LECTURE NOTES ON FORCING

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Abstract. Lectures on forcing from the summer 2014 in Bonn. We assume knowledge of iterated forcing and proper forcing.

Contents

1. Consistency of the proper forcing axiom

We give proofs of the consistency of the proper forcing axiom *PFA* from a supercompact cardinal and the consistency of the bounded proper forcing axiom *BPFA* from a reflecting cardinal.[1] April 07

1.1. Some Lemmas on forcing and names.

We begin with preliminary results on forcing names and on iterated forcing. These lemmas are used for the *BPFA* iteration, and they are very useful for other applications as well. Let $\mathbb{P}, \mathbb{Q}, \mathbb{R}, \mathbb{S}$ always denote partial orders and $\dot{\mathbb{P}}, \dot{\mathbb{Q}}, \dot{\mathbb{R}}, \dot{\mathbb{S}}$ names for partial orders. Recall that $H_\kappa = \{x \mid |tc(x)| < \kappa\}$, where κ is a cardinal.

Lemma 1.1. *If $\mathbb{P}$ is a forcing that does not collapse κ and $\dot{x} \in H_\kappa$, then $p \Vdash \dot{x} \in H_\kappa$ for any $p \in \mathbb{P}$.*

Proof. By induction on $\mathrm{rk}(\dot{x})$. The lemma holds for $\mathrm{rk}(\dot{x}) = 0$, so suppose that it is true for all names with rank smaller $r = \mathrm{rk}(\dot{x})$. Suppose that $\dot{x} \in H_\kappa$ and write $\dot{x} = \{(\dot{y}_i, p_i) \mid i \in I\}$ for some indexing set I woth $|I| = \kappa$. By the induction hypothesis, $\mathbb{1}_{\mathbb{P}} \Vdash \dot{y}_i \in H_\kappa$. Since $|\dot{x}| < \kappa$ and κ remains a cardinal, $\mathbb{1}_{\mathbb{P}} \Vdash |\dot{x}| < \kappa$. Thus $\mathbb{1}_{\mathbb{P}} \Vdash \dot{x} \in H_\kappa$. □

The reversal of this result is more interesting.

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[1]Most of this section is taken from Julian Schlöder's Master's thesis

Lemma 1.2. *(Goldstern) If κ is regular and $\mathbb{P} \subseteq H_\kappa$ and satisfies the κ-c.c., then for all $p \in \mathbb{P}$: If $p \Vdash \sigma \in H_\kappa$, there is $\dot\sigma \in H_\kappa$ with $p \Vdash \sigma = \dot\sigma$.*

Proof.

Claim. *For every $x \in H_\kappa$, there is some $\lambda < \kappa$ and a sequence $(x_\alpha \mid \alpha \leq \lambda)$, $x_\alpha \in H_\kappa$ such that: for all $\alpha \leq \lambda : x_\alpha \subseteq \{x_\beta \mid \beta < \alpha\}$ and $x = x_\lambda$.*

Proof. We prove this via induction on x, it is clear for $x = \emptyset$. Suppose that this holds for all $y \in x$ and take for each $y \in x$ an appropriate $\lambda^y < \kappa$ and one such sequence $(x^y_\alpha \mid \alpha \leq \lambda^y)$. Let $\lambda = \sup_{y \in x} \lambda^y$. $\lambda < \kappa$, since $|x| < \kappa$ and κ is regular. Let $(x_\alpha)_{\alpha<\lambda}$ be the concatenation of all the $(x^y_\alpha)_{\alpha \leq \lambda^y}$ and finally set $x_\lambda = x$. This works because every $y \in x$ is at some point in the sequence. □

Since $\mathbb{P}$ satisfies the κ-c.c., it does not collapse κ. Now suppose that $p \in \mathbb{P}$ and $p \Vdash \sigma \in H_\kappa$. Then we can find names $\dot\lambda$, $\dot x_\alpha$ for the sequence discussed above. There is an ordinal $\lambda < \kappa$ such that $p \Vdash \dot\lambda \leq \check\lambda$ and since, in $V[G]$, we may set $x_\alpha = \emptyset$ for all $\dot\lambda^G < \alpha < \check\lambda^G$, we can assume that $\dot\lambda = \check\lambda$.

We now inductively define $\dot\sigma_\alpha := \{(\dot\sigma_\beta, q) \mid \beta < \alpha \wedge q \leq p \wedge q \Vdash \sigma_\beta \in \sigma_\alpha\}$ and let $\dot\sigma = \dot\sigma_\lambda$. Then by induction, all $\dot\sigma_\alpha$ are in H_κ.

We now show that for all $\alpha < \lambda$, $p \Vdash \sigma_\alpha = \dot\sigma_\alpha$, in particular $p \Vdash \sigma = \dot\sigma$. To prove this by induction, suppose that for all $\beta < \alpha$, $p \Vdash \sigma_\beta = \dot\sigma_\beta$. Suppose that G is $\mathbb{P}$-generic with $p \in G$. Then

$$\begin{aligned}\dot\sigma^G_\alpha &= \left\{\dot\sigma^G_\beta \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \sigma_\beta \in \sigma_\alpha\right\} \text{ (by definition)}\\ &= \left\{\sigma^G_\beta \mid \beta < \alpha \wedge \exists q \leq p : q \in G \wedge q \Vdash \sigma_\beta \in \sigma_\alpha\right\} \text{ (by induction)}\\ &= \sigma^G_\alpha\end{aligned}$$

In the last equality "$\subseteq$" holds: If there is a $q \leq p$, $q \in G$, $q \Vdash \sigma_\beta \in \sigma_\alpha$, then $\sigma^G_\beta \in \sigma^G_\alpha$. In the last equaliy "$\supseteq$" holds: Suppose $V[G] \models \tau^G \in \sigma^G_\alpha$, then $\tau^G = \sigma^G_\beta$ for some $\beta < \alpha$. Hence there is $q \leq p$, $q \in G$ that forces $\tau = \sigma_\beta$. □

The following result shows that we can compute the forcing relation for a forcing $\mathbb{P} \in H_\kappa$ in H_κ.

Lemma 1.3. *If κ is regular and $\mathbb{P} \in H_\kappa$ then for any formula $\varphi(x_0, ..., x_n)$, any $p \in \mathbb{P}$ and any names $\sigma_0, ..., \sigma_n$ with $p \Vdash \sigma_0, ..., \sigma_n \in H_\kappa$, there are names $\dot\sigma_0, ..., \dot\sigma_n \in H_\kappa$ such that*

$$(p \Vdash H_\kappa \models \varphi(\sigma_0, ..., \sigma_n)) \Leftrightarrow (H_\kappa \models p \Vdash \varphi(\dot\sigma_0, ..., \dot\sigma_n)).$$

Proof. We assume that $n = 0$ and let $\sigma = \sigma_0$, $\dot\sigma = \dot\sigma_0$. We prove the claim by induction on the complexity of formulas. By Lemmas 1.2 and 1.1 we may set $\dot\sigma = \sigma$. The induction step for $\wedge$ is trivial.

We begin with atomic formulas. Let $\varphi(x, y) = x \in y$, since we can write $x = y$ equivalently as $\forall z : z \in x \leftrightarrow z \in y$ and H_κ satisfies Extensionality. Obviously, $p \Vdash$ "$H_\kappa \models \dot x \in \dot y$" iff $p \Vdash \dot x \in \dot y$. So it suffices to show $p \Vdash \dot x \in \dot y \Leftrightarrow H_\kappa \models p \Vdash \dot x \in \dot y$. We do an induction over the rank of $\dot y$: If $\mathrm{rk}(\dot y) = 0$, $\dot y$ is (a name for) the empty set, so both $p \Vdash \dot x \in \dot y$ and $H_\kappa \models p \Vdash \dot x \in \dot y$ are false. Now consider $\mathrm{rk}(\dot y) > 0$. Suppose $p \Vdash \dot x \in \dot y$. Then $D_{\dot x, \dot y} = \{r \mid \exists (\dot z, q) \in \dot y : r \leq q \wedge r \Vdash \dot x = \dot z\}$ is dense below p. We can write $D_{\dot x, \dot y}$ as $\{r \mid \exists (\dot z, q) \in \dot y : r \leq q \wedge \forall \dot a : (r \Vdash \dot a \in \dot x) \leftrightarrow (r \Vdash \dot a \in \dot z)\}$. So we can apply the inductive hypothesis and obtain $D^{H_\kappa}_{\dot x, \dot y} = D_{\dot x, \dot y}$ and hence $H_\kappa \models$ "$D_{\dot x, \dot y}$ is dense below p". Thus $H_\kappa \models p \Vdash \dot x \in \dot y$. The backwards direction follows since the statement is Σ_2.

Suppose that $\varphi = \neg\psi$ and that the lemma holds for ψ. For the backward direction suppose $H_\kappa \models p \Vdash \neg\psi$. If $p \Vdash \neg(H_\kappa \models \psi)$, we are done. Otherwise there is some $q \leq p$ that forces $H_\kappa \models \psi$, which by the induction hypothesis yields $H_\kappa \models q \Vdash \psi$, contradicting the assumption. The forward direction is similar.

Lastly assume $\varphi = \exists x \psi$ and that the lemma holds for ψ. Then:

$$\begin{aligned}
&p \Vdash H_\kappa \models \exists x \psi(x) \\
\Leftrightarrow\ &\exists \dot{x} \in H_\kappa : p \Vdash H_\kappa \models \psi(\dot{x}) && \text{(by Lemmas 1.2 , 1.1, the max. principle)} \\
\Leftrightarrow\ &\exists \dot{x} \in H_\kappa : H_\kappa \models p \Vdash \psi(\dot{x}) && \text{(by induction hypothesis)} \\
\Leftrightarrow\ &H_\kappa \models \exists \dot{x} : p \Vdash \psi(\dot{x}) \\
\Leftrightarrow\ &H_\kappa \models p \Vdash \exists x \psi(x) && \text{(by the maximality principle).}
\end{aligned}$$

□

Lemma 1.4. *Suppose that $\kappa > \omega_1$ is regular. Let $\mathbb{P}_\kappa$ be a countable support iteration of length κ such that all stages satisfy the κ-cc. Then $\mathbb{P}_\kappa$ satisfies the κ-cc.*

Proof. Assume $A = (p_\xi \mid \xi < \kappa)$ is an antichain in $\mathbb{P}_\kappa$. We may assume its indices have uncountable cofinality. Let $F(\xi) = \min\{\alpha \mid \mathrm{supp}(p_\xi) \cap \xi \subseteq \alpha\}$. Since $\mathbb{P}_\kappa$ has countable supports, F is regressive. By Fodor's Lemma, e.g., [**?**, Theorem 8.7], there is a stationary $S \subseteq \kappa$ and $\gamma < \kappa$ with $F[S] = \{\gamma\}$. Construct $\{\alpha_i \mid i \in S\} = S' \subseteq S$, $|S'| = \kappa$ with $\forall \xi < \zeta \in S' : \mathrm{supp}(p_\xi) \subseteq \zeta$ by recursion:

$$\alpha_i = \min(S \setminus (\sup_{j<i}(\mathrm{supp}(p_{\alpha_j}) \cup \alpha_j))).$$

Note that if $\xi < \zeta \in S'$, then $\mathrm{supp}(p_\xi) \subseteq \zeta$ and $\mathrm{supp}(p_\zeta) \cap \zeta \subseteq \gamma$, therefore $\mathrm{supp}(p_\xi) \cap \mathrm{supp}(p_\zeta) \subseteq \gamma$.

Since $\mathbb{P}_\gamma$ satisfies the κ-cc, there are $\xi < \zeta \in S'$ and $r' \in \mathbb{P}_\gamma$ such that $r' \leq p_\xi \restriction \gamma, p_\zeta \restriction \gamma$. Define a condition $q = (q(\alpha) \mid \alpha < \kappa) \in \mathbb{P}_\kappa$ by:

$$q(\alpha) = \begin{cases} r'(\alpha), \alpha < \gamma, \\ p_\xi(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\xi), \\ p_\zeta(\alpha), \alpha \geq \gamma \wedge \alpha \in \mathrm{supp}(p_\zeta), \\ \mathbb{1}, \quad \text{otherwise.} \end{cases}$$

This is well-defined, since above γ the supports of p_ζ and p_ξ are disjoint. But then $q \leq p_\xi$ and $q \leq p_\zeta$, i.e., A is no antichain, contradicting our assumption. □

The lemma is false for $\kappa = \omega_1$, in fact the countable support iteration of the forcing $\{p, q, 1\}$ with $p \perp q$ of length ω is not c.c.c. Moreover, it is an exercise to check that any countable support iteration of nonatomic forcings of length ω is not c.c.c.

Lemma 1.5. *Suppose that $((\mathbb{P}_\alpha)_{\alpha \leq \gamma}, (\dot{\mathbb{Q}}_\alpha)_{\alpha < \gamma})$ is an iteration and $\alpha < \gamma$. There is a $\mathbb{P}_\alpha$-name $\dot{\mathbb{Q}}$ such that $\mathbb{P}_\gamma$ is isomorphic to a dense subset of $\mathbb{P}_\alpha * \dot{\mathbb{Q}}$.*

Proof. Let $p^\alpha = p \restriction [\alpha, \gamma)$ for $p \in \mathbb{P}_\gamma$. Let $\mathbb{Q} = \{p^\alpha \mid p \in \mathbb{P}_\gamma\}$. If G is $\mathbb{P}_\alpha$-generic over V, let $f \leq g$ for $f, g \in \mathbb{Q}$ if there is some $p \in G$ with $p \cup f \leq p \cup g$. Let $\dot{\mathbb{Q}}$ denote a $\mathbb{P}_\alpha$-name for $\mathbb{Q}$.

Let $\pi \colon \mathbb{P}_\gamma \to \mathbb{P}_\alpha * \dot{\mathbb{Q}}$, $\pi(p) = (p \restriction \alpha, p^\alpha)$. If $p \leq q$, then $p \restriction \alpha \leq q \restriction \alpha$ and hence $p \leq (p \restriction \alpha) \cup q^\alpha$. So $p \restriction \alpha \Vdash_{\mathbb{P}_\alpha} p^\alpha \leq q^\alpha$ and hence $\pi(p) \leq \pi(q)$. Suppose that $\pi(p) \leq \pi(q)$. If $p \not\leq q$, then there is some $r \leq p$ which is incompatible with q. Then $r \restriction \alpha \leq q \restriction \alpha$ and $r \restriction \alpha \Vdash_{\mathbb{P}_\alpha} \exists s \in \dot{G}_{\mathbb{P}_\alpha} s \cup r^\alpha \leq s \cup q^\alpha$. So there is some common extension $t \leq (r \restriction \alpha), s$. Then $t \cup r^\alpha \leq t \cup q^\alpha \leq q$. The first inequality holds since $t \leq s$. The last inequality holds since $t \leq r \restriction \alpha \leq q \restriction \alpha$. This contradicts the assumption that r, q are incompatible.

To show that the image of π is dense, suppose that $(p, \dot{f}) \in \mathbb{P}_\alpha * \dot{\mathbb{Q}}$. There is some $q \leq p$ and some $f \in \mathbb{Q}$ with $q \Vdash \dot{f} = \check{f}$. Then $\pi(q \cup f) = (q, f) \leq (p, \dot{f})$. □

Note that a weaker version of the lemma, where isomorphism to a dense subset os replaced by the statement that the Boolean completions are equal, follows from general facts about quotient forcing.

One can show that the $\mathbb{P}$-name $\dot{\mathbb{Q}}$ defined above is in fact a name for an iteration (see Baumgartner's paper on iterated forcing).

