Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke Problem sheet 8

Problem 27 (6 Points)**.** Prove Lemma 5.16. from the lecture course: *Let* $\pi : \mathbb{P} \to \mathbb{Q}$ *be a dense embedding of partial orders. If* $\varphi(v_0, \dots, v_{n-1})$ *is an* $\mathcal{L}_\in$*-formula, then*

$$p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \dots, \tau_{n-1}) \iff \pi(p) \Vdash^*_{\mathbb{Q}} \varphi(\pi_*(\tau_0), \dots, \pi_*(\tau_{n-1}))$$

for all $p \in \mathbb{P}$ *and* $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$. (Hint: Use Problem 22 in the nontrivial direction of the $\exists$-case.)

Problem 28 (8 Points)**.** Prove the Product Lemma (Theorem 5.2.5. from the lecture course): *Let* M *be a transitive model of* ZFC *and* $\mathbb{P}_0, \mathbb{P}_1 \in M$ *be partial orders with maximal elements. Then the following statements are equivalent for every filter* $G_0 \times G_1$ *on* $\mathbb{P}_0 \times \mathbb{P}_1$.

(1) $G_0 \times G_1$ *is* $(\mathbb{P}_0 \times \mathbb{P}_1)$*-generic over* M.

(2) G_0 *is* $\mathbb{P}_0$*-generic over* M *and* G_1 *is* $\mathbb{P}_1$*-generic over* $M[G_0]$.

(3) G_1 *is* $\mathbb{P}_1$*-generic over* M *and* G_0 *is* $\mathbb{P}_0$*-generic over* $M[G_1]$.

(Hint: *Use Lemma 4.1.5. from the lecture course and Problem 22.*)

Problem 29 (*Mathias forcing*, 10 Points)**.** Fix a non-principal ultrafilter $\mathcal{U}$ on ω. Let $\mathbb{P}_{\mathcal{U}}$ denote the partial order consisting of conditions $p = (s_p, A_p)$ such that $s_p : n_p \longrightarrow \omega$ is a strictly increasing function with $n_p < \omega$ and $A_p \in \mathcal{U}$ ordered by

$$p \leq_{\mathbb{P}_{\mathcal{U}}} q \iff s_q \subseteq s_p \ \wedge \ A_p \subseteq A_q \ \wedge \ \forall k \in \mathrm{dom}(s_p) \setminus \mathrm{dom}(s_q) \ s_p(k) \in A_q.$$

(1) Show that $\mathbb{P}_{\mathcal{U}}$ satisfies the countable chain condition.

Let M be a transitive model of ZFC, $\mathcal{U}$ be a non-principal ultrafilter on ω in M and G be $\mathbb{P}_{\mathcal{U}}$-generic over M.

(2) Show that $s_G = \bigcup_{p \in G} s_p$ is a strictly increasing function with domain ω.

(3) Prove that

$$\mathcal{U} = \{A \in \mathcal{P}(\omega) \cap M \mid \mathrm{ran}(s_G) \setminus A \textit{ is finite}\}.$$

Please hand in your solutions on Wednesday, June 17, before the lecture.