\alpha \cup p'\}$. Since $\pi_{\sigma\tau}$ and π map locations cofinally this is also true for $\pi_0 := \pi \circ \pi_{\sigma\tau}$ (locations $\tilde{<} s(\sigma)$ are mapped to locations $\tilde{<} s'$). Hence without loss of generality, $s_0 = \pi_0(\bar{s}_0)$ where $\bar{s}_0 \tilde{<} s(\sigma)$. Therefore, $L_{s(\sigma)} \models$ "$L_{\bar{s}_0}\{\gamma_\sigma \cup p_\sigma\}$ is bounded below σ". This is preserved by $\pi_{\sigma\tau}$: $L_{s(\tau)} \models$ "$L_{\pi_{\sigma\tau}(\bar{s}_0)}\{\gamma_\tau \cup p_\tau\}$ is bounded below τ". Finally, this is preserved by π downwards: $L_{s'} \models$ "$L_{s_0}\{\alpha \cup p'\}$ is bounded below υ", contradicting the definition of $s(\upsilon) \tilde{<} s_0$.

Finally, we have to show that π^{-1} is Σ_1-preserving, then $\pi^{-1} = \pi_{\upsilon\tau}$ and $\pi_{\sigma\upsilon} = \pi_{\upsilon\tau}^{-1} \circ \pi_{\sigma\tau}$. First note that π is structure-preserving.

Σ_1 is preserved upwards by π^{-1} (i.e., from $L_{s(\upsilon)}$ to $L_{s(\tau)}\{\alpha \cup p_\upsilon\}$). For the other direction, assume $L_{s(\tau)} \models \exists x\, \phi(x, \vec{r})$, where ϕ is quantifier-free and $\vec{r} \in \operatorname{dom} \pi = L_{s(\tau)}\{\gamma_\upsilon \cup p_\tau\}$; we have to show $L_{s(\upsilon)} \models \exists x\, \phi(x, \pi(\vec{r}))$. As before, fix $s_0 \tilde{<} s(\sigma)$ s.t. $\vec{r} \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\upsilon \cup p_\tau\}$ and $w \in L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\}$ where w is the least witness for $\exists x\, \phi(x, \vec{r})$. Our aim is to show that γ_τ can be replaced by γ_υ in the latter hull.

Let $\pi_1 : L_{s_0}\{\gamma_\sigma \cup p_\sigma\} \cong L_{\bar{s}} = L_{\bar{s}}\{\gamma_\sigma \cup \bar{p}\}$ where $\bar{p} = \pi_1(p_\sigma)$. As above using type preservation, we shift π_1 to the γ_τ-level, let's call the resulting map π_2: $L_{\pi_{\sigma\tau}(s_0)}\{\gamma_\tau \cup p_\tau\} \cong L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$. Then we have $\pi_2(\vec{r}) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\upsilon \cup \pi_{\sigma\tau}(\bar{p})\}$ and $\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$: $L_{\pi_{\sigma\tau}(\bar{s})} \models \phi(\pi_2(w), \pi_2(\vec{r}))$

Further, also as above, there is a $\bar{\sigma} \prec \sigma$ s.t. $L_{\bar{s}} \in L_{\bar{\sigma}}$ with $\bar{\sigma} \multimap \bar{\upsilon} \multimap \bar{\tau} := \pi_{\sigma\tau}(\bar{\sigma})$ and $\pi_2(\vec{r})$, $\pi_{\sigma\tau}(\bar{s})$, $\pi_{\sigma\tau}(\bar{p}) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$. Therefore, $\pi_2(w) \in \operatorname{range} \pi_{\bar{\upsilon}\bar{\tau}}$ and hence by $\pi_{\bar{\upsilon}\bar{\tau}}$ being a morass map, we can replace γ_τ by γ_υ in "$\pi_2(w) \in L_{\pi_{\sigma\tau}(\bar{s})}\{\gamma_\tau \cup \pi_{\sigma\tau}(\bar{p})\}$". Applying π_2^{-1} we get $w \in \operatorname{range} \pi_{\upsilon\tau}$. This proves Σ_1-preservation. □

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