Set theory - Winter Semester 2014

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Problem 34 (6 points)**.** Suppose that κ is an uncountable cardinal.

(1) Suppose that κ is regular and $A \subseteq \kappa$ is not stationary in κ. Show that there is a regressive function $f\colon A \to \kappa$ such that $\{\alpha \mid f(\alpha) \leq \gamma\}$ is bounded for all $\gamma < \kappa$.

(2) Suppose that κ is singular. Show that there is a regressive function $f\colon \kappa \to \kappa$ such that $\{\alpha \mid f(\alpha) \leq \gamma\}$ is bounded for all $\gamma < \kappa$.

Problem 35 (8 points)**.** Suppose that κ is a singular cardinal of uncountable cofinality and $(\kappa_\alpha)_{\alpha<\mathrm{cof}(\kappa)}$ is a strictly increasing continuous cofinal sequence in κ. Prove the following statements.

(1) If $\mathcal{F} \subseteq \prod_{\alpha<\mathrm{cof}(\kappa)} A_\alpha$ is almost disjoint and $|A_\alpha| \leq \kappa_\alpha^{++}$ for all $\alpha < \mathrm{cof}(\kappa)$, then $|\mathcal{F}| \leq \kappa^{++}$.

(2) If $2^\mu \leq \mu^{++}$ for all infinite cardinals $\mu < \kappa$, then $2^\kappa \leq \kappa^{++}$.

(Hint: adapt the proof of Silver's Theorem from the lecture.)

Problem 36 (6 points)**.** Fix $\alpha, \beta \in \mathrm{Ord}$ and let ${}^{<\alpha}\beta$ denote the set of all functions $f : \bar{\alpha} \longrightarrow \beta$ with $\bar{\alpha} < \alpha$. Define $\mathbb{T}$ to be the partial order $({}^{<\alpha}\beta,\ \subseteq)$.

(1) Prove that the partial order $\mathbb{T}$ is a tree.

(2) Determine the following objects.
   (a) The height $\mathrm{ht}(\mathbb{T})$ of $\mathbb{T}$.
   (b) The α-th level $\mathbb{T}(\alpha)$ of $\mathbb{T}$ for every $\alpha \in \mathrm{Ord}$.
   (c) The set $\sigma\mathbb{T}$ of all branches through $\mathbb{T}$.
   (d) The set $\partial\mathbb{T}$ of all maximal branches through $\mathbb{T}$.
   (e) The set $[\mathbb{T}]$ of all cofinal branches through $\mathbb{T}$.

Due Wednesday, December 10, before the lecture.