1.2. **PFA.**

April 09

Axiom 1.6 (Proper Forcing Axiom (PFA)). If $(\mathbb{P},<)$ is a proper forcing notion and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$, is a collection of predense subsets of $\mathbb{P}$, then there exists a $\mathcal{D}$-generic filter on $\mathbb{P}$.

Definition 1.7. Let $\mathbb{P}$ be a forcing notion. A set $C \subseteq \mathbb{P}$ is called *centered* iff each finite set $A \subseteq C$ is compatible. C is *directed* iff for all $a, b \in C$ there is $c \in C$ with $c \leq a, b$.

Axiom 1.8 (Bounded Fragments of PFA). Let λ be a cardinal.

(i) PFA_λ is the following axiom: Let $(\mathbb{P},<)$ be a proper *preordered set* and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ be collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$. Then there exists a $\mathcal{D}$-generic *centered set* on $\mathbb{P}$.

(ii) PFA^*_λ is the following axiom: Let $(\mathbb{P},<)$ be a proper *Boolean algebra* and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ be collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$. Then there exists a $\mathcal{D}$-generic *filter* on $\mathbb{P}$.

Lemma 1.9. *Suppose that $\mathbb{P}$ is a Boolean algebra. If $C \subseteq \mathbb{P}$ is centered, then there is a filter $F \supseteq C$.*

Proof. Suppose that $C \subseteq \mathbb{P}$ is centered. We show that C extends to a directed set, since directed sets extend to filters by closing upwards. We inductively construct ω extensions of C. Let $C_0 = C$ and let $C_{n+1} = C_n \cup \{p \cdot q \mid p, q \in C\}$.

We show by induction that for each $n \in \omega$, C_n is centered. This holds for $n = 0$ by the assumption. Suppose that this is true for $n-1$ and let $A \subseteq_{<\omega} C_n$. For each $a \in A$ we find some $p_a, q_a \in C_{n-1}$ such that $a = p_a \cdot p_a$. If $a \in C_{n-1}$ then $p_a = q_a = a$. The set $A' = \{p_a, q_a \mid a \in A\} \subseteq C_{n-1}$ is still finite, so there is a lower bound r of A'. In particular, for each $a \in A$, $r \leq p_a, q_a$, so $r \leq p_a \cdot q_a = a$. Thus r is a lower bound for A.

To see that $C^\omega = \bigcup_{n<\omega} C_n \supseteq C$ is directed, let $p, q \in C^\omega$. Then $p, q \in C_n$ for some n, i.e. $p \cdot q \in C_{n+1} \subseteq C^\omega$. □

In fact, for the proof of the lemma, we have only used the existence of largest lower bounds for all $p, q \in \mathbb{P}$.

Definition 1.10. (i) An elementary embedding $j \colon V \to M$ is called λ-*supercompact* if M transitive, $M^\lambda \subseteq M$, and $\lambda < j(\kappa)$ for $\kappa = \mathrm{crit}(j)$.

(ii) A cardinal κ is λ-*supercompact* for some cardinal $\lambda \geq \kappa$ if and only if there is a λ-supercompact embedding j with $\kappa = \mathrm{crit}(j)$.

(iii) A cardinal κ is called *supercompact* if it is λ-supercompact for all $\lambda \geq \kappa$.

Supercompactness is very high in the large cardinal hierarchy. To see that supercompactness is expressible in the language of set theory, we need to express the existence of a λ-supercompact embedding j with $\mathrm{crit}(j) = \kappa$ by the existence of a normal filter on $P_\kappa(\lambda)$ (see Jech's book). We will omit this here, but note that the following results can be read as results about λ-supercompact cardinals for fixed λ, where this problem does not appear.

Lemma 1.11. *A cardinal κ is measurable if and only if it is κ-supercompact (see e.g. Jech Lemma 17.9).*

Proof. Suppose that $j\colon V \to ult(V,U)$ is an ultrapower with a $<\kappa$-complete ultrafilter on κ. If $[f_\alpha] \in ult(V,U)$ for $\alpha < \kappa$ and $[g] = \kappa$, let $h\colon \kappa \to V$, $h(\gamma) = (f_\alpha(\gamma))_{\alpha<g(\gamma)}$. We have $\alpha < h(\gamma)$ for almost all γ. Then $h(\gamma)(\alpha) = f_\alpha(\gamma)$ for almost all γ. Hence $[h] = (f_\alpha)_{\alpha<\kappa}$. □

Lemma 1.12. *Let M be a transitive model with* $\mathrm{Ord} \subseteq M$*, $\mathbb{P} \in M$ a λ^+-cc forcing notion, G some $\mathbb{P}$-generic filter on M and λ a cardinal. In $V[G]$, if $V \models M^\lambda \subseteq M$ then $M[G]^\lambda \subseteq M[G]$.*

Proof. We work in $V[G]$. Let $c = (c_\alpha \mid \alpha < \lambda)$ be a λ-sequence such that for all $\alpha < \lambda$, $c_\alpha \in M[G]$. For each $\alpha < \lambda$, let $\dot{c_\alpha}$ be a $\mathbb{P}$-name with $\dot{c_\alpha}^G = c_\alpha$. Let $\dot{a}$ be a $\mathbb{P}$-name with $\dot{a}^G = (\dot{c_\alpha} \mid \alpha < \lambda)$. Choose a $p \in G$ with $p \Vdash \forall \alpha < \check{\lambda} : \dot{a}(\alpha) \in M^{\mathbb{P}}$ in V.

Working in V, for each $\alpha < \lambda$, there is a maximal antichain A_α below p such that every $q \in A_\alpha$ decides $\dot{a}(\alpha)$, i.e., for some $x \in M$, $q \Vdash \dot{a}(\alpha) = \check{x}$. Define $\sigma = \{((\check{\alpha}, x), q) \mid \alpha < \lambda, q \in A_\alpha, q \Vdash \dot{a}(\alpha) = \check{x}\}$. Then $p \Vdash \sigma = \dot{a}$. Notice that $|\sigma| \leq \lambda$, since for each α, $|A_\alpha| \leq \lambda$. Thus $\sigma \in M$.

in $V[G]$ again, $(\dot{c_\alpha} \mid \alpha < \lambda) = \dot{a}^G = \sigma^G \in M[G]$. We can compute $c = (c_\alpha \mid \alpha < \lambda) = (\dot{c_\alpha}^G \mid \alpha < \lambda)$ from $(\dot{c_\alpha} \mid \alpha < \lambda)$ and G. Hence by Replacement, $c \in M[G]$. □

Lemma 1.13. *Let λ be a cardinal and $M^\lambda \subseteq M$ for some model M with* $\mathrm{Ord} \subseteq M$*. Then $H^M_{\lambda^+} \supseteq H_{\lambda^+}$.*

Proof. Let $x \in H_{\lambda^+}$ and set $a := |\mathrm{tc}(\{x\})| \leq \lambda$. Find a bijection $f : |\mathrm{tc}(\{x\})| \to \mathrm{tc}(\{x\})$ with $f(\emptyset) = x$. Now define a relation R on a^2 by $\alpha R \beta \leftrightarrow f(\alpha) \in f(\beta)$. Then, (a, R) has a transitive collapse in $a^2 \subseteq \lambda$. By assumption $M^\lambda \subseteq M$, i.e., $a^2, R \in M$. We can reconstruct x from a^2 and R. □

Definition 1.14. Suppose that $\{\mathbb{P}_\alpha \mid \alpha < \lambda\}$ is a set of forcing notions. The *lottery sum* of the $\mathbb{P}_\alpha$ is their disjoint union $\mathbb{P}$ with a new $\mathbb{1}$ such that $\mathbb{1} > p$ for all $p \in P_\alpha$, $\alpha < \lambda$.

Lemma 1.15. *Lottery sums of proper forcings are themselves proper.*

Proof. Let $\mathbb{P}$ be the lottery sum of $(\mathbb{Q}_\alpha \mid \alpha < \kappa)$. Let G be $\mathbb{P}$-generic. Since elements of G are pairwise compatible and if $p, q \in \mathbb{P}$, $p \in \mathbb{Q}_\alpha$, $q \in \mathbb{Q}_\beta$, $\alpha \neq \beta$, p, q are incompatible, $G \subseteq \mathbb{Q}_\alpha \cup \{\mathbb{1}\}$ for some α. A set D is clearly dense in $\mathbb{P}$ if and only if $D \cap \mathbb{Q}_\alpha$ is dense in $\mathbb{Q}_\alpha$ for all $\alpha < \kappa$. Hence G is a $\mathbb{Q}_\alpha$-generic filter for some α, i.e., stationary sets are preserved between V and $V[G]$. □

We can now define a general scheme for the iterations which we will use.

Definition 1.16. Suppose that $\kappa > \lambda > \omega$ are cardinals. The *minimal counterexample iteration* $\mathbb{P}_\kappa = \mathbb{P}_\kappa^{PFA_\lambda}$ for PFA_λ of length κ is the countable support iteration of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta \mid \alpha \leq \kappa,\ \beta < \kappa)$, where $\mathbb{P}_\alpha$ and $\dot{\mathbb{Q}}_\alpha$ are defined by induction: Let $\dot{\mathbb{Q}}_\alpha$ be a hereditarily minimal $\mathbb{P}_\alpha$-name for the lottery sum of all *proper* counterexamples to PFA_λ of minimal hereditary size smaller than κ.

We will only consider iterations of inaccessible length κ.

Lemma 1.17. *If κ is inaccessible and $\alpha < \kappa$, then $|\mathbb{P}_\alpha| < \kappa$.*

Proof. This is shown by induction on α. If $\alpha = 0$, then $\mathbb{P}_\alpha$ is a union of forcings of hereditary size $\gamma < \kappa$, so $\mathbb{P}_\alpha \subseteq H_{\gamma^+}$. Therefore $|\mathbb{P}_\alpha| \leq |H_{\gamma^+}| \leq 2^\gamma < \kappa$.

If $\alpha = \beta + 1$, then $\mathbb{P}_\beta$ forces that $\mathbb{P}_\alpha$ is a union of forcing notions with hereditary size $\gamma < \kappa$, so exactly as above, $\mathbb{1}_\beta \Vdash |\mathbb{Q}_\alpha| \leq |H_{\gamma^+}| \leq 2^\gamma$. Now, since $|\mathbb{P}_\beta| < \kappa$, there is some $\delta > \max\{\gamma, |\mathbb{P}_\beta|\}$, $\delta < \kappa$, i.e., $\mathbb{1}_\beta \Vdash 2^\gamma \leq 2^\delta = (2^\delta)^V$, by counting nice names for subsets of δ. Thus, since κ is inaccessible, $|\mathbb{P}_\alpha| \leq 2^\delta < \kappa$.

Suppose that $\gamma < \kappa$ is a limit and that for all $\alpha < \gamma$, $|\mathbb{P}_\alpha| < \kappa$. Since κ is regular, there is some λ such that for all $\alpha < \gamma$, $\lambda > |\mathbb{P}_\alpha|$. Notice that $|\mathbb{P}_\gamma| \leq \Pi_{\alpha<\gamma} |\mathbb{P}_\alpha|$, since $p \mapsto (p \restriction \alpha)_{\alpha<\gamma}$ is injective. Thus we conclude $\Pi_{\alpha<\gamma} |\mathbb{P}_\alpha| \leq \Pi_{\alpha<\gamma}\lambda = \lambda^\gamma < \kappa$. □

Moreover $\mathbb{P}_\kappa$ only depends on H_κ, i.e. if M is transitive with $H_\kappa \subseteq M$, then $\mathbb{P}^M_\kappa = \mathbb{P}_\kappa$, by the following lemma.

Lemma 1.18. *Suppose that κ is inaccessible. Then there is forcing $\mathbb{Q} \subseteq H_\kappa$ definable in H_κ which is isomorphic to $\mathbb{P}_\kappa$.*

Proof. The point is that if $\mathbb{P}$ is a forcing in H_κ, then it is proper if and only if it is proper in H_κ. Using this, we will show that the definition of the sequence $(\mathbb{P}_\alpha \mid \alpha < \kappa)$ is absolute between H_κ and V, where the $\mathbb{P}_\alpha$ are the initial segments of $\mathbb{P}_\kappa$.

Let us give a recursive definition of the sequence. If γ is a limit and we have defined $\mathbb{P}_\alpha$ for $\alpha < \gamma$, then we can define $\mathbb{P}_\gamma$ as a countable support limit.

Now let $\alpha = \beta + 1$ and suppose that $\mathbb{P}_\beta$ be defined. We can define $\varphi(\dot{Q}) =$"$\dot{Q}$ is a hereditarily minimal $\mathbb{P}_\beta$-name for the lottery sum of all proper counterexamples to PFA of minimal hereditary size". Find such $\dot{Q}$ and let $\mathbb{P}_\alpha = \mathbb{P}_\beta * \dot{Q}$. We now need to show that $\varphi(\dot{Q})$ holds in V.

We argue that it is sufficient to see that V also believes that $\dot{Q}$ is a name for a lottery sum of proper forcings. Note that $\dot{Q}$ is indeed a name for a lottery sum consisting of forcings with hereditary size smaller κ by Lemma 1.1. All other properties except properness in φ are easily absolute between H_κ and V because by Lemma 1.2 $V[G]$ and $H_\kappa[G]$ agree on the relevant witnesses.

We now deal with the properness of $\dot{Q}$ in a generic extension. We have assumed that $H_\kappa \models$ "$\mathbb{1}_\beta \Vdash \dot{Q}$ is a proper lottery sum". By Theorem 1.3, $\mathbb{1}_\beta \Vdash$ "$H_\kappa \models \dot{Q}$ is a proper lottery sum". As in the proof of Lemma 1.17, there is some regular $\lambda < \kappa$ such that $\mathbb{1}_\beta \Vdash 2^{|\dot{Q}|} < \check{\lambda}$. Since properness of $\dot{\mathbb{Q}}$ is absolute between H_λ and H_κ, $\dot{\mathbb{Q}}$ is a name for a proper forcing. □

Theorem 1.19. *If κ is λ-supercompact, then $\mathbb{P}^{\mathrm{PFA}}_\kappa$, forces that* PFA *holds for all proper forcings $\mathbb{P}$ with $2^{|\mathbb{P}|} \leq \lambda$.*

Proof. We follow Baumgartner's argument (see Jechs book), but avoid the use of Laver functions and instead work with lottery sums. The use of lottery sums in such iterations is an idea of Joel Hamkins and has been extensively used by Hamkins and Apter.

Let $j \colon V \to M$ be a λ-supercompact embedding with $\mathrm{crit}(j) = \kappa$, $\lambda < j(\kappa)$, $M^\lambda \subseteq M$. e that G is $\mathbb{P}_\kappa$-generic over V. We work in $V[G]$. Let $\mathbb{P}$ be a proper forcing violating PFA with $2^{|\mathbb{P}|} \leq \lambda$ of minimal hereditary size. Let $\mathcal{D} = \{D_\alpha \mid \alpha < \aleph_1\}$ witness this. We show that $\mathbb{P} \in M[G]$ by Lemma 1.13.

