Set theory - Winter Semester 2014

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Problem 18 (5 points)**.** Prove that $\kappa^{\mathrm{cof}(\kappa)} > \kappa$ for all infinite cardinals κ, without using König's Theorem (Lemma 2.3.4 (3) in the lecture).

(Hint: As in the proof of Cantor's Theorem (Lemma 2.1.3 (3) in the lecture), suppose that you have a list of length κ of functions $f\colon \mathrm{cof}(\kappa) \to \kappa$. Construct a function $g\colon \mathrm{cof}(\kappa) \to \kappa$ which is not on the list.)

Problem 19 (3 points)**.** Suppose that κ is an infinite cardinal. Show that the following sets have the same size.

(1) ${}^{\mathrm{cof}(\kappa)}\kappa$.

(2) The set of cofinal functions $f\colon \mathrm{cof}(\kappa) \to \kappa$.

(3) $\prod_{i<\mathrm{cof}(\kappa)} g(i)$, where $g\colon \mathrm{cof}(\kappa) \to \kappa$ is a strictly increasing cofinal function with $g(0) > 0$.

Problem 20 (4 points)**.** Suppose that $(L, <)$ is a linear order such that $|pred_<(x)| < \kappa$ for all $x \in L$. Show that $|L| \leq \kappa$.

Problem 21 (9 points)**.** Consider the *linear order topology* on an ordinal δ whose basic open sets are the intervals

$$(\alpha, \beta) := \{\gamma < \delta \mid \alpha < \gamma < \beta\}$$

for $\alpha < \beta \leq \delta$ and

$$[0, \alpha) = \{\gamma \mid \gamma < \alpha\}$$

for $\alpha \leq \delta$. A set $x \subseteq \delta$ is open if and only there is a cardinal μ and a sequence $(U_\alpha)_{\alpha<\mu}$ of basic open sets such that $x = \bigcup_{\alpha<\mu} U_\alpha$.

(1) Show that a set $x \subseteq \delta$ is closed (i.e. its complement is open) in the linear order topology if and only if $\sup(y) \in x$ for all nonempty $y \subseteq x$ which are *bounded* in δ ($y \subseteq \delta$ is bounded in δ if $\sup(y) < \delta$).

(2) Show that a strictly monotone function $f\colon \delta \to \delta$ is continuous in the sense of the lecture if and only if f is continuous with respect to the linear order topology.

(3) Show that for any $\alpha < \delta$, the set α is compact with respect to the linear order topology if and only if $\alpha = 0$ or α is a successor ordinal.

Due Wednesday, November 12, before the lecture.