Mathematical Logic. An Introduction

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by Peter Koepke

1 Introduction

Mathematics models real world phenomena like space, time, number, probability, games, etc. It proceeds from initial assumptions to conclusions by rigorous arguments. Its results are "universal" and "logically valid", in that they do not depend on external or implicit conditions which may change with time, nature or society.

It is remarkable that mathematics is also able to model itself: mathematical logic defines rigorously what mathematical statements and rigorous arguments are. The mathematical enquiry into the mathematical method leads to deep insights into mathematics, applications to classical field of mathematics, and to new mathematical theories. The study of mathematical language has also influenced the theory of formal and natural languages in computer science, linguistics and philosophy.

1.1 A simple proof

We want to indicate that rigorous mathematical proofs can be generated by applying simple text manipulations to mathematical statements. Let us consider a fragment of the elementary theory of functions which expresses that the composition of two surjective maps is surjective as well:

> Let f and g be *surjective*, i.e., for all y there is x such that $y = f(x)$, and for all y there is x such that $y = g(x)$.
> *Theorem.* $g \circ f$ is surjective, i.e., for all y there is x such that $y = g(f(x))$.
> *Proof*. Consider any y. Choose z such that $y = g(z)$. Choose x such that $z = f(x)$. Then $y = g(f(x))$. Thus there is x such that $y = g(f(x))$. Thus for all y there is x such that $y = g(f(x))$.
> *Qed*.

These statements and arguments are expressed in an austere and systematic language, which can be normalized further. Logical symbols like $\forall$ and $\exists$ abbreviate figures of language like "for all" or "there exists":

> Let $\forall y \exists x\, y = f(x)$.
> Let $\forall y \exists x\, y = g(x)$.
> Theorem. $\forall y \exists x\, y = g(f(x))$.
> Proof. Consider y.
> $\exists x\, y = g(x)$.
> Let $y = g(z)$.
> $\exists x\, z = f(x)$.
> Let $z = f(x)$.
> $y = g(f(x))$.
> Thus $\exists x\, y = g(f(x))$.
> Thus $\exists x\, y = g(f(x))$.
> Thus $\forall y \exists x\, y = g(f(x))$.
> Qed.

These lines can be considered as formal sequences of symbols. Certain sequences of symbols are acceptable as mathematical formulas. There are rules for the formation of formulas which are acceptable in a proof. These rules have a purely formal character and they can be applied irrespectively of the "meaning" of the symbols and formulas.

1.2 Formal proofs

In the example, $\exists x\, y = g(f(x))$ is inferred from $y = g(f(x))$. The rule of *existential quantification*: "put $\exists x$ in front of a formula" can usually be applied. It has the character of a left-multiplication by $\exists x$.

$$\exists x\,, \varphi \mapsto \exists x\, \varphi.$$

Logical rules satisfy certain algebraic laws like associativity. Another interesting operation is *substitution*: From $y = g(z)$ and $z = f(x)$ infer $y = g(f(x))$ by a "find-and-replace"-substitution of z by $f(x)$.

Given a sufficient collection of rules, the above sequence of formulas, involving "keywords" like "let" and "thus" is a *deduction* or *derivation* in which every line is generated from earlier ones by syntactical rules. Mathematical results may be provable simply by the application of formal rules. In analogy with the formal rules of the infinitesimal calculus one calls a system of rules a *calculus*.

1.3 Syntax and semantics

Obviously we do not just want to describe a formal derivation as a kind of domino but we want to *interpret* the occuring symbols as mathematical objects. Thus we let variables $x, y, \ldots$ range over some domain like the real numbers $\mathbb{R}$ and let f and g stand for functions F, G: $\mathbb{R} \to \mathbb{R}$. Observe that the symbol or "name" f is not identical to the function F, and indeed f might also be interpretated as another function F'. To emphasize the distinction between names and objects, we classify symbols, formulas and derivations as *syntax* whereas the interpretations of symbols belong to the realm of *semantics*.