Since $|\mathbb{P}_\kappa| \leq \kappa$, by Lemma 1.12 it remains to show that $\mathbb{P}_\kappa \in M$. Since $M[G]^\lambda \subseteq M[G]$ by Lemma 1.12, $\mathbb{P}_\kappa \subseteq M$ is sufficient. Let $p \in \mathbb{P}_\kappa$. Since $\mathbb{P}_\kappa$ is a countable support iteration, there is some $\gamma < \kappa$ such that $p(\alpha) = \mathbb{1}$ for all $\alpha > \gamma$. Since $j(\gamma) = \gamma$, $j(p)(\alpha) = \mathbb{1}$ for all $\alpha > \gamma$. Moreover $p(\alpha) = (p \restriction \gamma)(\alpha)$ for all $\alpha \leq \gamma$, hence

$$j(p)(\alpha) = j(p \restriction \gamma)(\alpha) = (p \restriction \gamma)(\alpha) = p(\alpha)$$

for all $\alpha \leq \gamma$. Thus $j(p) = p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1}$, i.e., $j(p) \restriction \kappa = p \in M$.

Claim (i). *In $M[G]$, $\mathbb{P}$ violates* PFA*, is of minimal hereditary size with that property and $\mathbb{P} \in H_{j(\kappa)}$.*

Proof. We first claim that $|\mathrm{tc}(\mathbb{P})| = |\mathbb{P}|$. Otherwise, take a bijection $f : \mathbb{P} \to \alpha = |\mathbb{P}|$ and define a relation $<_\alpha$ on α by $\beta <_\alpha \gamma$ iff $f^{-1}(\beta) <_{\mathbb{P}} f^{-1}(\gamma)$. $(\alpha, <_\alpha)$ is a forcing notion equivalent to $\mathbb{P}$ but of smaller hereditary size $\mathrm{tc}(\alpha) = \alpha$, contradicting the assumption.

We now show that $\mathbb{P}$ is proper in $M[G]$. Let $\mu = (|\mathbb{P}|)^+$. Since we now know $|\mathrm{tc}(\mathbb{P})| = |\mathbb{P}| < \mu$, $\mathbb{P} \in H_\mu$. Choose a club $C \subseteq [H_\mu]^\omega$ witnessing that $\mathbb{P}$ is proper in $V[G]$. Note that

$$|C| \leq |H_\mu| \leq 2^{<\mu} \leq 2^{|\mathbb{P}|} \leq \lambda$$

and therefore by Lemma 1.13, $C \in M[G]$ and hence C witnesses that $\mathbb{P}$ is proper in $M[G]$.

Also, $V[G]$ and $M[G]$ have the same $\aleph_1$, since $\mathbb{P}_\kappa$ is proper (as a countable support iteration of proper forcing notions). Hence, $|\mathcal{D}|^{M[G]} = \aleph_1^{M[G]}$. For all $\alpha < \omega_1$, $D_\alpha \subseteq \mathbb{P} \in M[G]$, $|D_\alpha| \leq |\mathbb{P}| \leq \lambda$, i.e., $D_\alpha \in M[G]$. Thus, since $\aleph_1 < \lambda$, $\mathcal{D} \in M[G]$.

Furthermore, $|\mathrm{TC}(\mathbb{P})| < \lambda < j(\kappa)$, so $\mathbb{P} \in H_{j(\kappa)}$. Finally, if there were a hereditary smaller counterexample in $M[G]$, it would be in $V[G]$ and be a counterexample to PFA there, because $M[G]$ is sufficiently closed to contain filters witnessing the contrary and clubs witnessing properness. Hence this would contradict the hereditarily minimality of $\mathbb{P}$. □

In M, the forcing $j(\mathbb{P}_\kappa)$ is, by elementarity, a countable support iteration of length $j(\kappa) > \lambda$ and $\mathbb{P}_\kappa$ is an initial segment of $j(\mathbb{P}_\kappa)$, since $\mathrm{crit}(j) = \kappa$ (i.e. $j \restriction H_\kappa = \mathrm{id}$ while $\mathbb{P}_\alpha \in H_\kappa$ for all $\alpha < \kappa$). By Lemma 1.5, $j(\mathbb{P}_\kappa)$ is forcing equivalent to an iteration $(\mathbb{P}_\kappa * \dot{\mathbb{P}}) * \dot{\mathbb{Q}}$ where $\dot{\mathbb{P}}^G$ is the lottery sum of all counterexamples to PFA in $M[G]$ of minimal hereditary size smaller $j(\kappa)$.

Let H be $\mathbb{P}$-generic over $V[G]$. Note that there is a $\dot{\mathbb{P}}^G$-generic $\tilde{H}$ over $M[G]$ with $M[G * H] = M[G * \tilde{H}]$. Let I be $\dot{\mathbb{Q}}^{G*\tilde{H}}$-generic over $V[G * \tilde{H}]$.

We now work in $V[(G * \tilde{H}) * I]$. Consider:

$$j^*\colon V[G] \to M[(G * \tilde{H}) * I],$$

$$j^*(\sigma^G) = j(\sigma)^{(G*\tilde{H})*I}.$$

Claim (ii)**.** *j^* is well-defined and elementary and extends j.*

Proof. To show that j^* is well-defined, let σ, τ be $\mathbb{P}_\kappa$-names with $\sigma^G = \tau^G$. Then there is $p \in G$ such that $p \Vdash \sigma = \tau$, i.e., $j(p) \Vdash j(\sigma) = j(\tau)$. $j(p)$ is an element of $(G * \tilde{H}) * I$: $p = (p_\alpha \mid \alpha < \kappa)$ with countable support, so there is some $\beta < \kappa$ with $p_\gamma = \mathbb{1}$ for all $\gamma \geq \beta$. $V[G] \models \forall \gamma < \beta : p(\gamma) = (p \restriction \beta)(\gamma)$, so

$$\forall \gamma < j(\beta) : j(p)(\gamma) = (j(p \restriction \beta))(\gamma).$$

Since $j \restriction H_\kappa = \mathrm{id}$, $\mathbb{P}_\gamma \in H_\kappa$ and $j(\beta) = \beta$, $j(p)(\gamma) = p(\gamma)$ below β and $\mathbb{1}$ otherwise. Therefore $j(p) = p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1} \in (G * \tilde{H}) * I$.

To show that j^* is elementary, let $\varphi = \varphi(x)$ be a formula, σ a $\mathbb{P}_\kappa$-name and suppose $V[G] \models \varphi(\sigma^G)$. Then there is some $p \in G$ with $p \Vdash \varphi(\sigma)$, i.e., $j(p) \Vdash \varphi(j(\sigma))$. As above $j(p) \in (G * \tilde{H}) * I$.

Moreover j^* extends j, since $j^*(x) = j^*(\check{x}^G) = j(\check{x})^G = \check{x}^G = x$ for $x \in V$. □

Suppose that $\mathcal{D}$ is a family of size $\aleph_1$ of dense subsets of $\mathbb{P}$ in $V[G]$. As in (i), $\mathcal{D}$ is a family of size $\aleph_1$ of dense subsets of $\mathbb{P}$ in $M[G]$. We show that there is a $(j^*(\mathbb{P}), j^*(\mathcal{D}))$-generic filter in $M[(G * \tilde{H}) * I]$. Notice that $j^* \restriction \mathbb{P} \in M[G]$, since $|\mathbb{P}| < \lambda$. $H \subseteq \mathbb{P}$ and therefore by Replacement $j^*[H] \in M[(G * \tilde{H}) * I]$.

Since $j^*(\omega_1) = \omega_1$, $j^*(\mathcal{D}) = \{j^*(D) \mid D \in \mathcal{D}\}$. Since H is $\mathbb{P}$-generic in $V[G]$, it intersects every $D \in \mathcal{D}$. Thus for every $D \in \mathcal{D}$ there is some $x_D \in H$ such that $V[G] \models x_D \in D$, so by elementarity, $M[(G * H) * I] \models j^*(x_D) \in j^*(D)$.

Therefore the filter on $j^*(\mathbb{P})$ generated by $j^*[H]$ in $M[(G*\tilde{H})*I]$ intersects every $D \in j^*(\mathcal{D})$. Hence, by elementarity, there is a filter on $\mathbb{P}$ in $V[G]$ which intersects every $D \in \mathcal{D}$. □

The classical result follows immediately.

Corollary 1.20. *If κ is a supercompact cardinal, then $\mathbb{P}_\kappa$ forces* PFA*, hence* PFA *is consistent relative to the existence of a supercompact cardinal.*

1.3. **BPFA.**

April 14 **Definition 1.21.** *BPFA* is defined as the axiom PFA_{ω_1}, i.e. it states that for any proper preordered set $(\mathbb{P},<)$ and $\mathcal{D}$, $|\mathcal{D}| = \aleph_1$ a collection of predense subsets of $\mathbb{P}$ such that for all $D \in \mathcal{D}$, $|D| \leq \lambda$, there exists a $\mathcal{D}$-generic centered set on $\mathbb{P}$.

There is a third (and possibly stronger) version of *BPFA*, besides PFA_{ω_1} and $PFA^*_{\omega_1}$, which asks for the existence of a filter in a partial order. We don't know whether the three versions can be separated.

Definition 1.22. A cardinal κ is *reflecting* if and only if it is regular and for any formula φ and any $a \in H_\kappa$, if there is a cardinal $\delta > \kappa$ with $H_\delta \models \varphi(a)$, then there is some cardinal $\gamma < \kappa$ with $a \in H_\gamma$ and $H_\gamma \models \varphi(a)$.

Remark 1.23. By adding any cardinal $\alpha < \kappa$ as a parameter to φ, we can make the γ provided by the reflecting property as large as we require.

Definition 1.24. A cardinal κ is a Mahlo cardinal if the set of inaccessible cardinals $\mu < \kappa$ is stationary in κ.

Lemma 1.25. *Suppose that there is a Mahlo cardinal. Then there is a model of* ZFC *with a reflecting cardinal.*

Proof. We claim that for κ is inaccessible, the set $\{\alpha < \kappa \mid V_\alpha \prec V_\kappa\}$ is club in κ. To show unboundedness, let $\alpha < \kappa$ be arbitrary and define a sequence $(\alpha_n)_{n\in\omega}$ by induction. Let $\alpha_0 = \alpha$. Let $\alpha_{n+1} \geq \alpha_n$ such that for all formulas φ and all $\bar{y} \in V_{\alpha_n}$, if $V_\kappa \models \exists x\varphi(x,\bar{y})$, then there is $\bar{x} \in V_{\alpha_{n+1}}$ such that $V_\kappa \models \varphi(\bar{x},\bar{y})$. Since κ is inaccessible, $|V_{\alpha_n}| < \kappa$, hence there are less than κ many such $\bar{x}$, so $\alpha_{n+1} < \kappa$.

Let $\gamma = \sup_{n<\omega} \alpha_n \geq \alpha$. Then $V_\gamma \prec V_\kappa$ by the Tarski-Vaught criterion. Closure is trivial, since if $V_{\gamma_n} \prec V_\kappa$, $n < \omega$, then again by Tarski-Vaught, $\bigcup_{n<\omega} V_{\gamma_n} \prec V_\kappa$.

Now let μ be Mahlo. In V_μ, there is a club $C \subseteq \mu$ such that $V_\alpha \prec V_\mu$ for all $\alpha \in C$. Let $\kappa \in C$ be inaccessible. Then V_μ models that κ is reflecting: If $a \in H_\kappa$, $\delta \in V_\mu$ and $V_\mu \models$ "$H_\delta \models \varphi(a)$", then $V_\mu \models \exists\delta :$ "$V_\delta \models \varphi(a)$". By elementarity, so does κ. □

Reflecting cardinals are indeed large.

Lemma 1.26. *If κ is a reflecting cardinal, then κ is inaccessible.*

Proof. Suppose that there is some $\delta < \kappa$ with $2^\delta \geq \kappa$. Then $\delta \in H_\kappa$ and we may reflect "2^δ exists". So, there is some $\gamma < \kappa$ such that $H_\gamma \models$ "2^δ exists". But $2^\delta \notin H_\gamma$, contradicting the assumption. □

Lemma 1.27. *A regular cardinal $\kappa > \omega$ is reflecting if and only if it is Σ_2-correct, i.e. $V_\kappa \prec_{\Sigma_2} V$.*

Proof. Suppose that κ reflecting, $a \in H_\kappa$, and φ is a Σ_2 formula with $\varphi(a)$. Then there is a cardinal $\gamma < \kappa$ with $a \in H_\gamma$ and $H_\gamma \vDash \varphi(a)$. We have $H_\gamma \prec_{\Sigma_1} V$ by Löwenheim-Skolem. Since φ is Σ_2, this implies $V_\kappa \vDash \varphi(a)$.

The other direction holds since the statement "$H_\delta \vDash \varphi(a)$" is Σ_2. □

The next result shows that reflecting cardinals are indestructible by small forcing.

Lemma 1.28. *Let $\mathbb{P} \in H_\kappa$. If κ is reflecting, then* $\mathbb{1}_{\mathbb{P}} \Vdash$ "κ is reflecting".

Proof. Let $\mathbb{P} \in H_\kappa$. Let φ be a formula with

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{"}\dot{a} \in H_\kappa \wedge \exists \delta > \kappa : H_\delta \models \varphi(\dot{a})\text{"}.$$

We may assume that $\dot{a} \in H_\kappa$ by Lemma 1.2. By Theorem 1.3 $H_\delta \models \mathbb{1}_{\mathbb{P}} \Vdash \varphi(\dot{a})$. Since κ is reflecting, there is $\gamma < \kappa$ such that $\mathbb{P}, \dot{a} \in H_\gamma$ and $H_\gamma \models \mathbb{1}_{\mathbb{P}} \Vdash \varphi(\dot{a})$. So by Lemma 1.3, $\mathbb{1}_{\mathbb{P}} \Vdash H_\gamma \models \varphi(\dot{a})$.

Since $\mathbb{P} \in H_\gamma$, κ remains a regular cardinal. Thus κ is reflecting in any $\mathbb{P}$-generic extension. □

Lemma 1.29. *If δ is a limit ordinal, $\mathbb{P}$ is a countable support forcing iteration of length δ, G is $\mathbb{P}$-generic, $\gamma < \mathrm{cof}(\delta)^{V[G]}$, $\mathrm{cof}(\delta)^{V[G]} \geq \omega_1$, and $X \in \mathcal{P}(\gamma)^{V[G]}$, then there is an ordinal $\alpha < \delta$ such that $X \in V[G_\alpha]$, where $G_\alpha = \{p \upharpoonright \alpha \mid p \in G\}$.*

Proof. Let $\dot{X}$ be a $\mathbb{P}$-name for X. For each $\alpha < \gamma$ we choose a $p_\alpha \in G$ which decides $\check{\alpha} \in \dot{X}$. Since the supports are countable and $\gamma < \mathrm{cf}(\delta)$, $\eta := \sup(\bigcup_{\alpha<\gamma} \mathrm{supp}(p_\alpha)) < \delta$. Now $X = \{\alpha < \eta \mid \exists p \in G_\eta : p^\frown \mathbb{1}^\frown \ldots ^\frown \mathbb{1} \Vdash \check{\alpha} \in \dot{X}\} \in V[G_\eta]$. □

The lemma is false if we weaken the assumption $\gamma < \mathrm{cf}(\delta)^{V[G]}$ to $\gamma < \mathrm{cf}(\delta)^V$; this can be seen by collapsing ω_1 in the first step.

We can now define the notion of *special counterexamples*. Note that a special counterexample no longer contains an actual (potentially large) notion of forcing. For convenience, we include minimality in the definition.

