Set theory - Winter Semester 2014

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Problem 37 (6 points)**.** Let I denote the set of all strictly increasing functions $f : \alpha \longrightarrow \mathbb{Q}$ with $\alpha < \omega_1$. Define $\mathbb{T}$ to be the partial order $(I, \subseteq)$.

(1) Prove that the partial order $\mathbb{T}$ is a tree.

(2) Determine the following objects.

   (a) The height $\mathrm{ht}(\mathbb{T})$ of $\mathbb{T}$.

   (b) The set $\partial\mathbb{T}$ of all maximal branches through $\mathbb{T}$.

   (c) The set $[\mathbb{T}]$ of all cofinal branches through $\mathbb{T}$.

Problem 38 (6 points)**.** Let κ be an infinite cardinal. Construct a tree $\mathbb{T}$ with $\mathrm{ht}(\mathbb{T}) = \kappa$, $[\mathbb{T}] = \emptyset$ and $|\mathbb{T}(\alpha)| \leq \mathrm{cof}(\kappa)$ for all $\alpha < \kappa$.

Problem 39 (2 points)**.** Construct a tree $\mathbb{T}$ with $\mathrm{ht}(\mathbb{T}) > \omega$ that is not extensional at limit levels.

Problem 40 (6 points)**.** Let κ be an infinite cardinal and $\mathbb{T}$ be a tree of height κ. Show that the following statements are equivalent.

(1) $[\mathbb{T}] \neq \emptyset$.

(2) There is a function $e : \sigma\mathbb{T} \setminus [\mathbb{T}] \longrightarrow \mathbb{T}$ with

$$b \subsetneq c \longrightarrow e(b) <_{\mathbb{T}} e(c)$$

for all $b, c \in \sigma\mathbb{T} \setminus [\mathbb{T}]$.

(Hint: To show (2) $\Rightarrow$ (1)*, first show that we may assume that* $[\mathbb{T}] = \emptyset$ *and* $\mathrm{lh}_{\mathbb{T}}(e(b)) = \mathrm{lh}_{\mathbb{T}}(b)$ *holds for all* $b \in \sigma\mathbb{T}$ *and then construct* $b \in [\mathbb{T}]$ *with* $b(\alpha) = e(\{b(\bar{\alpha}) \mid \bar{\alpha} < \alpha\})$ *for all* $\alpha < \kappa$*.)*

Due Wednesday, December 17, before the lecture.