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7.4. Simultaneous reflection

This section is devoted to the proof of the following result.

7.4.1. Theorem (Magidor): The following theories are equiconsistent:

(i) ZFC + "there is a weakly compact cardinal".

(ii) ZFC + "if $S_0, S_1 \subseteq S^{\omega_2}_{\omega}$ are stationary, then there is an $\alpha < \omega_2$ such that both $S_0 \cap \alpha$ and $S_1 \cap \alpha$ are stationary in α".

7.4.2. Lemma: Let $\kappa > \omega_1$ be a regular cardinal and $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ be a coherent C-sequence. Then the following statements are equivalent:

(i) $\vec{C}$ is a $\square(\kappa)$-sequence.

(ii) If S is a stationary subset of κ, then there are stationary sets $S_0, S_1 \subseteq S$ such that for every $\alpha \in \kappa \cap \mathrm{Lim}$ there is an $i < 2$ with $\mathrm{Lim}(C_\alpha) \cap S_i = \emptyset$.

Proof: Assume (i) fails. Pick a club C in κ that threads $\vec{C}$ and let S_0 and S_1 be stationary subsets of κ. Then there $\gamma, \gamma_0, \gamma_1 \in \mathrm{Lim}(C)$ such that $\gamma_0 < \gamma_1 < \gamma$ and $\gamma_i \in S_i$ for $i < 2$. Since $\gamma_0, \gamma_1 \in \mathrm{Lim}(C \cap \gamma) = \mathrm{Lim}(C_\gamma)$, we have $C_{\gamma_i} = \gamma_i \cap C_\gamma$ and $\gamma_i \in \mathrm{Lim}(C_\gamma) \cap S_i$ for $i < 2$. Hence (ii) fails in this case.

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Now, assume that (ii) fails for some stationary subset S of κ.

First, assume, towards a contradiction, that there is a $\gamma < \kappa$ such that the set

$$\{\alpha \in S \mid \operatorname{Lim}(C_\alpha) \cap (\gamma, \kappa) \neq \emptyset\}$$

is non-stationary. Then there are stationary subsets $S_0, S_1 \subseteq S$ such that the set

$$\bar{S} = \{\alpha \in S \setminus (\gamma+1) \mid \operatorname{Lim}(C_\alpha) \cap (\gamma, \kappa) = \emptyset\}$$

is equal to the disjoint union of S_0 and S_1. By our assumptions, there is an $\alpha \in \kappa \cap \operatorname{Lim}$ and $\beta_0, \beta_1 \in \operatorname{Lim}(C_\alpha) \cap \bar{S}$ with $\beta_0 < \beta_1$. Then $\beta_0 \in \operatorname{Lim}(C_{\beta_1}) \cap (\gamma, \kappa)$ and $\beta_1 \in S$, a contradiction.

This shows that for every $\gamma < \kappa$, the set

$$S_\gamma = \{\alpha \in S \mid \operatorname{Lim}(C_\alpha) \cap (\gamma, \kappa) \neq \emptyset\}$$

is stationary in κ. Given $\gamma \in \kappa \cap \operatorname{Lim}$, define

$$S^0_\gamma = \{\alpha \in S \setminus (\gamma+1) \mid \gamma \notin \operatorname{Lim}(C_\alpha)\}$$

and

$$S^1_\gamma = \{\alpha \in S \setminus (\gamma+1) \mid \gamma \in \operatorname{Lim}(C_\alpha)\}.$$

Given $\alpha, \gamma \in \kappa \cap \operatorname{Lim}$, $\operatorname{Lim}(C_\alpha) \cap S^i_\gamma \neq \emptyset$ implies $\alpha \in S^i_\gamma$ and, since $S^0_\gamma \cap S^1_\gamma = \emptyset$, this implies that either $\operatorname{Lim}(C_\gamma) \cap S^0_\gamma = \emptyset$ or $\operatorname{Lim}(C_\gamma) \cap S^1_\gamma = \emptyset$. By our assumption, this shows that for every $\gamma \in \kappa \cap \operatorname{Lim}$ there is an $i < 2$ such that S^i_γ is non-stationary.

Define

$$A = \{\gamma \in \kappa \cap \operatorname{Lim} \mid S^0_\gamma \text{ is non-stationary}\}.$$

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<u>Claim:</u> A is unbounded in κ.

<u>Proof of the Claim:</u> Fix $\gamma < \kappa$. Then

$$r : S_\gamma \longrightarrow \kappa \ ; \ \alpha \longmapsto \min(\mathrm{Lim}(C_\alpha) \cap (\gamma, \kappa))$$

is a regressive function and hence it is constant with value $\delta > \gamma$ on a stationary subset of S_γ. Then S^1_δ is stationary and, by the above remarks, this shows that S^0_δ is non-stationary. Hence $\delta \in A \cap (\gamma, \kappa)$. □ (Claim)

<u>Claim:</u> Given $\gamma_0, \gamma_1 \in A$ with $\gamma_0 < \gamma_1$, we have $C_{\gamma_0} = C_{\gamma_1} \cap \gamma_0$.