Definition 1.30. Let $\lambda > \omega$ be a cardinal. We call a triple $(D, D^*, \leq^*)$ a *special counterexample to PFA_λ* if there is a proper forcing $\mathbb{Q}$ such that

i. $(\mathbb{Q}, \leq)$ is a hereditarily minimal counterexample to PFA_λ,
ii. $\bigcup D \subseteq D^* \subseteq \mathbb{Q}$,
iii. $|D^*| \leq \lambda$, $|D| \leq \aleph_1$,
iv. all $A \in D$ are predense in $\mathbb{Q}$,
v. if $A \subseteq_{<\omega} D^*$ is compatible w.r.t. $\mathbb{Q}$, there is $a \in D^*$, $a \leq A$,
vi. $\leq^* = \leq \upharpoonright D^*$, and
vii. there is no generic centered set $G \subseteq \mathbb{Q}$ with $G \cap A \neq \emptyset$ for all $A \in D$.

Since the order $\leq^*$ is clear from the context in all cases, we implicitly include $\leq^*$ in D^* and consider special counterexamples as tuples (D, D^*).

Let us write $\Gamma^\lambda(D, D^*)$ if (D, D^*) is a special counterexample to PFA_λ. We will write $\Gamma^\lambda(D, D^*, \mathbb{Q})$ if $\Gamma^\lambda(D, D^*)$ and $\mathbb{Q}$ witnesses this.

The following lemma shows why this is the crucial notion for the treatment of bounded fragments of PFA.

Lemma 1.31. *Let $\lambda > \omega$, $(D, D^*, \leq^*)$ be a special counterexample to* PFA_λ *and let $\mathbb{P}$ and $\mathbb{Q}$ be forcing notions satisfying* ii.-vi. *in the definition of special counterexamples. Let G be a filter on $\mathbb{P}$. Then $G \cap D^*$ is centered in $\mathbb{Q}$.*

Proof. We show that $g = D^* \cap G$ is centered with respect to $\leq^*$ in the partial order D^*. Then g is also centered with respect to $\mathbb{Q}$ by the definition of special counterexamples. Let $A \subseteq g$ be finite. Because G is a filter, there is $r \in G$ that is a lower bound for A. Note that it is not clear that $r \in D^*$. But by *v.*, there is some such lower bound in D^*. □

On the other hand, we need to know that we can always find a special counterexample if we have a counterexample $\mathbb{Q}$.

Lemma 1.32. *Let $\lambda > \omega$. If $\mathbb{Q}$ is a counterexample to* PFA_λ*, then there are D, D^* satisfying* ii.-vii. *in the definition of special counterexamples to* PFA_λ*. In particular, if $\mathbb{Q}$ is some hereditarily minimal counterexample to* PFA_λ*, then $\Gamma^\lambda(D, D^*, \mathbb{Q})$.*

Proof. Let $\mathbb{Q}$ be a counterexample to PFA_λ and let D be a set of predense sets of $\mathbb{Q}$ witnessing this. For each compatible $A \subseteq_{<\omega} \mathbb{Q}$, choose $r_A \in \mathbb{Q}$ such that $r_a \leq A$.

Let $D_0 = \bigcup D$. If D_n is defined, let $D_{n+1} = \{r_A \mid A \subseteq_{<\omega} D_n \text{ compatible}\}$. Set $D^* = \bigcup_{n\in\omega} D_n$. This process adds at most λ conditions each step, so $|D^*| \leq \lambda$. Also, if $A \subseteq_{<\omega} D^*$ is compatible w.r.t. $\mathbb{Q}$, there is some $n \in \omega$ with $A \subseteq_{<\omega} D_n$, so $r_A \in D^*$. □

Furthermore, reflecting cardinals provide small witnesses to special counterexamples to BPFA. Combined with the previous result this will be the crucial step in our main argument.

Lemma 1.33. *Let κ be reflecting. If there is a special counterexample D, D^* to* BPFA*, $D, D^* \in H_\kappa$, then there is a forcing $\mathbb{Q}$ witnessing this in H_κ.*

Proof. Suppose there are $D, D^* \in H_\kappa$ with $\Gamma^{\aleph_1}(D, D^*)$. There is a cardinal δ such that

$$H_\delta \models \text{"}\Gamma^{\aleph_1}(D, D^*), \exists\lambda : \exists\mathbb{Q} \in H_\lambda : \Gamma^{\aleph_1}(D, D^*, \mathbb{Q}), 2^\lambda \text{ exists''}.$$

Since κ is reflecting, there is $\gamma < \kappa$ such that

$$H_\gamma \models \text{"}\Gamma^{\aleph_1}(D, D^*), \exists\lambda : \exists\mathbb{Q} \in H_\lambda : \Gamma^{\aleph_1}(D, D^*, \mathbb{Q}), 2^\lambda \text{ exists''}.$$

Choose such $\mathbb{Q}$ and λ, i.e., $H_\gamma \models$ "$\mathbb{Q} \in H_\lambda$, 2^λ exists, $\Gamma^{\aleph_1}(D, D^*, \mathbb{Q})$", and $\mathbb{Q}$ is really proper, since properness of $\mathbb{Q}$ is absolute between H_γ and V. All other properties of "special counterexample to BPFA" are obviously absolute. □

Now we can show the main result in this section.

Theorem 1.34. *If κ is reflecting, then the minimal counterexample iteration for* BPFA*, $\mathbb{P}_\kappa^{\mathrm{BPFA}}$, forces* BPFA.

Proof. Suppose not. Let p be some condition that forces $\Gamma^{\aleph_1} \neq \emptyset$. Let G be $\mathbb{P}$-generic over V, $p \in G$ and live in $V[G]$. Take witnesses (viz., a special counterexample to BPFA) D, D^* for $\Gamma^{\aleph_1}$. Note that $\omega_1^V = \omega_1^{V[G]}$ since $\mathbb{P}_\kappa$ is proper. Also $\mathbb{P}_\kappa$ does not collapse κ.

Since D, D^* are of size at most ω_1, we can code them by subsets of ω_1. By Lemma 1.29, there is some $\alpha < \kappa$ with $D, D^* \in V[G_\alpha]$, where $G_\alpha = \{q \upharpoonright \alpha \mid q \in G\}$. Since G_α is $\mathbb{P}_\alpha$-generic and $\mathbb{P}_\alpha \in H_\kappa$ by the construction of $\mathbb{P}_\kappa$, by Lemma 1.28, κ is reflecting in $V[G_\alpha]$. Now work in $V[G_\alpha]$.

Because $\mathbb{P}_\kappa$ is a countable support iteration, there is some $q \in H_\kappa^V \subseteq H_\kappa$ such that $p = q^\frown \mathbb{1}^\kappa$. The statement $\exists\lambda : q^\frown \mathbb{1}^\lambda \Vdash_{\mathbb{P}_\lambda} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$ holds (take $\lambda = \kappa$) and its parameters are in H_κ. So, since κ is reflecting, there are $\gamma < \delta < \kappa$ with $H_\delta \models q^\frown \mathbb{1}^\gamma \Vdash_{\mathbb{P}_\gamma} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$, and since this is Σ_2, $q^\frown \mathbb{1}^\gamma \Vdash_{\mathbb{P}_\gamma} \Gamma^{\aleph_1}(\check{D}, \check{D}^*)$ is true.

The forcing $\mathbb{P}_\gamma$ has hereditary size smaller κ, thus $q^\frown \mathbb{1}^\gamma$ also forces that κ is reflecting, thus it forces that there is a witness $\mathbb{Q}$ to $\Gamma^{\aleph_1}(D, D^*)$ with hereditary size smaller κ by Lemma 1.33. We may assume $\mathbb{Q}$ has minimal hereditary size; then there is some $r \leq q^\frown \mathbb{1}^\kappa = p$ choosing that $\mathbb{Q}$ from the lottery sum in the γ-th step.

Hence forcing with r adjoins a $\mathbb{Q}$-generic filter h, so $h \in V[G]$ for each generic G with $r \in G$. Then h intersects each $A \in D$ and is a filter on D^*. Thus by the construction of D^*, $h \cap D^*$ extends to a centered set on *any* witness to $\Gamma^{\aleph_1}(D, D^*)$. Hence r forces $\neg\Gamma^{\aleph_1}(D, D^*)$ contradicting the assumption that $p \Vdash \Gamma^{\aleph_1}(D, D^*)$. □

2. Solovay's model

April 16, 22, 29, 30 **Theorem 2.1.** *(Solovay) Suppose that κ is inaccessible. Suppose that G is $Col(\omega, \kappa)$-generic over V and $M = HOD({}^\omega Ord)^{V[G]}$. Then in M, every set $A \subseteq \mathbb{R}$ is Lebesgue measurable.*

3. Singularizing cardinals

May 6

In order to construct models with interesting cardinal arithmetic, the question arises how to make an uncountable regular cardinal κ singular without collapsing any cardinals $< \kappa$. The first interesting problem is to singularize ω_2 with cofinality ω without collapsing ω_1. This is achieved by Namba forcing. As we will see with the use of the covering lemma for K (without proof), there is no analogue for $\kappa > \omega_2$ to Namba forcing, unless we work with measurable cardinals. If κ is measurable, then we can singularize κ with cofinality ω without collapsing any cardinals below κ. We will later see an application of Prikry forcing to force the failure of the singular cardinal hypothesis.

3.1. Namba forcing.

Namba forcing is an analogue of Sacks forcing for trees of width ω_2. Namba forcing is to $Col(\omega, \omega_2)$ as Sacks forcing to Cohen forcing, in each case functions are replaced with trees.

Definition 3.1. (perfect trees) Suppose that κ is a cardinal.

(i) $T \subseteq {}^{<\omega}\kappa$ is a *tree* if T is closed under initial segments.
(ii) Let $t_T := \bigcup\{s \in T \mid \forall t \in T\ (s \subseteq t \vee t \subseteq s)\}$ denote the *trunk* of T.
(iii) If $s \in T$, let $T/s := \{t \in T \mid s \subseteq t \vee t \subseteq s\}$.
(iv) T is *κ-perfect* if every $s \in T$ ha κ many extensions $t \supseteq s$ in T.

Definition 3.2. (Cantor-Bendixson derivative) Suppose that $T \subseteq {}^{<\omega}\omega_2$ is a tree.

(i) Let $T' := \{s \in T \mid s \text{ has } \geq \omega_2 \text{ many extensions } t \supseteq s \text{ in } T\}$.
(ii) Let $T_0 = T$, $T_{\alpha+1} = T'_\alpha$, $T_\lambda = \bigcup_{\alpha<\lambda} T_\alpha$ for limits $\lambda < \omega_3$.
(iii) Let $\vartheta_T := \min\{\alpha < \omega_3 \mid T_\alpha = T_{\alpha+1}\}$.
(iv) Let $r_T(s) := \min\{\alpha < \omega_3 \mid s \notin T_\alpha\}$ when this exists.

Definition 3.3. (Namba forcing) Let $\mathbb{N}$ denote the set of ω_2-perfect trees $T \subseteq {}^{<\omega}\omega_2$. Let $S \leq T$ if $S \subseteq T$.

Lemma 3.4. *Suppose that G is $\mathbb{N}$-generic over V. Let$f_G := \bigcup_{T\in G} t_T$. Then $f_G \colon \omega \to \omega_2^V$ is cofinal.*

Proof. Let $D_\alpha := \{T \in \mathbb{N} \mid \sup(\text{range}(t_T)) \geq \alpha\}$ for $\alpha < \omega_2$. Then D_α is dense for every $\alpha < \omega_2$, since every $T \in \mathbb{N}$ is ω_2-perfect. □

Theorem 3.5. *Suppose that CH holds and that G is $\mathbb{N}$-generic over V. Then $\omega_1^V = \omega_1^{V[G]}$ and $\text{cof}(\omega_2^V)^{V[G]} = \omega$.*

Proof. Suppose that $T \Vdash \dot{f} \colon \omega \to 2$. We construct $(T_s)_{s\in {}^{<\omega}\omega_2}$ such that

(i) $T_\emptyset = T$,
(ii) if $s \subseteq t$, then $T_t \subseteq T_s$,
(iii) if $|s| = n$, then T_s decides $\dot{f} \restriction n$,
(iv) for each n, $(t_{T_s})_{s\in\omega_2^n}$ are pairwise incomparable.

Suppose that T_s is defined for $s \in \omega_2^n$. Let $t \supseteq s$ and $I \subseteq \omega_2$, $|I| = \omega_2$ with $t^\frown i \in T_s$ for all $i \in I$. Let $T_{s^\frown i} \leq T/(t^\frown i)$ decide $\dot{f}(n)$.

Let $(\dot{f} \restriction n)^S := s$ if $S \Vdash (\dot{f} \restriction n) = \check{s}$, and otherwise undefined, for $S \in \mathbb{N}$.

If $\vec{i} \in 2^n$, let $T(\vec{i}) := \bigcup\{T_s \mid s \in {}^{<\omega}\omega_2,\ (\dot{f} \restriction n)^{T_s} = \vec{i}\}$.

If $x \in {}^\omega 2$, let $T(x) := \bigcap_{n\in\omega} T(x \restriction n)$.

Claim 3.6. $T(\vec{i}) \Vdash (\dot{f} \restriction n) = \vec{i}$ for all $\vec{i} \in 2^n$.

Proof. Suppose that $S \leq T(\vec{i})$. There is $S' \leq S$ such that S' decides $(\dot{f} \restriction n)$ and $t_{S'} \supseteq t_{T_s}$ for some $s \in \omega_2^n$ with $(\dot{f} \restriction n)^{T_s} = \vec{i}$. Then $S' \subseteq T_s$. So $S' \Vdash (\dot{f} \restriction n) = \vec{i}$. □

Claim 3.7. There is some $x \in {}^{\omega}2$ such that $T(x)$ has an ω_2-perfect subtree.

Proof. Suppose not. Then for all $x \in {}^{\omega}2$, $r_{T(x)} \colon T(x) \to \omega_3$ is a function with

(i) $r_{T(x)}(t) \leq r_{T(x)}(s)$ if $s \subseteq t$.
(ii) for every $s \in T(x)$, there are $\leq \omega_1$ many $t \supseteq s$ with $r_{T(x)}(s) = r_{T(x)}(t)$.

We construct $s_0 \subseteq s_1 \subseteq ...$, $s_n \in \omega_2^n$. Suppose that s_n is defined. Since $2^\omega = \omega_1$, there is some $\alpha < \omega_2$ such that $r_{T(x)}(t_{T_{s_n^\frown \alpha}}) < r_{T(x)}(t_{T_{s_n}})$, for all $x \in {}^{\omega}2$ with $t_{T_{s_n^\frown \alpha}}, t_{T_{s_n}}) \in T(x)$. Let $s_{n+1} = s_n^\frown \alpha$.

Let $x(n) = i$ if $T_{s_{n+1}} \Vdash \dot{f}(n) = i$.

Subclaim 3.8. $t_{T_{s_{n+1}}} \in T(x)$ for all $n \in \omega$.

Proof. We have $T_{s_{n+1}} \subseteq T(x \restriction n)$, since $(\dot{f} \restriction n)^{T_{s_{n+1}}} = x \restriction n$. So for $m \geq n$, $t_{T_{s_n}} \subseteq t_{T_{s_m}} \in T(x \restriction m)$. □

Then $r_{T(x)}(t_{T_{s_0}}) > r_{T(x)}(t_{T_{s_1}}) > r_{T(x)}(t_{T_{s_2}}) > ...$ is a strictly decreasing sequence of ordinals. □

Hence $T(x) \leq T$ decides $\dot{f}(n)$ for all n. Hence $\mathbb{N}$ does not add new reals, so $\omega_1^V = \omega_1^{V[G]}$. □

Problem 3.9. *Show that Namba forcing is not proper.*

Definition 3.10. An *inner model* is a transitive model M of ZFC with $Ord \subseteq M$.

