Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke Problem sheet 4

Problem 15 (4 Points)**.** Prove Lemma 3.2.3. from the lecture course: *Assume* ZF^- *and let* $\varphi(v_0, \dots, v_{n-1})$ *be an* $\mathcal{L}_\in$*-formula. The following statements are equivalent for every partial order* $\mathbb{P}$*,* $p \in \mathbb{P}$ *and* $\tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$.

(1) $p \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \dots, \tau_{n-1})$.
(2) $q \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \dots, \tau_{n-1})$ for every $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p$.
(3) The set $\{q \in \mathbb{P} \mid q \Vdash^*_{\mathbb{P}} \varphi(\tau_0, \dots, \tau_{n-1})\}$ is dense below p in $\mathbb{P}$.

Problem 16 (9 Points)**.** Prove Lemma 4.1.5. from the lecture course: *Assume* ZF^- *and let* $\varphi(v_0, \dots, v_n)$ *be an* $\mathcal{L}_\in$*-formula,* $\mathbb{P}$ *be a partial order,* $p \in \mathbb{P}$ *and* $\tau, \tau_0, \dots, \tau_{n-1} \in \mathrm{V}^{\mathbb{P}}$*. Prove the following statements.*

(1) $p \Vdash^*_{\mathbb{P}} \forall x \ \varphi(x, \tau_0, \dots, \tau_{n-1})$ if and only if $p \Vdash^*_{\mathbb{P}} \varphi(\sigma, \tau_0, \dots, \tau_{n-1})$ for all $\sigma \in \mathrm{V}^{\mathbb{P}}$.
(2) $p \Vdash^*_{\mathbb{P}} \exists x \in \tau \ \varphi(x, \tau_0, \dots, \tau_{n-1})$ if and only if the set
$$\{q \in \mathbb{P} \mid \exists (\rho, r) \in \tau \ [q \leq_{\mathbb{P}} r \wedge q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \dots, \tau_{n-1})]\}$$
is dense below p in $\mathbb{P}$.
(3) $p \Vdash^*_{\mathbb{P}} \forall x \in \tau \ \varphi(x, \tau_0, \dots, \tau_{n-1})$ if and only if $q \Vdash^*_{\mathbb{P}} \varphi(\rho, \tau_0, \dots, \tau_{n-1})$ holds for all $(\rho, r) \in \tau$ and $q \in \mathbb{P}$ with $q \leq_{\mathbb{P}} p, r$.

Problem 17 (6 Points)**.** Let $\mathrm{Fn}(\omega, \omega, \omega)$ denote the partial order consisting of all finite partial functions $p : \omega \xrightarrow{par} \omega$ ordered by reversed inclusion.

(1) Construct a dense subset D of the Cohen forcing $\mathbb{C}$ and a dense embedding of the partial order $(D, \supseteq)$ into $\mathrm{Fn}(\omega, \omega, \omega)$.
(2) Let $\mathbb{P}$ be a countable atomless partial order with a maximal element $\mathbb{1}_{\mathbb{P}}$. Prove that there is a dense embedding of $\mathrm{Fn}(\omega, \omega, \omega)$ into $\mathbb{P}$ (Hint: *Use a previous exercise to show that there is an infinite antichain below every condition in* $\mathbb{P}$*. Fix an enumeration* $\langle p_n \mid n < \omega \rangle$ *of* $\mathbb{P}$ *and define a function* π *by recursion. Set* $\pi(\emptyset) = \mathbb{1}_{\mathbb{P}}$*. If* $\pi(s)$ *is defined for some* $s : n \to \omega$*, then extend* π *in a way such that* $\{\pi(s^\frown \langle m \rangle) \mid m < \omega\}$ *is a maximal antichain below* $\pi(s)$ *in* $\mathbb{P}$*. Moreover, if the conditions* p_n *and* $\pi(s)$ *are compatible in* $\mathbb{P}$*, then ensure that there is an* $m < \omega$ *with* $\pi(s^\frown \langle m \rangle) \leq_{\mathbb{P}} p_n$*. Show that the resulting function is a dense embedding.*).

Problem 18 (5 Points)**.** Let M be a transitive model of ZFC and $\mathbb{P}, \mathbb{Q} \in M$ be partial orders.

(1) Assume that there is a dense embedding $\pi : \mathbb{Q} \longrightarrow \mathbb{P}$ contained in M. Show that every $\mathbb{P}$-generic extension of M is also a $\mathbb{Q}$-generic extension of M and vice versa (*Hint: Use Problem 13 and the minimality of generic extensions*).

(2) Assume that $\mathbb{P}$ is atomless and countable in M. Show that every $\mathbb{P}$-generic extension of M is also a $\mathbb{C}$-generic extension of M and vice versa.

Please hand in your solutions on Wednesday, May 13, before the lecture.