By interpreting x, y, ... and f, g, ... in a structure like $(\mathbb{R}, F, G)$ we can define straightforwardly whether a formula like $\exists x\, g(f(x))$ is *satisfied* in the structure. A formula is *logically valid* if it is satisfied under *all* interpretations. The fundamental theorem of mathematical logic and the central result of this course is GÖDEL's completeness theorem:

Theorem. *There is a calculus with finitely many rules such that a formula is derivable in the calculus iff it is logically valid.*

1.4 Set theory

In modern mathematics notions can usually be reduced to set theory: non-negative integers correspond to cardinalities of finite sets, integers can be obtained via pairs of non-negative integers, rational numbers via pairs of integers, and real numbers via subsets of the rationals, etc. Geometric notions can be defined from real numbers using analytic geometry: a point is a pair of real numbers, a line is a set of points, etc. It is remarkable that the basic set theoretical axioms can be formulated in the logical language indicated above. So mathematics may be understood abstractly as

Mathematics = (first-order) logic + set theory.

Note that we only propose this as a reasonable abstract viewpoint corresponding to the logical analysis of mathematics. This perspective leaves out many important aspects like the applicability, intuitiveness and beauty of mathematics.

1.5 Circularity

We shall use *sets* as symbols which can then be used to formulate the axioms of *set* theory. We shall *prove* theorems about *proofs*. This kind of circularity seems to be unavoidable in comprehensive foundational science: linguistics has to *talk* about *language*, *brain research* has to be carried out by brains. Circularity can lead to paradoxes like the liar's paradox: "I am a liar", or "this sentence is false". Circularity poses many problems and seems to undermine the value of foundational theories. We suggest that the reader takes a *naive* standpoint in these matters: there are sets and proofs which are just as obvious as natural numbers. Then theories are formed which abstractly describe the naive objects.

A closer analysis of circularity in logic leads to the famous *incompleteness theorems* of GÖDEL's:

Theorem. *Formal theories which are strong enough to "formalize themselves" are not complete, i.e., there are statements such that neither it nor its negation can be proved in that theory. Moreover such theories cannot prove their own consistency.*

It is no surprise that these results, besides their initial mathematical meaning had a tremendous impact on the theory of knowledge outside mathematics, e.g., in philosophy, psychology, linguistics.

2 Set theoretic preliminaries

To model the mathematical method, we have to formalize mathematical language and general structures by mathematical objects. The most basic mathematical objects seem to be *sets*. We briefly present some facts from set theory which are used in the sequel.

In line with our introductory remarks on circularity we initially treat set theory *naively*, i.e., we view sets and set theoretic operations as concrete mental constructs. We shall later introduce a powerful axiom system for sets. From an axiomatic standpoint most of our arguments can be carried out under weak set theoretical hypotheses. In particular it will not be necessary to use sets of high cardinality.

The theory of *finite* sets is based on the *empty set* $\emptyset = \{\}$ and operations like

$$x \mapsto \{x\};\ x, y \mapsto \{x, y\};\ x, y \mapsto x \cup y;\ x, y \mapsto x \cap y;\ x, y \mapsto x \setminus y.$$

The operation $x, y \mapsto \{\{x\}, \{x, y\}\}$ defines the *ordered pair* of x and y. Its crucial property is that

$$\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\} \text{ if and only if } x = x' \text{ and } y = y'.$$

The ordered pair $\{\{x\}, \{x, y\}\}$ is denoted by (x, y). Ordered pairs allow to formalize (binary) relations and functions:

- a *relation* is a set R of ordered pairs;
- a *function* is a relation f such that for all x, y, y' holds: if $(x, y) \in f$ and $(x, y') \in f$ then $y = y'$. Then $f(x)$ denotes the unique y such that $(x, y) \in f$.

We assume standard notions and notations from relation theory, see also Definition 2 below. For binary relations R we can use the *infix* notation aRb instead of $(a, b) \in R$.

If a function maps the elements of a set a into a set b we write

$$f\colon a \to b.$$

In case we do not want to specify the target set b, we can also write $f\colon a \to V$ where V is understood to be the *universe* of all sets. We assume the usual notions of function theory like *injective*, *surjective*, *bijective*, etc.

It is natural to formalize the integer n by some set with n elements. We shall later see that the following formalization can be carried out uniformly in set theory:

$$\begin{aligned}
0 &= \emptyset \\
1 &= \{0\} \\
2 &= \{0,1\} \\
&\vdots \\
n+1 &= \{0,1,\dots,n\} = \{0,1,\dots,n-1\} \cup \{n\} = n \cup \{n\} \\
&\vdots \\
\mathbb{N} = \omega &= \{0,1,\dots\}
\end{aligned}$$

These integers satisfy the usual laws of complete induction and recursion.