Fact 3.11. (Dodd-Jensen) There is a formula $\varphi_K(x)$ such that $K = \{x \mid \varphi_K(x)$ is an inner model and

(i) $K \vDash GCH$.
(ii) $K^K = K$.
(iii) $K^{V[G]} = K^V$ for any generic extension $V[G]$ of V, i.s. for all $x \in V[G]$, $V[G] \vDash \varphi_K(x)$ if and only if $x \in V \wedge V \vDash \varphi_K(x)$.

Fact 3.12. (Dodd-Jensen covering lemma) Suppose that there is no inner model with a measurable cardinal. For every set $X \subseteq Ord$, there is a set $Y \subseteq Ord$ in K with $|X| \leq |Y| + \omega_1$.

Remark 3.13. In any $\mathbb{N}$-generic extension $V[G]$ of V, there is a set $X \subseteq Ord$ such that there is no set $Y \subseteq Ord$ in V with $|X|^{V[G]} = |Y|^{V[G]}$.

Proof. Let $X = \text{range}(f_G)$. Then $X \subseteq \omega_2^V$ is cofinal and $|X|^{V[G]} = \omega$. Suppose that $Y \in V$, $Y \supseteq X$, $|Y|^{V[G]} = |X|^{V[G]} = \omega$. Since $Y \subseteq \omega_2^V$ is cofinal, there is some $\alpha < \omega_2^V$ with $|Y \cap \alpha|^V = \omega_1^V = \omega_1^{V[G]}$. So $|Y|^{V[G]} \geq \omega_1^{V[G]}$. □

Remark 3.14. Suppose that there is no inner model with a measurable cardinal.

(i) Suppose that $\kappa \geq \omega_2$ is a cardinal. There is no generic extension $V[G]$ of V with the same cardinals $\leq \kappa$ ant $\text{cof}((\kappa^+)^V)^{V[G]} < \kappa$.
(ii) Suppose that $\kappa > \omega$ is a regular limit cardinal. There is no generic extension $V[G]$ of V with the same cardinals $\leq \kappa$ and $\text{cof}(\kappa)^{V[G]} < \kappa$.

Proof. (i) Suppose that $C \subseteq (\kappa^+)^V$ is cofinal, $C \in V[G]$, $otp(C) < \kappa$. There is some $D \supseteq C$, $D \in K^{V[G]} = K^V \subseteq V$ with $|D|^{V[G]} = |C|^{V[G]} + \omega_1^{V[G]} < \kappa$ by the covering lemma. But $|D|^V = (\kappa^+)^V$, so κ and κ^+ are collapsed.

(ii) Suppose that $C \subseteq \kappa$ is cofinal, $C \in V[G]$, $otp(C) < \kappa$. There is some $D \supseteq C$, $D \in K^{V[G]} = K^V \subseteq V$ with $|D|^{V[G]} = |C|^{V[G]} + \omega_1^{V[G]} < \kappa$ by the covering lemma. But $|D|^V = \kappa$, so κ is collapsed. □

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3.2. **Prikry forcing.** Before we define Prikry forcing, let us first review a few results on measurable cardinals.

Definition 3.15. Suppose that $\kappa < \omega$ is a cardinal.

(i) An ultrafilter U on κ is *$<\kappa$-complete* if $\bigcap_{\alpha<\gamma} X_\alpha \in U$ for all $X_\alpha \in U$, $\gamma < \kappa$.
(ii) κ is *measurable* if there is a $<\kappa$-complete nonprincipal ultrafilter on κ.
(iii) An ultrafilter U on κ is *normal* if U is nonprincipal and

$$\bigtriangleup_{\alpha<\kappa} X_\alpha = \{\beta < \kappa \mid \beta \in \bigcap_{\alpha<\beta} X_\alpha\} \in U$$

for all $(X_\alpha)_{\alpha<\kappa}$ with $X_\alpha \in U$ for all $\alpha < \kappa$.

Lemma 3.16. *Suppose that $\kappa > \omega$ is regular. A nonprincipal ultrafilter U on κ is* normal *if every regressive $f\colon \kappa \to \kappa$ is constant on a set in U.*

Proof. Suppose that $f\colon \kappa \to \kappa$ is regressive. Let $X_\alpha = \kappa \setminus f^{-1}(\{\alpha\})$. Suppose that $X_\alpha \in U$ for all $\alpha < \kappa$. Then $\bigtriangleup_{\alpha<\kappa} X_\alpha \in U$. Let $\beta \in \bigtriangleup_{\alpha<\kappa} X_\alpha$. Then $\beta \notin f^{-1}(\{\alpha\})$ for all $\alpha < \beta$.

Let $f(\beta) = \min\{\alpha < \beta \mid \beta \notin X_\alpha\}$ when this exists, and $f(\beta) = 0$ otherwise. Then

$$f^{-1}(\{\alpha\}) = (\kappa \setminus X_\alpha) \setminus (\alpha + 1)$$

for $\alpha \geq 1$, and

$$f^{-1}(\{0\}) = (\bigtriangleup_{\alpha<\kappa} X_\alpha) \cup ((\kappa \setminus X_0) \setminus 1).$$

Since f is constant on a set in U, this implies $\bigtriangleup_{\alpha<\kappa} X_\alpha \in U$. □

Lemma 3.17. *If κ is measurable, then there is a normal ultrafilter on κ.*

Proof. Suppose that U is a nonprincipal $<\kappa$-complete ultrafilter on κ. Let $Ult_U(V) = \{[f] \mid f\colon \kappa \to V\}$, $[f] = [g]$ if $\{\alpha < \kappa \mid f(\alpha) = g(\alpha)\} \in U$, $[f] \in [g]$ if $\{\alpha < \kappa \mid f(\alpha) \in g(\alpha)\} \in U$.

Then $j\colon V \to Ult_U(V)$, $j(x) = [c_x]$, $c_x\colon \kappa \to \kappa$, $c_x(\alpha) = x$, is elementary by Los' theorem.

Claim 3.18. $Ult_U(V)$ is wellfounded.

Proof. Suppose that $[f_0] \ni [f_1] \ni ....$ Let $X_n = \{\alpha < \kappa \mid f_n(\alpha) \ni f_{n+1}(\alpha)\} \in U$. Let $\alpha \in \bigcap_{n\in\omega} X_n$. Then $f_0(\alpha) \ni f_1(\alpha) \ni ....$ □

Let $X \in U'$ if $X \subseteq \kappa$ and $\kappa \in j(X)$. Then U' is a nonprincipal ultrafilter on κ. If $X_\alpha \in U'$ for all $\alpha < \kappa$, then $\kappa \in j(X_\alpha)$ for $\alpha < \kappa$. So $\kappa \in j(\bigtriangleup_{\alpha<\kappa} X_\alpha)$ and hence $\bigtriangleup_{\alpha<\kappa} X_\alpha) \in U'$. Therefore U' is a normal ultrafilter on κ, as required. □

Lemma 3.19. *Suppose that U is a normal ultrafilter on κ and $F\colon [\kappa]^{<\kappa} \to \mu < \kappa$. Then there is a set $H \in U$ such that $F \upharpoonright [H]^n$ is constant for all $n \in \omega$.*

Proof. We prove that for each $n \in \omega$, there is a set $H \in U$ such that $F \upharpoonright [H]^n$ is constant. This is clear for $n = 1$.

For $\alpha < \kappa$, let $F_\alpha\colon [\kappa \setminus \{\alpha\}]^n \to \mu$, $F_\alpha(x) = F(x \cup \{\alpha\})$. By the inductiion hypothesis, there are $X_\alpha \in U$ for $\alpha < \kappa$ such that $F_\alpha \upharpoonright [X_\alpha]^n$ is constant with value $\mu_\alpha < \mu$.

There is some $\nu < \mu$ and $H \subseteq X$, $H \in U$ with $\mu_\alpha = \nu$ for all $\alpha \in H$. Then $X := \bigtriangleup_{\alpha<\kappa} X_\alpha \in U$.

If $\alpha_0 < \alpha_1 < ... < \alpha_n$ are in X, then $\{\alpha_1, ..., \alpha_n\} \in [X_{\alpha_0}]^n$, so $F(\{\alpha_0, ..., \alpha_n\}) = F_{\alpha_0}(\{\alpha_1, ..., \alpha_n\}) = \mu_{\alpha_0}$. Then $F(x) = \nu$ for all $x \in [H]^{n+1}$. □

Definition 3.20. (Prikry forcing) Suppose that κ is measurable and U is a normal ultrafilter on κ. Let $\mathbb{P}_U$ denote the set of pairs (s, A) with $s \in [\kappa]^{<\omega}$ and $A \in U$. Let $(s, A) \leq (t, B)$ if t is an initial segment of s and $A \cup (s \setminus t) \subseteq B$. But $s, t \in [B]^n$ and B is homogeneous.

Lemma 3.21. *$\mathbb{P}_U$ satisfies the κ^+-c.c.*

Proof. Suppose that $A \subseteq \mathbb{P}_U$ is an antichain with $|A| = \kappa^+$. Then there are $(s, A), (t, B) \in A$ with $s = t$. Then $(s, A \cap B) \leq (s, A), (t, B)$, contradicting the assumption. □

Lemma 3.22. *Suppose that G is $\mathbb{P}_U$-generic. Let $x = \bigcup\{s \mid \exists A\ (s, A) \in G\}$. Then $x \subseteq \kappa$ is cofinal and $otp(x) = \omega$.*

Proof. If $\alpha < \kappa$, then $D_\alpha = \{(s, A) \in \mathbb{P}_U \mid \max(\text{range}(s)) \geq \alpha\}$ is dense in $\mathbb{P}_U$. So $x \subseteq \kappa$ is cofinal.

Suppose that $otp(x) > \omega$. Then there are $(s, A), (t, B) \in G$ with $\max(s) < \max(t)$, $s \not\subseteq t$. Then $(s, A), (t, B)$ are incompatible. □

Lemma 3.23. *(Prikry lemma) Suppose that $(s, A) \in \mathbb{P}_U$ and φ is a formula. There is some $B \subseteq A$, $B \in U$ such that (s, B) decides φ.*

Proof. Let $\delta = \max(s) + 1$. Let

$$S_0 = \{t \in [\kappa \setminus \delta]^{<\omega} \mid \exists X \in U\ (t, X) \Vdash \varphi\}$$

$$S_1 = \{t \in [\kappa \setminus \delta]^{<\omega} \mid \exists X \in U\ (t, X) \Vdash \neg\varphi\}$$

$$S_2 = [\kappa \setminus \delta]^{<\omega} \setminus (S_0 \cup S_1).$$

By the previous lemma, $[B]^n \subseteq S_i$ for some $i < 2$.

Suppose that (s, B) does not decide φ. Then there are $t, u \in [B]^{<\omega}$, $X, Y \subseteq B$ with $(s \cup t, X) \Vdash \varphi$ and $(s \cup u, Y) \Vdash \neg\varphi$. We can assume that $|t| = |u| = n$. This contradicts the assumption that B is homogeneous. □

Lemma 3.24. *P_U does not add bounded subsets of κ.*

Proof. Suppose that $\dot{f}$ is a name for a function $\dot{f} \colon \mu \to Ord$ with $\mu < \kappa$ below a condition (s, A). There is a decreasing sequence $(A_n)_{n\in\omega}$ in U with $A_0 \subseteq A$ and such that (s, A_n) decides $\dot{f}(n)$. Let $B = \bigcap_{n\in\omega} A_n$. Then (s, B) decides all values of $\dot{f}$. □

Lemma 3.25. *Suppose that M is a countable transitive model of ZFC, κ is measurable in M, and U is a normal measure on κ in M. Then for any set $X \subseteq \kappa$ of order type ω, $X = \bigcup\{s \mid \exists A\ (s, A) \in G\}$ for some $\mathbb{P}_U^M$-generic filter G over M if and only if for every $A \in U$, $X \setminus A$ is finite.*

Proof. Suppose that G is $\mathbb{P}_U$-generic over M and $X = \bigcup\{s \mid \exists A\ (s, A) \in G\}$. Suppose that $A \in U$. Then for every (s, B), $(s, A \cap B)$ forces that every $\alpha \in X$ above $\max(s)$ is in A. So $X \setminus A$ is finite.

Suppose that $X \subseteq \kappa$ has order type ω and that $X \setminus A$ is finite for all $A \in U$. We claim that

$$G = \{(s, A) \in \mathbb{P}_U \mid s \text{ is an initial segment of } X \text{ and } X \setminus s \subseteq A\}$$

is $\mathbb{P}_U$-generic over M.

Suppose that $D \subseteq \mathbb{P}_U$, $D \in M$ is dense open. For $s \in [\kappa]^{<\omega}$, let $F_s \colon [\kappa]^{<\omega} \to 2$ with $F_s(t) = 1$ if and only if $\max(s) < \min(t)$ and $\exists X(s \cup t, X) \in D$. Suppose that $A_s \in U$ is homogeneous for F_s. If there is an Y such that $(s, Y) \in D$, let

$B_s = A_s \cap Y$. Let $B_s = A_s$ otherwise. Let $B_\alpha := \bigcap\{B_s \mid s \in [\kappa]^{<\omega},\ \max(s) = \alpha\}$
Let

$$A := \bigtriangleup_{s\in[\kappa]^{<\omega}} B_s := \{\beta < \kappa \mid \beta \in \bigcap_{\max(s)<\beta} B_s\} = \bigtriangleup_{\alpha} B_{\alpha<\kappa} \in U.$$

Since D is open dense, we have the following for all $s \in [\kappa]^{<\omega}$. +

Claim 3.26. If $\exists Y\ (s, Y) \in D$, then $(s, A \setminus s) \in D$.

X has an initial segment s such that $X \setminus s \subseteq A$.

Since D is dense, there is some $t \in [B \setminus s]^{<\omega}$ and Y with $(s \cup t, Y) \in D$. Let $u \subseteq X \setminus s$ be such that $|u| = |t|$. Since $A \setminus s \subseteq A_s$ is homogeneous for F_s, there is some Z with $(s \cup u, Z) \in D$.

Then $(s \cup u, A \setminus (s \cup u)) \in D$ by Claim 3.26. Since $(s \cup u, A \setminus (s \cup u)) \in G$, $G \cap D \neq \emptyset$. □

4. Indestructibility

May 13

Large cardinals are useful in many forcing constructions, as we have seen for instance in the iteration for PFA. It is often useful to know that large cardinal properties are preserved under certain forcings. For example, most large cardinals κ are preserved under forcings of size $< \kappa$. For forcings of size $\geq \kappa$, κ could be collapsed, so we can only consider forcings which meet additional requirements. By a result of Laver, any supercompact cardinal κ can be made indestructible under $< \kappa$-directed closed forcing. To prove this, we will first characterize supercompact cardinals by the existence of normal measures on $P_\kappa(\lambda)$. We will also need some lemmas on how to factor an iterated forcing into an initial segment of the original iteration and a name for an iteration, with similar properties as the original iteration.

4.1. Supercompact cardinals and filters on $P_\kappa(\lambda)$.

Definition 4.1. Suppose that $\kappa \leq \lambda$ are cardinals.

(1) Let $P_\kappa(\lambda) = \{A \subseteq \lambda \mid |A| < \kappa\}$.
(2) An ultrafilter on $P_\kappa(\lambda)$ is *fine* if $\hat{x} = \{y \in P_\kappa(\lambda) \mid x \subseteq y\} \in U$ for all $x \in P_\kappa(\lambda)$.
(3) An ultrafilter on $P_\kappa(\lambda)$ is *normal* if it is fine and for every (regressive) function $f\colon P_\kappa(\lambda) \to \lambda$ with $f(x) \in x$ for almost all x, f is constant on a set in the filter.

