Set theory - Winter Semester 2014

Test Exam	Dr. Philipp Schlicht
3 hours	Dr. Philipp Lücke

Problem A.

(i) (3 points) State the Replacement Scheme in the language of set theory.

(ii) (3 points) Define the cofinality function that sends a limit ordinal α to $\mathrm{cof}(\alpha)$.

Problem B. (5 points) Show that the class of regular cardinals is a proper class.

Problem C. Let κ be an infinite cardinal.

(i) (5 points) Show that $\mathrm{cof}(\kappa) > \omega$ if and only if for every *increasing* function $f : \kappa \longrightarrow \kappa$ (i.e. $\alpha \leq \beta < \kappa$ implies $f(\alpha) \leq f(\beta)$), there is some $\gamma < \kappa$ with $f[\gamma] \subseteq \gamma$.

(ii) (5 points) Prove that $\mathrm{cof}(\kappa) > \omega$ if and only if the set

$$\{A \subseteq \kappa \mid \exists C \subseteq A \;\; C \textit{ is closed and unbounded in } \kappa\}$$

is a filter.

(iii) (5 points) Assume that $\mathrm{cof}(\kappa) > \omega$. We call a subset A of κ *stationary* if it meets every closed unbounded subset of κ. Show that every stationary subset of κ has cardinality at least $\mathrm{cof}(\kappa)$ and that there is a stationary subset of κ of cardinality $\mathrm{cof}(\kappa)$.

Problem D. (6 points) Use the inequality $\sum_i \mu_i < \prod_i \lambda_i$ for all nonempty sets I and infinite cardinals μ_i and λ_i with $\mu_i < \lambda_i$ to prove that $\mathrm{cof}(2^\kappa) > \kappa$ holds for every infinite cardinal κ.

Problem E. (6 points) Work in the theory ZF and show that Zorn's Lemma implies the Well-ordering Principle (i.e. the statement that every set can be well-ordered).

Problem F. (6 points) Show that an ordinal α is countable if and only if there is an order-preserving injection from $(\alpha, \leq)$ into $(\mathbb{Q}, \leq)$.

Problem G. (8 points) Let $\mathbb{T}$ be an Aronszajn tree and $\preceq$ be a suitable linear ordering of $\mathbb{T}$. Show that there is no order-preserving injection from $(\omega_1, \leq)$ into $(\partial\mathbb{T}, \preceq_{lex})$.

Problem H.

(i) (3 points) Define the notion of a coherent C-sequence and a $\square(\kappa)$-sequence.

(ii) (5 points) Show that there is a coherent C-sequence for every uncountable regular cardinal κ.