A *finite sequence* is a function $w\colon n \to V$ for some integer $n \in \mathbb{N}$ which is the *length* of w. We write w_i instead of $w(i)$, and the sequence w may also be denoted by $w_0 \dots w_{n-1}$. Note that the empty set $\emptyset$ is the unique finite sequence of length 0.

For finite sequences $w = w_0 \dots w_{m-1}$ and $w' = w'_0 \dots w'_{n-1}$ let $w \hat{\ } w' = w_0 \dots w_{m-1} w'_0 \dots w'_{n-1}$ be the *concatenation* of w and w'. $w \hat{\ } w'\colon m+n \to V$ can be defined by

$$w \hat{\ } w'(i) = \begin{cases} w(i), \text{ if } i < m; \\ w'(i-m), \text{ if } i \geqslant m. \end{cases}$$

We also write ww' for $w \hat{\ } w'$. This operation is a *monoid* satisfying some cancellation rules:

Proposition 1. *Let w, w', w'' be finite sequences. Then*

a) $(w \hat{\ } w') \hat{\ } w'' = w \hat{\ } (w' \hat{\ } w'')$.

b) $\emptyset \hat{\ } w = w \hat{\ } \emptyset = w$.

c) $w \hat{\ } w' = w \hat{\ } w'' \to w' = w''$.

d) $w' \hat{\ } w = w'' \hat{\ } w \to w' = w''$.

Proof. We only check the associative law a). Let $n, n', n'' \in \mathbb{N}$ such that $w = w_0 \dots w_{n-1}$, $w' = w'_0 \dots w'_{n'-1}$, $w'' = w''_0 \dots w''_{n''-1}$. Then

$$\begin{aligned}
(w \hat{\ } w') \hat{\ } w'' &= (w_0 \dots w_{n-1} w'_0 \dots w'_{n'-1}) \hat{\ } w''_0 \dots w''_{n''-1} \\
&= w_0 \dots w_{n-1} w'_0 \dots w'_{n'-1} w''_0 \dots w''_{n''-1} \\
&= w_0 \dots w_{n-1} \hat{\ } (w'_0 \dots w'_{n'-1} w''_0 \dots w''_{n''-1}) \\
&= w_0 \dots w_{n-1} \hat{\ } (w'_0 \dots w'_{n'-1} \hat{\ } w''_0 \dots w''_{n''-1}) \\
&= w \hat{\ } (w' \hat{\ } w'').
\end{aligned}$$

The trouble with this argument is the intuitive but vague use of the *ellipses* "...". In mathematical logic we have to ultimately eliminate such vaguenesses. So we show that for all $i < n + n' + n''$

$$((w \hat{\ } w') \hat{\ } w'')(i) = (w \hat{\ } (w' \hat{\ } w''))(i).$$

Case 1: $i < n$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= (w\hat{\ }w')(i)\\ &= w(i)\\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

Case 2: $n \leqslant i < n+n'$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= (w\hat{\ }w')(i)\\ &= w'(i-n)\\ &= (w'\hat{\ }w'')(i-n)\\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

Case 3: $n+n' \leqslant i < n+n'+n''$. Then

$$\begin{aligned}((w\hat{\ }w')\hat{\ }w'')(i) &= w''(i-(n+n'))\\ &= w'\hat{\ }w''(i-(n+n')+n') = w'\hat{\ }w''(i-n)\\ &= (w\hat{\ }(w'\hat{\ }w''))(i-n+n)\\ &= (w\hat{\ }(w'\hat{\ }w''))(i).\end{aligned}$$

□

A set x is *finite*, if there is an integer $n \in \mathbb{N}$ and a surjective function $f\colon n \to x$. The smallest such n is called the *cardinality* of the finite set x and denoted by $n = \operatorname{card}(x)$. The usual cardinality properties for finite sets follow from properties of finite sequences.

A set x is *denumerable* or *countable* if there is a surjective function $f\colon \mathbb{N} \to x$. If the set is not finite, it is *countably infinite*. Its cardinality is ω, written as $\omega = \operatorname{card}(x)$. Under sufficient set theoretical assumptions, the union

$$\bigcup_{n\in\omega} x_n$$

where each x_n is countable is again countable.

If a set x is not countable, it is *uncountable*. Within set theory one can develop an efficient notion of cardinality for uncountable sets.

The theory of infinite sets usually requires the *axiom of choice* which is equivalent to ZORN's lemma.