Lemma 4.2. *Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$ and that j_U is the ultrapower.*

(1) For all $X \in P_\kappa(\lambda)$, $X \in U$ if and only if $[id] \in j(X)$.
(2) $[id] = j[\lambda]$.

Proof. If $\gamma < \lambda$, then $\gamma \in x$ for almost all $x \in P_\kappa(\lambda)$ and hence $j(\gamma) \in [id]$.

Suppose that $[f] \in [id]$. Then $f(x) \in x$ for almost all x. Since U is normal, there is some $\gamma < \lambda$ such that $[f] = j(\gamma)$. □

Lemma 4.3. *A cardinal κ is λ-supercompact for $\lambda \geq \kappa$ if and only if there is a normal filter on $P_\kappa(\lambda)$.*

Proof. Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$ and $j_U\colon V \to Ult_U(V)$ is the ultrapower. We claim that j is λ-supercompact.

Since $Ult_U(V)$ is wellfounded, we can assume that it is transitive. Since U is $< \kappa$-complete, $j \restriction \kappa = id$. We have $j(\kappa) > [id] \geq \kappa$.

Let $M = Ult_U(V)$. We claim that $M^\lambda \subseteq M$. Suppose that $(a_\alpha)_{\alpha<\lambda}$ is a sequence with $a_\alpha = [f_\alpha] \in M$ for all $\alpha < \kappa$. It is sufficient to show that $\{a_\alpha \mid \alpha < \kappa\} \in M$.

Let $f\colon P_\kappa(\lambda) \to V$, $f(x) = \{f_\alpha(x) \mid \alpha \in x\}$.

Claim 4.4. $[f] = \{a_\alpha \mid \alpha < \lambda\}$.

Proof. If $\alpha < \lambda$, then $\alpha \in x$ for almost all x, so $[f_\alpha] \in [f]$.

If $[g] \in [f]$, then for almost all x, $g(x) = f_\alpha(x)$ for some $\alpha \in x$. Since U is normal, there is some $\alpha < \lambda$ such that $g(x) = f_\alpha(x)$ for almost all x. Hence $[g] = a_\alpha$. □

Now suppose that $j\colon V \to M$ is λ-supercompact. Let $X \in U$ if $X \subseteq P_\kappa(\lambda)$ and $j[\lambda] \in j(X)$.

It is straightforward that U is $< \kappa$-complete.

U is fine since for every $\alpha < \lambda$, $\{x \mid \alpha \in x\} \in U$.

To see that U is normal, suppose that $f\colon P_\kappa(\lambda) \to \lambda$ is regressive. Then $(j(f))(j[\lambda]) \in j[\lambda]$. Hence $(j(f))(j[\lambda]) = j(\alpha)$ for some $\alpha < \lambda$, so $f(x) = \alpha$ for almost all x. □

Corollary 4.5. *There is a first order formula φ such that $\varphi(\kappa)$ holds if and only if κ is supercompact.*

Corollary 4.6. *Suppose that $\kappa \leq \lambda$ are cardinals and κ is λ-supercompact. Then there is a λ-supercompact embedding $j\colon V \to M$ with $crit(j) = \kappa$ and $|(2^{j(\kappa)})^M|^V \leq |2^{(\lambda^{<\kappa})}|^V$.*

Proof. Suppose that U is a normal ultrafilter on $P_\kappa(\lambda)$ and j_U is the ultrapower. The elements of $P(j(\kappa))^{Ult_U(V)}$ are represented by functions $f\colon P_\kappa(\lambda) \to P(\kappa)$. There are $((2^\kappa)^{\lambda^{<\kappa}})^V = (2^{(\lambda^{<\kappa})})^V$ many such functions. □

4.2. **Indestructibility under $<\kappa$-directed closed forcing.**

Definition 4.7. Suppose that κ is a cardinal. A forcing $\mathbb{P}$ is $<\kappa$-directed closed if for every directed set $A \subseteq \mathbb{P}$ with $|A| < \kappa$, there is some $p \in \mathbb{P}$ with $p \leq q$ for all $q \in A$.

Lemma 4.8. *Suppose that $\kappa > \omega$ is regular. If $\mathbb{P}$ is $<\kappa$-directed closed and $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $<\kappa$-directed closed, then $\mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-directed closed.*

Proof. Suppose that $D \subseteq \mathbb{P} * \dot{\mathbb{Q}}$ is $<\kappa$-directed closed. Then $E := \{p \in \mathbb{P} \mid \exists \dot{q} \mid (p, \dot{q}) \in D\}$ is $<\kappa$-directed closed. Suppose that $p_0 \in \mathbb{P}$ with $p_0 \leq p$ for all $p \in E$. Let $F := \{\dot{q} \mid \exists p \mid (p, \dot{q}) \in D\}$. Then p_0 forces that F is $<\kappa$-directed closed. There is a name $\dot{q}_0$ such that p_0 forces that $\dot{q}_0 \leq \dot{q}$ for all $\dot{q} \in E$. Then $(p_0, \dot{q}_0) \leq (p, \dot{q})$ for all $(p, \dot{q}) \in D$. □

Lemma 4.9. *Suppose that $\kappa, \eta > \omega$ are regular cardinals. Suppose that $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta) \mid \alpha \leq \kappa,\ \beta < \kappa)$ is a forcing iteration such that*

(i) *$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is $<\eta$-directed closed for all $\alpha < \kappa$ and*
(ii) *all limits are direct or inverse, and inverse limits are taken at every limit stage with cofinality less than η.*

Then $\mathbb{P}_\kappa$ is $<\eta$-directed closed.

Proof. As in the previous proof. □

Lemma 4.10. *Suppose that κ is inaccessible and $\mathbb{P}_\kappa$ is a forcing iteration of length κ such that*

(i) *$\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha \in V_\kappa$ for all $\alpha < \kappa$,*
(ii) *a direct limit is taken at κ, and*
(iii) *a direct limit is taken at a stationary set of limit stages below κ.*

Then $\mathbb{P}_\kappa$ satisfies the κ-c.c.

Proof. Suppose that $A = \{p_\alpha \mid \alpha < \kappa\}$ is an antichain in P_κ of size κ. Let $f\colon \kappa \to \kappa$, $f(\alpha) = \sup(supp(p_\alpha))$.

Then f is regressive on a stationary subset of κ. Then there is a stationary set $S \subseteq \kappa$ and $\gamma < \kappa$ such that $f(\alpha) = \gamma$ for all $\alpha \in S$.

We can assume that for all $\alpha < \beta$ in S, $supp(p_\alpha) \subseteq \beta$, by thinning out S.

Since $\mathbb{P}_\gamma$ satisfies the κ-c.c., there are $\alpha < \beta$ in S such that $p_\alpha \upharpoonright \gamma$ and $p_\beta \upharpoonright \gamma$ are compatible. Then p_α, p_β are compatible. □

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Definition 4.11. Suppose that $\mathbb{P}$ is a forcing. Let $\langle\rangle_{\mathbb{P}} = \langle\rangle$ denote the canonical function which maps $\mathbb{P}$-names σ, τ to a $\mathbb{P}$-name for the pair (σ, τ).

Lemma 4.12. *Suppose that $(\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$ is a 3-step iteration and η is a regular cardinal. Then there is a $\mathbb{P}$-name $\dot{\mathbb{S}}$ for a 2-step iteration $\dot{\mathbb{Q}} * \bar{\mathbb{R}}$ such that*

(i) *There is an isomorphism $\pi\colon (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \to \mathbb{P} * \dot{\mathbb{S}}$.*

(ii) *If $1_{\mathbb{P}} \Vdash \dot{\mathbb{Q}}$ is $< \eta$-directed closed and $1_{\mathbb{P}*\dot{\mathbb{Q}}} \Vdash \dot{\mathbb{R}}$ is $< \eta$-directed closed, then $1_{\mathbb{P}} \Vdash (1_{\dot{\mathbb{Q}}} \Vdash \bar{\mathbb{R}}$ is $< \eta$-directed closed).*

Proof. We define maps $\xi, \bar{\xi}$ which convert a $\mathbb{P}*\dot{\mathbb{Q}}$-name into a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name and conversely. By induction in $rk(\tau)$ let

$$\xi(\tau) = \{(\langle \xi(\sigma), \dot{q}\rangle, p) \mid (\sigma, (p, \dot{q})) \in \tau\}$$
$$\bar{\xi}(\tau) = \{(\bar{\xi}(\sigma), (o, \dot{q}) \mid (\langle \sigma, \dot{q}\rangle, p) \in \tau\}.$$

Let $\bar{\mathbb{R}} := \xi(\dot{\mathbb{R}})$.

Claim 4.13. $\bar{\xi} = \xi^{-1}$. Suppose that τ is a $\mathbb{P}$-name, $G \times H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V. Then $\tau^{G\times H} = (\xi(\tau)^G)^H$.

Proof. By induction on $rk(\tau)$, $x \in \tau^{G\times H} \Leftrightarrow \exists (p, \dot{q}) \in G\times H\ (\sigma, (p, \dot{q})) \in \tau$, $\sigma^{G\times H} = x \Leftrightarrow \exists p \in G\ \exists \dot{q}\ (\dot{q}^G = q \wedge (\langle \xi(\sigma), \dot{q}\rangle, p) \in \xi(\tau) \wedge (\xi(\sigma)^G)^H = x \Leftrightarrow x \in (\xi(\tau)^G)^H$. □

Claim 4.14. Suppose that τ is a $\mathbb{P}$-name for a $\dot{\mathbb{Q}}$-name and $G * H$ is $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V. Then $(\tau^G)^H = \bar{\xi}(\tau)^{G\times H}$.

Proof. $\bar{\xi}(\tau)^{G*H} = (\xi(\bar{\xi}(\tau)^G)^H = (\tau^G)^H$ by the previous claim. □

Claim 4.15. If τ is a full $\mathbb{P} * \dot{\mathbb{Q}}$-name, then $\Vdash_{\mathbb{P}} \xi(\tau)$ is a full $\dot{\mathbb{Q}}$-name.

Proof. Suppose that $\Vdash_{\mathbb{P}} (\Vdash_{\dot{\mathbb{Q}}} \bar{\sigma} \in \xi(\tau))$. Let $\sigma = \bar{\xi}(\bar{\sigma})$. Then $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \sigma \in \bar{\xi}(\xi(\tau)) = \tau$. There is $\nu \in dom(\tau)$ with $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \sigma = \nu$. Let $\bar{\nu} = \xi(\nu)$.

Then $\Vdash_{\mathbb{P}} (\Vdash_{\dot{\mathbb{Q}}} \xi(\sigma) = \xi(\bar{\xi}(\bar{\sigma})) = \bar{\sigma} = \xi(\nu) = \bar{\nu})$ by a claim above.

To see that $\Vdash_{\mathbb{P}} \bar{\nu} \in dom(\xi(\tau))$, note that $\nu \in dom(\tau)$ and $\xi(\tau) = \{(\langle \xi(\sigma), \dot{q}\rangle, p) \mid (\mu, (p\dot{q})) \in \tau\}$. For $\mu = \nu$, $\xi(\mu) = \xi(\nu) = \bar{\nu}$. This implies the claim. □

Let $\dot{\mathbb{S}}$ denote a $\mathbb{P}$-name for $\mathbb{Q} * \bar{\mathbb{R}}$. Let $\pi\colon (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}} \to \mathbb{P} * \dot{\mathbb{S}}$, $\pi((p, \dot{q}), \dot{r}) = (p, \langle \dot{q}, \xi(\dot{r}\rangle)$.

Claim 4.16. For all $a, b \in (\mathbb{P} * \dot{\mathbb{Q}}) * \dot{\mathbb{R}}$, $a \leq b$ if and only if $\xi(a) \leq \xi(b)$.

Proof. By induction on $rk(p)$, $rk(q)$. Let $((p, \dot{q}), \dot{r}) \leq ((u, \dot{v}), \dot{w}) \Leftrightarrow G * H$ $\mathbb{P} * \dot{\mathbb{Q}}$-generic over V below $(p, \dot{q})$, $\dot{r}^{G*H} \leq \dot{w}^{G*H} \Leftrightarrow$ for all $\mathbb{P}$-generic G below p over V and $\dot{\mathbb{Q}}^G$-generic H below $\dot{q}^G$ over $V[G]$, $(\xi(\dot{r})^G)^H \leq (\xi(\dot{w}^G)^H \Leftrightarrow (\dot{p}, \langle \dot{q}, \xi(\dot{r})\rangle) \leq (\dot{u}, \langle \dot{v}, \xi(\dot{w})\rangle)$. □

Claim 4.17. For all $a, b \in \mathbb{P} * \dot{\mathbb{S}}$, $a \leq b$ if and only if $\bar{\xi}(a) \leq \bar{\xi}(b)$.

Proof. $\alpha = \xi(\bar{\xi}(\alpha)) \leq b = \xi(\bar{\xi}(b))$ if and only if $\bar{\xi}(a) \leq \bar{\xi}(b)$ by the previous claim. □

Claim 4.18. Suppose that η is a cardainl. If $\Vdash_{\mathbb{P}} \dot{\mathbb{Q}}$ is $< \eta$-directed closed and $\Vdash_{\mathbb{P}*\dot{\mathbb{Q}}} \dot{\mathbb{R}}$ is $< \eta$-directed closed, then $\Vdash_{\mathbb{P}} (\Vdash_{\dot{\mathbb{Q}}} \bar{\mathbb{R}}$ is $< \eta$-directed closed.

Proof. By a lemma above. □

□

Lemma 4.19. *Suppose that $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha \leq \gamma+\delta}$ is an iteration of separative forcings such that $\mathbb{P}_\beta$ is a direct limit for $\beta \in D$, and an inverse limit for $\beta \in I$, where $\gamma + \delta = D \sqcup I$.*

Suppose that $\gamma + \alpha \in D$, i.e. $\mathbb{P}_{\gamma+\alpha}$ is a direct limit, for every limit ordinal $\alpha \leq \delta$ with $\operatorname{cof}(\alpha) \leq |\mathbb{P}_\gamma|$. Suppose that η is regular and inverse limits are taken at every limit of cofinalty $< \eta$.

Then there is a $\mathbb{P}_\gamma$-name for an iteration $(\dot{\mathbb{P}}^{(\gamma)}_\alpha, \dot{\mathbb{Q}}^{(\gamma)}_\alpha)_{\alpha \leq \delta}$ with the following properties.

(i) *for each $\alpha \leq \delta$, there is an isomorphism $\pi^{\gamma,\alpha} \colon \mathbb{P}_{\gamma+\alpha} \to \mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_\alpha$.*
(ii) *$1_{\mathbb{P}_\gamma}$ forces that $\dot{\mathbb{P}}^{(\gamma)}_\alpha$ is a direct limit if $\gamma + \alpha \in D$, and an inverse limit if $\gamma + \alpha \in I$.*
(iii) *If $1_{\mathbb{P}_\alpha} \Vdash \dot{\mathbb{Q}}_\alpha$ is $< \eta$-directed closed for all $\alpha < \gamma + \delta$, then $1_{\mathbb{P}_\alpha}$ forces that $1_{\dot{\mathbb{P}}^{(\alpha)}_\gamma} \Vdash$ "$\dot{\mathbb{Q}}^{(\alpha)}_\gamma$" is $< \eta$-directed closed for all $\alpha < \delta$.*

Proof. In the successor step, we need to construct $\pi^{\gamma,\alpha+1} \colon \mathbb{P}_{\gamma+\alpha+1} \to \mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_{\alpha+1}$. Note that $\mathbb{P}_{\gamma+\alpha} \cong \mathbb{P}_{\gamma+\alpha} * \dot{\mathbb{Q}}_{\gamma+\alpha} \cong (\mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_\alpha) * \dot{\mathbb{Q}}_{\gamma+\alpha}$.

