Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke — Problem sheet 7

Problem 25 (16 Points)**.** Let $\mathbb{B} = \langle \mathbb{B}, \leq, \wedge, \vee, 0, 1, ' \rangle$ be a complete boolean algebra, $\mathbb{B}^*$ denote the corresponding partial order and $\mathcal{U}$ be an ultrafilter on $\mathbb{B}$ (i.e. there is an homomorphism $\pi_{\mathcal{U}}$ of boolean algebras from $\mathbb{B}$ to the unique boolean algebra $\{0,1\}$ with two elements such that $\mathcal{U} = \pi^{-1}"\{1\}$). Define a relation $\equiv_{\mathcal{U}}$ on $\mathrm{V}^{\mathbb{B}^*}$ by setting

$$\sigma \equiv_{\mathcal{U}} \tau \iff [\![\text{"}\sigma = \tau\text{"}]\!]_{\mathbb{B}} \in \mathcal{U}$$

for all $\sigma, \tau \in \mathrm{V}^{\mathbb{B}^*}$.

(1) Show that $\equiv_{\mathcal{U}}$ is an equivalence relation on $\mathrm{V}^{\mathbb{B}^*}$.

Given $\tau \in \mathrm{V}^{\mathbb{B}^*}$, define

$$[\tau]_{\mathcal{U}} \;=\; \{\sigma \in \mathrm{V}^{\mathbb{B}^*} \mid \sigma \equiv_{\mathcal{U}} \tau \;\wedge\; \forall \rho \in \mathrm{V}^{\mathbb{B}^*} \; [\rho \equiv_{\mathcal{U}} \tau \longrightarrow \mathrm{rank}(\rho) \geq \mathrm{rank}(\sigma)]\} \in \mathrm{V}.$$

Let $\mathrm{V}/\,\mathcal{U}$ denote the class $\{[\tau]_{\mathcal{U}} \mid \tau \in \mathrm{V}^{\mathbb{B}^*}\}$ and define a relation $\in_{\mathcal{U}}$ on $\mathrm{V}/\,\mathcal{U}$ by setting

$$[\sigma]_{\mathcal{U}} \in_{\mathcal{U}} [\tau]_{\mathcal{U}} \;\Leftrightarrow\; [\![\text{"}\sigma \in \tau\text{"}]\!]_{\mathbb{B}} \in \mathcal{U}$$

for all $\sigma, \tau \in \mathrm{V}^{\mathbb{B}^*}$.

(2) Show that the relation $\in_{\mathcal{U}}$ is well-defined.

(3) Show that

$$(\mathrm{V}/\,\mathcal{U}, \in_{\mathcal{U}}) \models \varphi([\tau_0]_{\mathcal{U}}, \ldots, [\tau_{n-1}]_{\mathcal{U}}) \iff [\![\varphi(\tau_0, \ldots, \tau_{n-1})]\!]_{\mathbb{B}} \in \mathcal{U}$$

holds for every $\mathcal{L}_{\in}$-formula $\varphi(v_0, \ldots, v_{n-1})$ and all $\tau_0, \ldots, \tau_{n-1} \in \mathrm{V}^{\mathbb{B}^*}$.

(4) Show that $(\mathrm{V}/\,\mathcal{U}, \in_{\mathcal{U}})$ is a model of ZFC.

Problem 26 (8 Points)**.** Let M be a countable transitive model of ZFC, κ be an uncountable regular cardinal in M and $\mathbb{P} \in M$ be a separative partial order (see Problem 21). Show that, if $\mathbb{P}$ is not $<\kappa$-distributive in M, then there is G $\mathbb{P}$-generic over M and $f : \lambda \longrightarrow \mathrm{Ord}$ with $\lambda < \kappa$ and $f \in M[G] \setminus M$. (Hint: *Work in M and fix a sequence $\langle D_\alpha \mid \alpha < \lambda \rangle$ of dense open subsets of $\mathbb{P}$ such that $\lambda < \kappa$ and $\bigcap_{\alpha<\lambda} D_\alpha$ is not dense in $\mathbb{P}$. Given $\alpha < \lambda$, let $\langle a^\alpha_\beta \mid \beta < \nu_\alpha \rangle$ enumerate a maximal antichain in D_α. Define*

$$\sigma \;=\; \{(\mathsf{op}(\check{\alpha}, \check{\beta}), a^\alpha_\beta) \mid \alpha < \lambda,\ \beta < \nu_\alpha\} \in M^{\mathbb{P}}$$

and find G $\mathbb{P}$-generic over M such that $\sigma^G : \lambda \longrightarrow \mathrm{Ord}$ with $\sigma^G \notin M$).

Please hand in your solutions on Wednesday, June 10, before the lecture.