Definition 2. *Let A be a set and $\leqslant$ be a binary relation. Define*

a) $(A,\leqslant)$ *is* transitive *if for all $a,b,c \in A$*

$$a \leqslant b \text{ and } b \leqslant c \text{ implies } a \leqslant c.$$

b) $(A,\leqslant)$ *is* reflexive *if for all $a \in A$ holds $a \leqslant a$.*

c) $(A,\leqslant)$ *is a* partial order *if $(A,\leqslant)$ is transitive and reflexive and $A \neq \emptyset$.*

So let $(A,\leqslant)$ is be a partial order.

a) $z \in A$ *is a* maximal element *of A if there is no $a \in A$ with $z \leqslant a$ and $z \neq a$.*

b) *If $X \subseteq A$ then u is an* upper bound *for X if for all $x \in X$ holds $x \leqslant u$.*

c) $I \subseteq A$ *is* linear *if for all $a,b \in I$*

$$a \leqslant b \text{ or } b \leqslant a.$$

d) $(A,\leqslant)$ *is* inductive *if every linear subset of A has an upper bound.*

ZORN's *lemma* states

Theorem 3. *Every inductive partial order has a maximal element.*

3 Symbols and words

Intuitively and also in our theory a word is a finite sequence of symbols. A symbol has some basic information about its role within words. E.g., the symbol $\leqslant$ is usually used to stand for a binary relation. So we let symbols include such type information. We provide us with a sufficient collection of symbols.

Definition 4. *The* basic symbols *of first-order logic are*

a) $\equiv$ *for* equality*,*

b) $\neg$, $\rightarrow$, $\perp$ *for the logical operations of* negation*,* implication *and the truth value* false*,*

c) $\forall$ *for* universal quantification*,*

d) (*and*) *for auxiliary* bracketing*.*

e) *variables* v_n *for* $n \in \mathbb{N}$*.*

Let $\mathrm{Var} = \{v_n | n \in \mathbb{N}\}$ *be the set of variables and let* S_0 *be the set of basic symbols.*

An n*-ary* relation symbol*, for* $n \in \mathbb{N}$*, is (a set) of the form* $R = (x, 0, n)$*; here* 0 *indicates that the values of a relation will be* truth values*.* 0*-ary relation symbols are also called* propositional constant symbols*. An* n*-ary* function symbol*, for* $n \in \mathbb{N}$*, is (a set) of the form* $f = (x, 1, n)$ *where* 1 *indicates that the values of a function will be elements of a structure.* 0*-ary function symbols are also called* constant symbols*.*

A symbol set *or a* language *is a set of relation symbols and function symbols.*

We assume that the basic symbols are pairwise distinct and are distinct from any relation or function symbol. For concreteness one could for example set $\equiv = 0$*,* $\neg = 1$*,* $\rightarrow = 2$*,* $\perp = 3$*,* $(= 4$*,* $) = 5$*, and* $v_n = (1, n)$ *for* $n \in \mathbb{N}$*.*

An n-ary relation symbol is intended to denote an n-ary relation; an n-ary function symbol is intended to denote an n-ary function. A symbol set is sometimes called a *type* because it describes the type of structures which will later interpret the symbols. We shall denote variables by letters like $x, y, z, \ldots$, relation symbols by $P, Q, R, \ldots$, functions symbols by $f, g, h, \ldots$ and constant symbols by $c, c_0, c_1, \ldots$ We shall also use other typographical symbols in line with standard mathematical practice. A symbol like $<$, e.g., usually denotes a binary relation, and we could assume for definiteness that there is some fixed set theoretic formalization of $<$ like $<=(999, 0, 2)$. Instead of the arbitrary 999 one could also take the number of $<$ in some typographical font.

Example 5. The *language of group theory* is the language

$$S_{\mathrm{Gr}} = \{\circ, e\},$$

where $\circ$ is a binary (= 2-ary) function symbol and e is a constant symbol. Again one could be definite about the coding of symbols and set $S_{\mathrm{Gr}} = \{(80, 1, 2), (87, 1, 0)\}$, e.g., but we shall not care much about such details. As usual in algebra, one also uses an *extended language of group theory*

$$S_{\mathrm{Gr}} = \{\circ, ^{-1}, e\}$$

to describe groups, where $^{-1}$ is a unary (= 1-ary) function symbol.

Definition 6. *Let S be a language. A* word *over S is a finite sequence*

$$w\colon n \to S_0 \cup S\,.$$

Let S^* be the set of all words over S. The empty set $\emptyset$ is also called the empty word.