<u>Proof of the Claim:</u> By our assumptions, there are clubs C_0 and C_1 in κ such that $C_i \cap S^0_{\gamma_i} = \emptyset$ for $i < 2$. Pick $\alpha \in \mathrm{Lim}(C_0 \cap C_1) \cap S$. Then $\gamma_i \in \mathrm{Lim}(C_\alpha)$ for $i < 2$ and this implies $C_{\gamma_0} = C_\alpha \cap \gamma_0 = C_{\gamma_1} \cap \gamma_0$ □ (Claim).

The above computations show that $C = \bigcup_{\gamma \in A} C_\gamma$ is a club in κ that threads $\vec{C}$. Hence (ii) fails. □

7.4.3. Corollary: Let $\kappa > \omega_1$ be a regular cardinal. If all pairs of stationary subsets of S^κ_ω reflect at a common point, then κ is a weakly compact cardinal in L.

Proof: Assume, towards a contradiction, that κ is not weakly compact in L. Then there is a $\square(\kappa)$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$. By Lemma 7.4.2., there are stationary subsets $S_0, S_1 \subseteq S^\kappa_\omega$ such that for every $\alpha \in \kappa \cap Lim$ there is an $i < 2$ with $Lim(C_\alpha) \cap S_i = \emptyset$. By our assumption, there is an $\alpha \in \kappa \cap Lim$ such that S_0 and S_1 both reflect at α. This implies $cof(\alpha) > \omega$ and $Lim(C_\alpha) \cap S_0 \neq \emptyset \neq Lim(C_\alpha) \cap S_1$, a contradiction. $\square$

Next, we show how weakly compact cardinals can be used to produce models in which all pairs of stationary subsets of $S^{\omega_2}_\omega$ reflect at a common point.

7.4.4. Lemma (Baumgartner): Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$, $\delta = \kappa^+$ and $S \subseteq S^\delta_{<\kappa}$ be a stationary subset of δ. If $\mathbb{P}$ is a $<\kappa$-closed partial order, then
$\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{S}$ is stationary in $\check{\delta}$".

Proof: Let $\dot{C}$ be a $\mathbb{P}$-name for a club in δ and p be a condition in $\mathbb{P}$. There is $\mu < \kappa$ regular such that $\bar{S} = S \cap S^{\delta}_{\mu}$ is stationary. Fix a sufficiently large regular cardinal θ and pick an elementary submodel M of $H(\theta)$ with $\alpha = M \cap \delta \in \bar{S}$, $|M| = \kappa$, $\kappa, \delta, \mu, \dot{C}, S, \mathbb{P} \in M$ and ${}^{<\mu}M \subseteq M$.

We inductively construct a decreasing sequence $\langle p_\xi \mid \xi < \mu \rangle$ of conditions in $\mathbb{P} \cap M$ and an increasing sequence $\langle \alpha_\xi \mid \xi < \mu \rangle$ of ordinals that is cofinal in α such that $p_{\xi+1} \Vdash$ "$\check{\alpha}_\xi \in \dot{C}$". Pick a condition p in $\mathbb{P}$ with $p \leq_{\mathbb{P}} p_\xi$ for all $\xi < \mu$. Then $p \Vdash$ "$\check{\alpha} \in \mathrm{Lim}(\dot{C})$" and hence $p \Vdash$ "$\check{\alpha} \in \dot{C} \cap \check{\bar{S}}$". ☒

7.4.5. Theorem (Baumgartner): Assume GCH. Let κ be an uncountable regular cardinal, $\delta > \kappa$ be a weakly compact cardinal and G be $\mathrm{Col}(\kappa, \delta)$-generic over V. In $V[G]$, every pair of stationary subsets of $S^{\delta}_{<\kappa}$ reflects at a common point.

Proof: Let $\dot{S}_0$ and $\dot{S}_1$ be $\mathrm{Col}(\kappa, \delta)$-nice names for stationary subsets of $S^{\delta}_{<\kappa}$. Then $\mathrm{Col}(\kappa, \delta), \dot{S}_0, \dot{S}_1 \subseteq V_\delta$ and the Π^1_1-indescribability of δ implies that there is an inaccessible cardinal $\kappa < \bar{\delta} < \delta$ such that $\dot{S}_i \cap V_{\bar{\delta}}$ is a $\mathrm{Col}(\kappa, \bar{\delta})$-name for a stationary subset of $\bar{\delta}$. Set $S_i = \dot{S}_i^{G}$ and $\bar{S}_i = (\dot{S}_i \cap V_{\bar{\delta}})^{\bar{G}}$, where $\bar{G}$ is the filter on $Col(\kappa, \bar{\delta})$ induced by G. Since $V[G]$ is a $Col(\kappa, [\bar{\delta}, \delta))$-generic extension of $V[\bar{G}]$, Lemma 7.4.4 implies that $\bar{S}_i$ is a stationary subset of $\bar{\delta}$ in $V[G]$ for all $i < 2$. Moreover, we have $\bar{S}_i \subseteq S_i \cap \delta$ for $i < 2$ and this implies that the stationary subsets S_0 and S_1 both reflect at $\bar{\delta}$. $\Box$