By the previous lemma, there are a $\mathbb{P}_\gamma$-name $\dot{\mathbb{Q}}^{(\gamma)}_\alpha$ for a $\dot{\mathbb{P}}^{(\gamma)}_\alpha$-name, a $\mathbb{P}_\gamma$-name $\dot{\mathbb{S}}$ for $\dot{\mathbb{P}}^{(\gamma)}_\alpha * \dot{\mathbb{Q}}^{(\gamma)}_\alpha$, and an isomorphism $(\mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_\alpha) * \dot{\mathbb{Q}}_{\gamma+\alpha} \to \mathbb{P}_\gamma * \dot{\mathbb{S}}$. This yields $\pi^{\gamma,\alpha+1}$.

In limit steps λ, we define $\dot{\mathbb{P}}^{(\gamma)}_\lambda$ as a $\mathbb{P}_\gamma$-name for the inverse or direct limit of $(\dot{\mathbb{P}}^{(\gamma)}_\alpha)_{\alpha<\lambda}$. Let $\pi^{\gamma,\lambda} \colon \mathbb{P}_{\gamma+\lambda} \to \mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_\lambda$, $\pi^{\gamma,\lambda}(p) = (p \restriction \gamma) * p^\gamma$, where p^γ is a name for $(\pi^{\gamma,\alpha}(p)_1)_{\alpha<\lambda}$.

In the inverse limit case, it follows by induction that $\pi^{\gamma,\lambda}$ is an isomorphism. In the direct limit case, suppose that $p \in \mathbb{P}_{\gamma+\lambda}$ and $\sup(supp(p)) = \delta$. Then $p \restriction \gamma \Vdash_{\mathbb{P}_\gamma} \sup(supp(\pi^{\gamma,\lambda}(p)_1)) \leq \delta$. So $\pi^{\gamma,\lambda}$ is well-defined. To see that $\pi^{\gamma,\lambda}$ is an isomorphism, suppose that $(p, \dot{q}) \in \mathbb{P}_\gamma * \dot{\mathbb{P}}^{(\gamma)}_\lambda$. Then by definition of $\dot{\mathbb{P}}^{(\gamma)}_\lambda$, $p \Vdash_{\mathbb{P}_\gamma} \sup(supp(\dot{q})) < \lambda$. Since $|\mathbb{P}_\gamma| < cof(\lambda)$ by our assumption, there is some $\delta < \lambda$ such that $p \Vdash_{\mathbb{P}_\gamma} \sup(supp(\dot{q}) \leq \delta$. □

Definition 4.20. A forcing iteration has *Easton support* if direct limits are taken at all regular limit cardinals and inverse limits are taken at singular limit ordinals.

Definition 4.21. (1) $(\mathbb{Q}, \vartheta, \kappa)$ is a *counterexample* if
  (i) $\mathbb{Q}$ is a $< \kappa$-directed closed forcing,
  (ii) κ is ϑ-supercompact, and
  (iii) $1 \Vdash_{\mathbb{Q}}$ "κ is not ϑ-supercompact".

(2) $(\mathbb{Q}, \vartheta, \kappa)$ is a *minimal counterexample* if (ϑ, η) is lexicographically least such that $(\mathbb{Q}, \vartheta, \kappa)$ is a counterexample and $|tc(\mathbb{Q})| = \eta$.

Definition 4.22. We define an Easton support iteration $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\kappa}$ and a sequence $(\vartheta_\alpha, \eta_\alpha)_{\alpha<\kappa}$ as follows. Suppose that $\mathbb{P}_\gamma$ is defined and that $\vartheta_\alpha, \eta_\alpha$ are defined for all $\alpha < \gamma$.

(i) If $\gamma < \vartheta_\alpha, \eta_\alpha$ for all $\alpha < \gamma$, let $\dot{\mathbb{Q}}_\gamma$ denote a $\mathbb{P}_\gamma$-name for the lottery sum of all forcings $\mathbb{Q}$ with $|tc(\mathbb{Q}| < \kappa$ such that $(\mathbb{Q}, \vartheta, \gamma)$ is a minimal counterexample for some $\vartheta < \kappa$. Let $\vartheta_\gamma = \vartheta$, $\eta_\gamma := |tc(\mathbb{Q})|$ for such $\vartheta, \mathbb{Q}$.

(ii) Let $\dot{\mathbb{Q}}_\gamma$ denote a $\mathbb{P}_\gamma$-name for the trivial forcing $\{1\}$ otherwise.

Theorem 4.23. *(Laver) Suppose that κ is supercompact and that G is $\mathbb{P}_\kappa$-generic over V. Suppose that $\mathbb{Q}$ is a $<\kappa$-directed closed forcing in $V[G]$ and that h is $\mathbb{Q}$-generic over $V[G]$. Then κ is supercompact in $V[G*h]$.*

Proof. Suppose that in $V[G]$, $(\mathbb{Q}, \vartheta, \kappa)$ is a minimal counterexample of size η. Let $\mu := \max\{2^{\vartheta^{<\kappa}}, \eta\}$.

Suppose that $j\colon V \to M$ is μ-supercompact. By the factor lemma, $M \vDash j(\mathbb{P}_\kappa) \cong \mathbb{P}_\kappa * \dot{\mathbb{Q}}_\kappa * \dot{\mathbb{R}}$ for some $\dot{\mathbb{Q}}_\kappa$, $\dot{\mathbb{R}}$ as in the factor lemma.

Claim 4.24. There is $\dot{\mathbb{Q}}_\kappa^G$-generic filter g over $V[G]$ which chooses $\mathbb{Q}$.

Proof. Since $M^\mu \subseteq M$ and $|\mathbb{Q}| \leq \mu$, $\mathbb{Q} \in M$. Then $P(P_\kappa(\vartheta))^{V[G*h]} = P(P_\kappa(\vartheta))^{M[G*h]}$ for every $\mathbb{Q}$-generic filter h over $V[G]$. So $\mathbb{Q}$ destroys the ϑ-supercompactness of κ over $M[G]$.

Moreover, since $M^\mu \subseteq M$, for every forcing $\mathbb{R} \in M$ with $|\mathbb{R}| < |\mathbb{Q}|$ and every $\mathbb{R}$-generic filter i over $V[G]$, $P(P_\kappa(\vartheta))^{V[G*i]} = P(P_\kappa(\vartheta))^{M[G*i]}$. So $\mathbb{Q}$ has minimal size.

Suppose that H is $\dot{\mathbb{R}}^{G*g}$-generic over $V[G*g]$. Let $j_0\colon V[G] \to M[G*g*H]$, $j_0(\sigma^G) := j(\sigma)^{G*g*H}$.

Claim 4.25. j_0 is well-defined and elementary.

Proof. Suppose that $\sigma^G = \tau^G$. THen $p \Vdash \sigma = \tau$ for some $p \in G$. Then $p^\frown 1^{j(\kappa)} = j(p) \Vdash j(\sigma) = j(\tau)$. Since $p^\frown 1^{j(\kappa)} \in G*g*H$, this implies that $j_0(\sigma^G) = j(\sigma)^{G*g*H} = j(\tau)^{G*g*H} = j_0(\tau^G)$.

The elementarity is proved similarly. □

We have $j \restriction \mathbb{Q} \in M[G]$, since $M[G]^\mu \subseteq M[G]$ by Lemma 1.12. So $j[g] \in M[G*g]$. insert tag
Since $j[g] \subseteq \dot{\mathbb{Q}}_\kappa^G$ is directed and $j(\mathbb{Q})$ is $< j(\kappa)$-directed closed in $M[G*g*H]$, there is a master condition $q_0 \in j(\mathbb{Q})$ with $q_0 \leq q$ for all $q \in j[g]$. Suppose that $\dot{q}_0$ is a $j(\mathbb{P}_\kappa)$-name with $\dot{q}_0^{G*g*H} = q_0$.

Let $j_1\colon V[G*g] \to M[G*g*H*h]$, $j_1(\sigma^{G*g}) := j(\sigma)^{G*g*H*h}$.

Claim 4.26. j_1 is well-defined and elementary.

Proof. Suppose that $\sigma^{G*g} = \tau^{G*g}$. Then $(p, \dot{q}) |Vdashn \sigma = \tau$ for some $(p, \dot{q}) \in G*g$. Then $(p^\frown 1^{j(\kappa)}, \dot{q}_0) \leq (p^\frown 1^{j(\kappa)}, j(\dot{q})) = (j(p), j(\dot{q})) \Vdash j(\sigma) = j(\tau)$. Since $(p^\frown 1^{j(\kappa)}, \dot{q}_0^{j(\mathbb{P}_\kappa)}) \in G*g*H*h$, this implies that $j_1(\sigma^{G*g}) = j_1^{G*g})$.

The elementarity of j_1 is proved similarly. □

Let $U := \{X \subseteq P_\kappa(\vartheta) \mid X \in V[G*g],\ j[\vartheta] \in j_1(X)\}$.

Claim 4.27. U is normal with respect to regressive functions $f\colon P_\kappa(\vartheta) \to \vartheta$, $f \in V[G*g]$.

Proof. Analogous to the proof that the ultrafilter induces by a ϑ-supercompact embedding is normal. □

In $M[G*g]$, the forcing $\dot{\mathbb{R}}^{G*g} * j(\mathbb{Q})$ is $<\mu$-closed, by the definition of $(\mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha)_{\alpha<\kappa}$ and by the factor lemma.

Since $M[G*g]^\mu \subseteq M[G*g]$ in $V[G*g]$ by Lemma 1.12, $\dot{\mathbb{R}}^{G*g} * j(\mathbb{Q})$ is μ-closed in tag lemma
$V[G*g]$. Hence κ is ϑ-supercompact in $V[G*g]$, contradicting the assumption. □

Problem 4.28. *Suppose that κ is weakly compact and consider the Easton support iteration which forces with $Add(\lambda, 1)$ at all regular cardinals $\lambda < \kappa$. Show that in the generic extension and in every further extension by $Add(\kappa, 1)$, κ is weakly compact.*

4.3. **SCH.**

Definition 4.29. (i) Suppose that κ is a singular strong limit cardinal. The *singular cardinal hypothesis* (SCH) holds at κ if $2^\kappa = \kappa^+$.
(ii) The *singular cardinal hypothesis* (SCH) holds if SCH holds at every singular strong limit cardinal.

Corollary 4.30. *Suppose that κ is supercompact. There is a generic extension $V[G]$ of V in which*
(i) *κ is a strong limit cardinal,*
(ii) $\mathrm{cof}(\kappa) = \omega$*, and*
(iii) $2^\kappa > \kappa^+$.

Proof. Suppose that g is $\mathbb{P}_\kappa$-generic over V and h is $Add(\kappa, \kappa^+ +)$-generic over $V[G]$. Then κ is supercopmact in $V[G * h]$.

Suppose that in $V[G * h]$, U is a normal ultrafilter on κ and i is $\mathbb{P}_U$-generic over $V[G * h]$, where $\mathbb{P}_U$ denotes Prikry forcing. Then $V[g * h(i]$ satisfies the requirements. □

4.4. $<\kappa$**-closed forcings.** The following lemmas show that no measurable cardinal is indestructible under $<\kappa$-closed forcing.

Definition 4.31. Suppose that $\kappa > \omega$ is regular. A *thin κ-Kurepa tree* is a tree $(T, \leq_T)$ such that
(i) $0 < |\mathrm{Lev}_\alpha(T)| \leq |\alpha|$ for all $\alpha < \kappa$ and
(ii) $|[T]| > \kappa$,
where $[T]$ denotes that set of branches $b \in T^\kappa$ of length κ through T.

Lemma 4.32. *Suppose that κ is measurable. Then there is no thin κ-Kurepa tree.*

Proof. We can assume that $T \subseteq V_\kappa$. Suppose that $j\colon V \to M$ is elementary with M transitive and $\mathrm{crit}(j) = \kappa$. Since $|\mathrm{Lev}_\alpha(T)| \leq |\alpha| < \kappa$ for $\alpha < \kappa$, $\mathrm{Lev}_\alpha(j(T)] = \mathrm{Lev}_\alpha(T)$ for $\alpha < \kappa$. If $b \in [T]$, then $M \vDash b \subseteq j(b) \in [j(T)]$ and $j(b)(\kappa)$ extends b. So $|\mathrm{Lev}_\kappa(j(T))| > \kappa$. □

Lemma 4.33. *Suppose that κ is measurable. Then there is a $<\kappa$-closed forcing which adds a thin κ-Kurepa tree.*

Proof. Let $\mathbb{P}$ denote the following forcing.
(1) (a) Conditions are pairs (p, f) where
 (i) $p = (\alpha_p, \leq_p)$ is a tree such that
 (b) (i) $|p| < \kappa$,
 (ii) $|\mathrm{Lev}_\alpha(p)| \leq |\alpha|$ for all $\alpha < \kappa$,
 (iii) every $\xi < \alpha_p$ has two incompatible extensions in p, unless $\mathrm{ht}(p) = \beta + 1$ and $\xi \in \mathrm{Lev}_\beta(p)$,
 (iv) if $\beta < \mathrm{ht}(p)$ is a limit ordinal and b is a branch in p of height β, then b has at most one extension in $\mathrm{Lev}_\beta(p)$.
 (The last two conditions define a *normal* tree, and will make sure that the generic κ-Kurepa tree is normal. This is not necessary for our claim).
 (c) $f\colon \mathrm{dom}(f) \to [p]$ is injective with
 (i) $\mathrm{dom}(f) \leq \kappa^+$,
 (ii) $|\mathrm{dom}(f)| < \kappa$.
(2) The ordering is defined by $(p, f) \leq (q, g)$ if
 (a) p is an end extension of q, i.e.
 (i) $\alpha_p \leq \alpha_q$,
 (ii) $\leq_q = \leq_[\cap (\alpha_q \times \alpha_q)$,

(iii) if $\xi \in \alpha_p \setminus \alpha_q$, then $\mathrm{ht}_p(\xi) \supseteq \mathrm{ht}(q)$.

(b) $\mathrm{dom}(f) \supseteq \mathrm{dom}(g)$ and for every $\alpha \in \mathrm{dom}(g)$, the branch $g(\alpha)$ is an initial segment of $f(\alpha)$.

Claim 4.34. $\mathbb{P}$ is $< \kappa$-closed.

Proof. Suppose that $(p_\beta, f_\beta)_{\beta<\gamma}$ is decreasing in $\mathbb{P}$ with $\gamma < \kappa$. Let

$$p := (\bigcup_{\beta<\gamma} \alpha_{p_\beta}, \bigcup_{\beta<\gamma} \leq_{p_\beta}).$$

Let f denote the function with domain $\bigcup_{\beta<\gamma} \mathrm{dom}(f_\beta)$ such that $f(\delta)$ is the unique branch $b \in [p]$ of length $\mathrm{ht}(p)$ with $f_\beta(\delta) \subseteq f(\delta)$ for all β with $\delta \in \mathrm{dom}(f_\beta)$. Then $(p, f) \in \mathbb{P}$ is a lower bound. □

Claim 4.35. $\mathbb{P}$ satisfies the κ^+-c.c.