Let S be a symbol set. We want to formalize how a word like $\exists x\, y = g(f(x))$ can be produced from a word like $y = g(f(x))$.

Definition 7. *A relation $R \subseteq (S^*)^n \times S^*$ is called a* rule *(over S). A* calculus *(over S) is a set $\mathcal{C}$ of rules (over S).*

We work with rules which *produce* words out of given words. A rule

$$\{(\text{arguments, production})|...\}$$

is usually written as a *production rule* of the form

$$\frac{\text{arguments}}{\text{production}} \quad \text{or} \quad \frac{\text{preconditions}}{\text{conclusion}}\,.$$

For the existential quantification mentioned in the introduction we may for example write

$$\frac{\varphi}{\exists x\, \varphi}$$

where the production is the concatenation of $\exists x$ and φ.

Definition 8. *Let $\mathcal{C}$ be a calculus over S. Let $R \subseteq (S^*)^n \times S^*$ be a rule of $\mathcal{C}$. For $X \subseteq S^*$ set*

$$R[X] = \{w \in S^* \mid \textit{there are words } u_0, ..., u_{n-1} \in X \textit{ such that } R(u_0, ..., u_{n-1}, w) \textit{ holds}\}.$$

Then the product *of $\mathcal{C}$ is the smallest subset of S^* closed under the rules of $\mathcal{C}$:*

$$\mathrm{Prod}(\mathcal{C}) = \bigcap \{X \subseteq S^* \mid \textit{for all rules } R \in \mathcal{C} \textit{ holds } R[X] \subseteq X\}.$$

The product of a calculus can also be described "from below" by:

Definition 9. *Let $\mathcal{C}$ be a calculus over S. A sequence $w^{(0)}, ..., w^{(k-1)} \in S^*$ is called a* derivation *in $\mathcal{C}$ if for every $l < k$ there exists a rule $R \in \mathcal{C}$, $R \subseteq (S^*)^n \times S^*$ and $l_0, ..., l_{n-1} < l$ such that*

$$R(w^{(l_0)}, ..., w^{(l_{n-1})}, w^{(l)}).$$

This means that every word of the derivation can be derived from earlier words of the derivation by application of one of the rules of the calculus. We shall later define a calculus such that the sequence of sentences

Let $\forall y \exists x\, y = f(x)$.
Let $\forall y \exists x\, y = g(x)$.
Consider y.
$\exists x\, y = g(x)$.
Let $y = g(z)$.
$\exists x\, z = f(x)$.
Let $z = f(x)$.
$y = g(f(x))$.
Thus $\exists x\, y = g(f(x))$.

Thus $\exists x\, y = g(f(x))$.
Thus $\forall y \exists x\, y = g(f(x))$.
Qed.

is basically a derivation in that calculus.

Everything in the product of a calculus can be obtained by a derivation.

Proposition 10. *Let $\mathcal{C}$ be a calculus over S. Then*

$$\mathrm{Prod}(\mathcal{C}) = \{w \,|\, \textit{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \textit{ in } \mathcal{C}\}.$$

Proof. The equality of sets can be proved by two inclusions.
($\subseteq$) The set

$$X = \{w \,|\, \text{there is a derivation } w^{(0)}, \ldots, w^{(k-1)} = w \text{ in } \mathcal{C}\}$$

satisfies the closure property $R[X] \subseteq X$ for all rules $R \in \mathcal{C}$. Since $\mathrm{Prod}(\mathcal{C})$ is the intersection of all such sets, $\mathrm{Prod}(\mathcal{C}) \subseteq X$.
($\supseteq$) Consider $w \in X$. Consider a derivation $w^{(0)}, \ldots, w^{(k-1)} = w$ in $\mathcal{C}$. We show by induction on $l < k$ that $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Let $l < k$ and assume that for all $i < l$ holds $w^{(i)} \in \mathrm{Prod}(\mathcal{C})$. Take a rule $R \in \mathcal{C}$, $R \subseteq (\mathbb{A}^*)^n \times \mathbb{A}^*$ and $l_0, \ldots, l_{n-1} < l$ such that $R(w^{(l_0)}, \ldots, w^{(l_{n-1})}, w^{(l)})$. Since $\mathrm{Prod}(\mathcal{C})$ is closed under application of R we get $w^{(l)} \in \mathrm{Prod}(\mathcal{C})$. Thus $w = w^{(k-1)} \in \mathrm{Prod}(\mathcal{C})$. □

Exercise 1. (Natural numbers 1) Consider the symbol set $S = \{\,|\,\}$. The set $S^* = \{\emptyset, |, ||, |||, \ldots\}$ of words may be identified with the set $\mathbb{N}$ of natural numbers. Formulate a calculus $\mathcal{C}$ such that $\mathrm{Prod}(\mathcal{C}) = S^*$.

