Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke Problem sheet 9

Problem 30 (4 Points). Let M be a transitive model of ZFC and G be $\mathrm{Fn}(\omega_1, 2, \omega)^M$-generic over M. Show that there is an $A \in \mathcal{P}(\omega_1^M) \cap M[G]$ with the property that, whenever $\omega \leq \gamma < \omega_1^M$ and $G_0 \times G_1$ is the filter on $(\mathrm{Fn}(\gamma, 2, \omega) \times \mathrm{Fn}(\omega_1 \setminus \gamma, 2, \omega))^M$ induced by G, then $A \notin M[G_0] \cup M[G_1]$.

Problem 31 (*Almost disjoint coding forcing*, 8 Points). Given a subset A of ${}^\omega\omega$, we define $\mathbb{P}_A$ to be the partial order whose conditions are pairs $p = (c_p, X_p)$ with

- X_p is a finite subset of A.
- $c_p : {}^{<\omega}\omega \xrightarrow{part} 2$ is a finite partial function.

such that $p \leq_{\mathbb{P}_A} q$ holds if and only if the following statements hold.

(1) $c_q \subseteq c_p$ and $X_q \subseteq X_p$.

(2) If $x \in X_q$ and $l < \omega$ with $x \restriction l \in \mathrm{dom}(c_p) \setminus \mathrm{dom}(c_q)$, then $c_p(x \restriction l) = 1$.

Prove the following statements.

(a) Given $x \in A$ and $t \in {}^{<\omega}\omega$, the set

$$\{p \in \mathbb{P}_A \mid x \in X_p \ \wedge \ t \in \mathrm{dom}(c_p)\}$$

is dense in $\mathbb{P}_A$.

(b) $\mathbb{P}_A$ satisfies the countable chain condition.

(c) Given $y \in {}^\omega\omega \setminus A$ and $k < \omega$, the set

$$\{p \in \mathbb{P}_A \mid \exists l < \omega \ [k < l \ \wedge \ y \restriction l \in \mathrm{dom}(c_p) \ \wedge \ c_p(y \restriction l) = 0]\}$$

is dense in $\mathbb{P}_A$.

Let M be a transitive model of ZFC, $A \in \mathcal{P}({}^\omega\omega) \cap M$ and G be $\mathbb{P}_A^M$-generic over M.

(d) There is a function $C : {}^{<\omega}\omega \longrightarrow 2$ in $M[G]$ such that the equivalence

$$x \in A \ \longleftrightarrow \exists N < \omega \ \forall n < \omega \ [N < n \ \longrightarrow \ C(x \restriction n) = 1]$$

holds for every $x \in \mathcal{P}({}^\omega\omega) \cap M$.

Problem 32 (5 Points). Let K be a countable field. Equip the set $\mathrm{Aut}(K)$ of all automorphisms of K with the topology whose basic open sets are of the form

$$N_s \ = \ \{\pi \in \mathrm{Aut}(K) \mid s \subseteq \pi\}$$

for some finite partial function $s : K \xrightarrow{partial} K$.

(1) Show that the resulting topological space is a Polish space (Hint: *View* $\mathrm{Aut}(K)$ *as a subset of* ${}^\omega\omega$ *and let* d *denote the metric defined in Proposition 6.1.10.(iv). Consider the function* $d_*(\pi, \sigma) = d(\pi, \sigma) + d(\pi^{-1}, \sigma^{-1})$).

(2) Show the function

$$\circ : \mathrm{Aut}(K) \times \mathrm{Aut}(K) \longrightarrow \mathrm{Aut}(K); (\pi, \sigma) \mapsto \pi \circ \sigma$$

is continuous with respect to this topology.

Problem 33 (7 Points)**.** Prove Proposition 6.1.3. from the lecture course:

(1) *Given a Polish space X, there is a complete metric d on X such that d is compatible with X and $d(x,y) \leq 1$ for all $x, y \in X$.* (Hint: *Pick a compatible complete metric d and conider the function $d_*(x,y) = \frac{d(x,y)}{1+d(x,y)}$*)

(2) *The product of countably-many Polish spaces is a Polish space.* (Hint: *Given a sequence $\langle X_n \mid n < \omega \rangle$ of Polish spaces, pick a sequence $\langle d_n \mid n < \omega \rangle$ of compatible complete metrics with $d_n(x,y) \leq 1$ for all $n < \omega$ and $x, y \in X_n$ and consider the function $d_*(\vec{x}, \vec{y}) = \sum_{n<\omega} 2^{-n-1} \cdot d_n(\vec{x}(n), \vec{y}(n))$*)

Please hand in your solutions on Wednesday, June 24, before the lecture.