Proof. Straightforward with the Δ-system lemma. □

Suppose that G is $\mathbb{P}$-generic over V. Let $T := \bigcup\{\alpha_p \mid \exists f\ (p, f) \in G\} = \kappa$ and $\leq_T := \bigcup\{\leq_p \mid \exists f\ (p, f) \in G\}$.

Claim 4.36. $(T, \leq_T)$ is a thin κ-Kurepa tree.

Proof. $|\mathrm{Lev}_\alpha(T)| < \kappa$ for $\alpha < \kappa$, since the conditions are ordered by end extension. Moreover, the function $\bigcup\{f \mid \exists p\ (p, f) \in G\} \colon \kappa^+ \to [T]$ is injective. □

This completes the proof of the lemma. □

Problem 4.37. (i) *Decide whether the forcing to add a κ-Kurepa tree for a regular cardinal κ with $\kappa^{<\kappa} = \kappa$ is $< \kappa$-directed closed.*

(ii) *Is there a $< \kappa$-closed but not $< \kappa$-directed closed forcing for every $\kappa > \omega$?*

5. Generic ultrapowers

We will now consider some large cardinal properties of small cardinals. For example if PFA holds, then ω_2 has some properties resembling a supercompact cardinal. In the following, we will consider generic elementary embeddings with critical point ω_1, so that ω_1 resembles a measurable cardinal.

Definition 5.1. Suppose that $\kappa > \omega$ is regular. An ideal on κ is a set $I \subsetneq P(\kappa)$ such that

(i) $\emptyset \in I$,
(ii) $\kappa \notin I$,
(iii) I is downwards closed,
(iv) I is closedd under finite unions.

An ideal I is κ-complete if I is closed under unions of size $< \kappa$.

Definition 5.2. Suppose that $\kappa > \omega$ is regular and I is an ideal on κ.

(i) $I^+ := \{X \subseteq \kappa \mid X \notin I\}$.
(ii) $I^* := \{X \subseteq \kappa \mid \kappa \setminus X \in I\}$ the filter dual to I.
(iii) $\mathbb{P}_I := I^+$, $X \leq Y$ if $X \subseteq Y$.

The separative quotient of $\mathbb{P}_I$ is isomorphic to $\mathbb{B}^* = \mathbb{B} \setminus \{0_{\mathbb{B}}\}$, where $\mathbb{B} := P(\kappa)/I$.

Lemma 5.3. *An ideal I on κ is* κ-complete *if and only if for every partition $\kappa = \bigsqcup_{\alpha<\gamma} X_\alpha$ with $\gamma < \kappa$, $X_\alpha \in I^+$ for some $\alpha < \gamma$.*

Proof. If I is κ-complete, $\kappa = \bigsqcup_{\alpha<\gamma} X_\alpha$, $\gamma < \kappa$, and $X_\alpha \in I$ for all $\alpha < \gamma$, then $\kappa \in I$.

Suppose that $(X_\alpha)_{\alpha<\gamma}$ is increasing with $X_\alpha \in I$ and $\gamma < \kappa$. Let $Y_\beta = X_\beta \setminus \bigcup_{\alpha<\beta} X_\alpha$ for $\beta < \gamma$ and $Y_\gamma = \kappa \setminus \bigcup_{\alpha<\gamma} X_\alpha$. Then $\kappa = \bigsqcup_{\alpha<\gamma+1} Y_\alpha$. Since $Y_\alpha \in I$ for $\alpha < \gamma$, $Y_\gamma \in I^+$. □

Definition 5.4. Suppose that M is a transitive model of a fragment of ZFC.

(i) An *M-ultrafilter* is a filter U on $P(\kappa)^M$ such that for every $X \in P(\kappa)^M$, $X \in U$ or $\kappa \setminus X \in U$.

(ii) An M-ultrafilter U is κ-complete if it is κ-complete with respect to sequences $(X_\alpha)_{\alpha<\gamma} \in M$.

Lemma 5.5. *Suppose that G is $\mathbb{P}_I$-generic over V. Then*

(1) *G is a V-ultrafilter on κ and $I^* \subseteq G$.*

(2) *If I is κ-complete, the G is a κ-complete V-ultrafilter.*

Proof. (1) If $X \in P(\kappa)^V$, then $\{Y \in \mathbb{P}_I \mid Y \subseteq X \vee Y \subseteq \kappa \setminus X\}$ is dense in $\mathbb{P}_I$. So G is a V-ultrafilter. If $X \in I^*$, then $\{Y \in \mathbb{P}_I \mid Y \subseteq X\}$ is dense in $\mathbb{P}_I$. So $X \in G$.

(2) Suppose that $\kappa = \bigsqcup_{\alpha<\gamma} Y_\alpha$ with $\gamma < \kappa$ and $(Y_\alpha)_{\alpha<\gamma} \in V$. Since$I$ is κ-complete, the set $\{X \in \mathbb{P}_I \mid X \subseteq Y_\alpha$ for some $\alpha < \gamma\}$ is dense in $\mathbb{P}_I$. So $Y_\alpha \in G$ for some $\alpha < \gamma$. Therefore G is κ-complete. □

Definition 5.6. Suppose that $\kappa > \omega$ is regular and i is a κ-complete nonprincipal ideal on κ. Suppose that G is $\mathbb{P}_I$-generic over V. Then $Ult_G(V)$ is called a *generic ultrapower*.

Lemma 5.7. (1) *The ultrapower map $j\colon V \to Ult_G(V)$, $j(x) = [c_x]$, is elementary.*

(2) *Los' theorem holds for $Ult_G(V)$, i.e. for all $f_i\colon \kappa \to V$ with $f_i \in V$, $Ult_G(V) \models \varphi([f_0], ..., [f_n])$ if and only if $\{\alpha < \kappa \mid V \models \varphi(f_0(\alpha), ..., f_n(\alpha))\} \in G$.*

(3) *$crit(j) = \kappa$.*

Proof. (3) We have $j(\alpha) = \alpha$ for all $\alpha < \kappa$¡ since G is κ-complete. We have $j(\kappa) \neq \kappa$, since $[c_\alpha] < [id] < c_\kappa$ for all $\alpha < \kappa$. □

Definition 5.8. Suppose that $\kappa > \omega$ is regular and I is a κ-complete ideal on κ. I is *precipitous* if for every $\mathbb{P}_I$-generic filter G, the generic ultrapower $Ult_G(V)$ is wellfounded.

We now aim for a combinatorial definition of precipitous ideals.

Definition 5.9. Suppose that I is an ideal on κ.

(i) An *I-partition* of a set $S \in I^+$ is a maximal family $W \subseteq S$ with $X \cap Y \in I$ for all $X, Y \in W$, $X \neq Y$.

(ii) A *functional* F on a set $S \in I^+$ is a set of function f with ordinal values such that $W_F := \{dom(f) \mid f \in F\}$ is an I-partition of S and $f \neq g$ implies $dom(f) \neq dom(g)$.

(iii) If F, G are functionals, let $F < G$ if
 (a) W_F refines W_G ($W_F \leq W_G$), i.e. for every $X \in W_F$, there is some $Y \in W_G$ with $X \subseteq Y$.
 (b) If $f \in F$, $g \in G$, and $dom(f) \subseteq dom(g)$, then $f(\alpha) < g(\alpha)$ for all $\alpha \in dom(f)$.

Lemma 5.10. *Suppose that I is an ideal on κ. The following conditions are equivalent.*

(i) *I is precipitous.*

(ii) *There is no set $S \in I^+$ and no strictly decreasing sequence $F_0 > F_1 > ...$ of functionals on S.*

Proof. Suppose that $F_0 > F_1 > ...$ and $dom(F_n) = S$ for all n. Let $\dot{f}_n$ be a $\mathbb{P}_I$-name such that $X \Vdash \dot{f}_n \upharpoonright X = \check{f}$ for all $f \in F$, $X = dom(f)$. For each n, $D_n := \{X \leq S \mid \exists f \in F_n,\ X \subseteq dom(f)\}$ is dense open below S. If $f \in F_n$, $X \subseteq dom(f)$, then $X \Vdash \dot{f}_n \upharpoonright \check{X} = \check{f} \upharpoonright \check{X}$. Since $D_n \cap D_{n+1}$ is dense open below S and $F_{n+1} < F_n$, $S \Vdash [\dot{f}_{n+1}] < [\dot{f}_n]$, contradicting the assumption that I is precipitous.

For the other direction, suppose that $S \Vdash [\dot{f}_0] > [\dot{f}_1] > ....$ For each n, $D_n := \{X \leq S \mid X \text{ decides } \dot{f}_n \upharpoonright X\}$ is dense open below S. Let $A_n \subseteq D_n$ be a maximal antichain in D_n. Let $F_n = \{f \mid X = \mathrm{dom}(()f) \in A_n,\ X \Vdash \dot{f}_n \upharpoonright \check{X} = \check{f} \upharpoonright \check{X}\}$. By partitioning and shrinking the sets inf W_{F_n}, we obtain $F_0 > F_1 > ....$ □

Definition 5.11. I is *λ-saturated* if $\mathbb{P}_I$ satisfies the λ-c.c.

Lemma 5.12. *Suppose that $\kappa > \omega$ is regular. If I is κ^+-saturates, then I is precipitous.*

Proof. Suppose that $S \in I^+$ and $F_0 > F_1 > ...$ is a strictly decreasing sequence of functionals on S. Let $W_n = W_{F_n}$. Since $\mathbb{P}_I$ is κ^+-c.c., we can make each W_n disjoint and obtain W_n'. We can refine W_n' to W_n'' such that $W_0'' \geq W_1'' \geq ...$ and $S_n := \bigcup W_n'' = (\bigcup W_0) \cap ... \cap (\bigcup W_n)$. Since $S \setminus S_n \in I$, $\bigcup_n S_n \neq \emptyset$. Suppose that $\alpha \in \bigcup_n S_n$. For each n, there is a unique $X_n \in W_n'$ with $\alpha \in X_n$. Find $f_n \in F_n$ with $dom(f_n) = X_n$. Then $f_0(\alpha) > f_1(\alpha) > ...$, a contradiction. □

Lemma 5.13. (1) *Suppose that $j\colon V \to M$ is elementary. Let $U = \{X \subseteq \kappa \mid \kappa \in j(X)\}$. Then $k\colon Ult_U(V) \to M$, $k([f]) = (f)$ is elementary and $j = kj_U$.*
(2) *If U is a normal ultrafilter on κ, then $Ult_U(V) = \{j(f)(\kappa) \mid f\colon \kappa \to V,\ f \in V\}$.*

Proof. k is elementary by Los' theorem and $kj_U(x) = k([c_x]) = j(c_x)(\kappa) = c_{j(x)}(\kappa) = j(x)$. For the second part, let $j = j_U$. □

Let $Col(\omega, X) = \{p\colon \omega \times X \to Ord \text{ partial}|\ |p| < \omega,\ \forall \gamma \in X \forall n\ p(n,\gamma) < \gamma\}$.

Lemma 5.14. *Suppose that U is a normal ultrafilter on κ and $j = j_U$ Let $\mathbb{P} = Col(\omega, <\kappa)$, $\mathbb{Q} = Col(\omega, [\kappa, j(\kappa))$, $G \times H$ $\mathbb{P} \times \mathbb{Q}$-generic over V. Then*
(1) *$j(\mathbb{P} \cong \mathbb{P} \times \mathbb{Q}$, we will identify $j(\mathbb{P})$ and $\mathbb{P} \times \mathbb{Q}$.*
(2) *In $V[G \times H]$, j extends to $j_G\colon V[G] \to M[G \times H]$, $j(\sigma^G) = j(\sigma)^{G \times H}$ and $j_G(G) = G \times H$.*
(3) *$M[G \times H] = \{j_G(f)(\kappa) \mid f \in V[G]\} = Ult_{U_G}(V[G])$, where $U_G = \{X \in P(\kappa)^{V[G]} \mid \kappa \in j_G(x)\}$.*

Proof. The second part holds since $j[G] \subseteq G \times H$. For the last part, suppose that $x = \sigma^{G \times H} \in M[G \times H]$, $\sigma = j(f)(\kappa) \in Ult_U(V)$. Let $g\colon \kappa \to V[G]$, $g(\alpha) = f(\alpha)^G$, $g \in V[G]$. Then $j(g0(\kappa) = \sigma^{G \times H}$. □

Theorem 5.15. *Suppose that κ is measurable and G is $Col(\omega, <\kappa)$-generic over V. Then in $V[G]$, there is a precipitous ideal on ω_1.*

Proof. We work in $V[G]$. Let $I := \{X \subseteq \kappa \mid \Vdash_{\mathbb{Q}}^{V[G]} \kappa \notin j_G(X)\}$.

Let $h\colon P(\kappa) \to ro(\mathbb{Q})$, $h(X) = [\![\check{X} \in \dot{U}_G]\!]$, where $\dot{U}_G$ is a $\mathbb{P}$-name for U_G. Then h induces a homomorphism $i\colon P(\kappa)/I \to ro(\mathbb{Q})$ of Boolean algebras.

Claim 5.16. $i\colon P(\kappa)/I \to ro(\mathbb{Q})$ is a dense embedding.

Proof. Suppose that $q \in \mathbb{Q}$. Then $q = j(F)(\kappa)$ for some $F\colon \kappa \to V$, $F \in V$, $F(\alpha) \in Col(\omega, [\alpha, \kappa)$. We work in $V[G]$.

Let $X := \{\alpha < \kappa \mid F(\alpha) \in G \restriction [\alpha, \kappa)\}$. If H is $\mathbb{Q}$-generic over $V[G]$, then $h(X) \in H \Leftrightarrow X \in U_G \Leftrightarrow \kappa \in j_G(X) = \{\alpha < j(\kappa) \mid j_G(F)(\alpha) \in j_G(G) \restriction [\alpha, j(\kappa))\} \Leftrightarrow q \in j_G(G) \restriction [\kappa, j(\kappa)) = H$. Then $h(X) = q$, since $\mathbb{Q}$ is separative. □

Since $h(X) \in H \Leftrightarrow X \in U_G$, U_G is $\mathbb{P}_I$-generic over $V[G]$.

For every $A \in \mathbb{P}_I$, there is a $\mathbb{P}_I$-generic filter U_G with $A \in U_G$ as above, and $Ult_{U_G}(V) = M[G \times H]$ is wellfounded. Hence I is precipitous.

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Claim 5.17. In $V[G]$, I is generated by $(U^*)^V$.

Proof. Suppose that $p \Vdash_{\mathbb{P}} \dot{X} \in \dot{I}$. Let $A := \{\alpha < \kappa \mid p \Vdash_{\mathbb{P}} \alpha \notin \dot{X}\}$. Then $p \Vdash \dot{X} \cap A = \emptyset$.

Suppose that $A \notin U$. Let $q\colon \kappa \setminus A \to \mathbb{P}$ such that $q(\alpha) \leq q$ and $q(\alpha) \Vdash \alpha \in \dot{X}$. Let $r := j(q)(\kappa)$. Then $r \Vdash \kappa \in j_G(\dot{X})$.

Suppose that $G \times H$ is $\mathbb{P} \times \mathbb{Q}$-generic over V with $r \in G \times H$, so $p \in G$. Then $\kappa \in j_G(\dot{X}^G)$, so $\dot{X}^G \in U_G$. □

This completes the proof of the theorem. □

Definition 5.18. (i) Let SSP denote the class of forcings which preserve stationary subsets of ω_1.
(ii) $MM := FA_{\omega_1}(SSP)$.

Fact 5.19. We can force MM from a supercompact cardinal.

Theorem 5.20. *MM implies that the nonstationary ideal on ω_1 is ω_2-saturated.*

Proof. Pages 688-689 in Jech's book. □

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