4 Induction and recursion on calculi

Derivations in a calculus have finite length so that one can carry out inductions and recursions along the lengths of derivations. We formulate appropriate induction and recursion theorems which generalize *complete induction* and *recursion* for natural numbers. Note the recursion is linked to induction but requires stronger hypothesis.

Theorem 11. (Induction Theorem) *Let $\mathcal{C}$ be a calculus over S and let $\varphi(-)$ be a property which is inherited along the rules of $\mathcal{C}$:*

$\forall R \in \mathcal{C}, R \subseteq (S^*)^k \times S^* \forall w^{(1)}, \ldots, w^{(k)}, w \in S^*, R(w^{(1)}, \ldots, w^{(k)}, w)\ (\varphi(w^{(1)}) \wedge \ldots \wedge \varphi(w^{(k)}) \rightarrow \varphi(w))$.

Then

$$\forall w \in \mathrm{Prod}(\mathcal{C})\ \varphi(w).$$

Proof. By assumption, $\{w \in S^* | \varphi(w)\}$ is closed under the rules of $\mathcal{C}$. Since $\mathrm{Prod}(\mathcal{C})$ is the intersection of all sets which are closed under $\mathcal{C}$,

$$\mathrm{Prod}(\mathcal{C}) \subseteq \{w \in S^* | \varphi(w)\}.$$ □

Definition 12. *A calculus $\mathcal{C}$ over S is* uniquely readable *if for every $w \in \mathrm{Prod}(\mathcal{C})$ there are a unique rule $R \in \mathcal{C}, R \subseteq (S^*)^k \times S^*$ and unique $w^{(1)}, \ldots, w^{(k)} \in S^*$ such that*

$$R(w^{(1)}, \ldots, w^{(k)}, w).$$

Theorem 13. (Recursion Theorem) *Let $\mathcal{C}$ be a calculus over S which is uniquely readable and let $(G_R|R\in\mathcal{C})$ be a sequence of* recursion rules*, i.e., for $R\in\mathcal{C}$, $R\subseteq(S^*)^k\times S^*$ let $G_R\colon V^k\to V$ where V is the universe of all sets. Then there is a uniquely determined function $F\colon \mathrm{Prod}(\mathcal{C})\to V$ such that the following* recursion equation *is satisfied for all $R\in\mathcal{C}$, $R\subseteq(S^*)^k\times S^*$ and $w^{(1)},\ldots,w^{(k)},w\in\mathrm{Prod}(\mathcal{C})$, $R(w^{(1)},\ldots,w^{(k)},w)$:*

$$F(w)=G_R(F(w^{(1)}),\ldots,F(w^{(k)})).$$

We say that F is defined by recursion along $\mathcal{C}$ *by the recursion rules $(G_R|R\in\mathcal{C})$.*

Proof. We define $F(w)$ by complete recursion on the length of the shortest derivation of w in $\mathcal{C}$. Assume that $F(u)$ is already uniquely defined for all $u\in\mathrm{Prod}(\mathcal{C})$ with shorter derivation length. Let w have shortest derivation $w^{(0)}$, ..., $w^{(l-1)}$. By the unique readability of $\mathcal{C}$ there are $R\in\mathcal{C}$, $R\subseteq(S^*)^k\times S^*$ and $w^{(i_0)},\ldots,w^{(i_{k-1})}$ with $i_0,\ldots,i_{k-1}<l-1$ such that

$$R(w^{(i_0)},\ldots,w^{(i_{k-1})},w).$$

Then we can uniquely define

$$F(w)=G_R(F(w^{(i_0)}),\ldots,F(w^{(i_{k-1})})).$$ □

Remark 14. The previous Theorem states the existence of a function F as a set of ordered pairs, but the proof argues that F can be defined (by some intuitive "procedure"). To complete the argument one would have to use the recursion theorem from set theory which says that definitions of a certain kind correspond to certain functions in the set theoretic universe.