Set theory - Winter Semester 2014

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Problem 14 (6 points)**.** The *Axiom of Dependent Choices* DC is the following statement:

> *Suppose that R is a (binary) relation on a set a with $dom(R) = a$. Then for every $x \in a$, there is a sequence $(x_n)_{n\in\omega}$ such that $x_0 = x$, $x_n \in a$, and $x_n R x_{n+1}$ for all n.*

Prove the following statements in ZF.

(1) The Axiom of Choice AC implies DC.

(2) Suppose that DC holds and $<_0$ is a (binary) relation on a set b. Then $(b, <_0)$ is wellfounded if and only if there is no sequence $(x_n)_{n\in\omega}$ with $x_{n+1} <_0 x_n$ for all n.

Problem 15 (12 points)**.** Suppose that κ is an infinite cardinal. Determine the cardinality of the following sets.

(1) The set ${}^{<\omega}\kappa = \bigcup_{n\in\omega} {}^{n}\kappa$ of tuples in κ.

(2) The set of finite subsets of κ.

(3) The set of rational numbers.

(4) The set of functions $f : \kappa \to \kappa$.

(5) The set of bijections $f : \kappa \to \kappa$.

(6) The set of strictly monotone functions $f : \kappa \to \kappa$.

Problem 16 (6 points)**.** Prove the following statements for all ordinals α and all limit ordinals β.

(1) $\mathrm{cof}(\aleph_\beta) = \mathrm{cof}(\beta)$.

(2) $\mathrm{cof}(\alpha +_{Ord} \beta) = \mathrm{cof}(\beta)$.

(3) $\mathrm{cof}(\alpha \cdot_{Ord} \beta = \mathrm{cof}(\beta)$ if $\alpha \geq 1$.

Problem 17 (6 points)**.** An infinite cardinal κ is *regular* if $\mathrm{cof}(\kappa) = \kappa$ and *singular* if $\mathrm{cof}(\kappa) < \kappa$. Prove the following statements.

(1) If $\kappa > \omega$ is regular, then every continuous strictly monotone function $f : \kappa \to \kappa$ has a fixed point.

(2) The $\aleph$-function has a fixed point.

(3) If κ is a regular fixed point of the $\aleph$-function, then there is a singular fixed point $\mu < \kappa$ of the $\aleph$-function.

Due Wednesday, November 05, before the lecture.