Models of Set Theory I. - Summer 2015

Dr. Philipp Lücke — Problem sheet 2

Problem 6 (5 Points)**.** Prove Proposition 2.2.1. from the lecture course: *There is a* Δ^{ZF^-}*-formula* $\varphi(v_0, v_1)$ *such that the axioms of* $\mathsf{ZF}^- - (\textit{Infinity})$ *prove that the relation* $\prec_\omega = \{(a,b) \mid \varphi(a,b)\}$ *is a well-ordering of* V_ω *of order-type* ω.

Problem 7 (4 Points)**.** Prove Proposition 2.2.2. from the lecture course: *Given* $a_0, a_1 \in {}^{<\omega}\mathrm{Ord}$*, we define*

$$\begin{aligned} a_0 \prec^* a_1 \iff & \max(\mathrm{ran}(a_0)) < \max(\mathrm{ran}(a_1)) \\ & \vee \big(\max(\mathrm{ran}(a_0)) = \max(\mathrm{ran}(a_1)) \ \wedge \ \mathrm{dom}(a_0) < \mathrm{dom}(a_1)\big) \\ & \vee \big(\max(\mathrm{ran}(a_0)) = \max(\mathrm{ran}(a_1)) \ \wedge \ \mathrm{dom}(a_0) = \mathrm{dom}(a_1) \\ & \quad \wedge \ \exists n \in \mathrm{dom}(a_0) \ [a_0 \restriction n = a_1 \restriction n \ \wedge \ a_0(n) < a_1(n)]\big). \end{aligned}$$

Then the relation $\prec^*$ *well-orders the class* ${}^{<\omega}\mathrm{Ord}$.

Problem 8 (*Richard's paradox: The undefinability of definability*, 6 Points)**.** Show that there is no $\mathcal{L}_\in$-formula $\varphi(v_0, v_1)$ with

$$\mathrm{ZFC} \ \vdash \ \forall k \in \mathrm{Fml} \ \forall x, y \ \ [(\varphi(k,x) \wedge \varphi(k,y)) \ \longrightarrow \ x = y]$$

and

$$\mathrm{ZFC} \ \vdash \ \exists x \ \forall y \ \ [\psi(y) \ \leftrightarrow \ x = y] \ \longrightarrow \ \forall y \ \ [\varphi(\ulcorner \psi \urcorner, y) \ \leftrightarrow \ \psi(y)]$$

for every $\mathcal{L}_\in$-formula $\psi(v)$ (Hint: *Show that there is an ordinal* α *with* $\neg\varphi(k,\alpha)$ *for all* $k \in \mathrm{Fml}$ *and consider the least ordinal with this property*).

Problem 9 (5 Points)**.** Let $\varphi(v_0, v_1)$ be an $\mathcal{L}_\in$-formula. Assume that ZF holds and the class $\{(a,b) \mid \varphi(a,b)\}$ is a well-ordering of V.

(1) Construct an $\mathcal{L}_\in$-formula $\psi(v_0, v_1)$ such that the class $\{(a,b) \mid \psi(a,b)\}$ is a well-ordering of V of order-type Ord.

(2) Show that the statement "V = HOD" holds.

Please hand in your solutions on Wednesday, April 29, before the lecture.