Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke | Problem sheet 11

Problem 37 (14 Points)**.** A tree $\mathbb{T}$ of height ω_1 is a *Kurepa tree* if $|[\mathbb{T}]| > \omega_1$ and $|\mathbb{T}(\alpha)| < \omega_1$ for all $\alpha < \omega_1$.

Define $\mathbb{P}$ to be the partial order whose conditions are pairs $p = \langle T_p, f_p \rangle$ such that there is an $\alpha_p < \omega_1$ with

(a) $T_p \subseteq {}^{<\alpha_p}2$ is countable and closed under initial segments.

(b) $f_p : \omega_2 \xrightarrow{part} {}^{\alpha_p}2$ is a countable partial function with $f(\gamma) \upharpoonright \alpha \in T_p$ for all $\alpha < \alpha_p$ and $\gamma \in \mathrm{dom}(f_p)$.

and whose ordering $p \leq_{\mathbb{P}} q$ is given by

$$\alpha_q \leq \alpha_p \ \wedge \ T_q = T_p \cap {}^{<\alpha_q}2 \ \wedge \ \mathrm{dom}(f_q) \subseteq \mathrm{dom}(f_p) \ \wedge \ \forall \gamma \in \mathrm{dom}(f_q) \ f_q(\gamma) = f_p(\gamma) \upharpoonright \alpha_q.$$

(1) Show that $\mathbb{P}$ is $<\omega_1$-closed.

(2) Show that, if (CH) holds, then $\mathbb{P}$ satisfies the ω_2-chain condition. (Hint: *Show that the set* $\{T_p \mid p \in \mathbb{P}\}$ *has cardinality* ω_1 *in this case and then use the* Δ*-system lemma*)

(3) Given $\alpha < \omega_1$ and $\gamma < \delta < \omega_2$, show that the set

$$D_{\alpha,\gamma,\delta} \ = \ \{p \in \mathbb{P} \mid \alpha < \alpha_p, \ \gamma, \delta \in \mathrm{dom}(f_p), \ f_p(\gamma) \neq f_p(\delta)\}$$

is dense in $\mathbb{P}$.

(4) Prove that the consistency of ZF implies the consistency of the theory

$$\text{ZFC} + \text{"}\textit{There is a Kurepa tree}\text{"}.$$

Problem 38 (4 Points)**.** Show that a subset A of $\mathcal{N}$ contains a perfect subset if and only if there is a sequence $\vec{t} = \langle t_s \in {}^{<\omega}\omega \mid s \in {}^{<\omega}2 \rangle$ with the following properties.

(1) If $s \in {}^{<\omega}2$, then $t_s \subsetneq t_{s^\frown\langle 0 \rangle}$, $t_s \subsetneq t_{s^\frown\langle 1 \rangle}$, $\mathrm{lh}(t_{s^\frown\langle 0 \rangle}) = \mathrm{lh}(t_{s^\frown\langle 1 \rangle})$ and $t_{s^\frown\langle 0 \rangle} \neq t_{s^\frown\langle 1 \rangle}$.

(2) If $x \in \mathcal{C}$, then $\bigcup_{n<\omega} t_{x \upharpoonright n} \in A$.

Problem 39 (6 Points)**.** Show that every uncountable Polish space contains a subset without the perfect set property. (Hint: *First show that every uncountable closed subset of* $\mathcal{N}$ *contains a subset without the perfect set property by using an enumeration* $\langle \vec{t}_\gamma \mid \gamma < 2^\omega \rangle$ *of all sequences with the properties listed in Problem 38*)

Problem 40 (8 Extra Points)**.** Work in the theory

$$\text{ZF} + \text{"}\textit{Every subset of } \mathcal{N} \textit{ has the perfect set property}\text{"}$$

and prove that there is no injection $i : \omega_1 \longrightarrow {}^\omega\omega$.

Please hand in your solutions on Wednesday, July 08, before the lecture.