7.2. Club shooting

<u>7.2.1. Definition</u>: Let κ be an uncountable regular cardinal. A subset S of κ is <u>fat stationary</u> if for every club subset C of κ and every $\lambda < \kappa$, the set $C \cap S$ contains a closed subset of order-type λ.

<u>7.2.2. Lemma:</u> Let κ be an uncountable regular cardinal, $\mu < \kappa$ and S be a subset of κ such that for every club C in κ, the set $C \cap S$ contains a closed subset of order-type $\mu + 1$. If C is a club in κ and $\tau < \mu^+$, then $C \cap S$ contains a club of order-type τ.

<u>Proof:</u> Fix a club C in κ. We prove the above statement by induction on τ. Assume that the statement holds for all $\bar{\tau} < \tau$. We may also assume that $\tau \in \mu^+ \cap \mathrm{Lim}$. Then there is a club D in κ such that for all $\alpha \in D$, $\beta < \alpha$ and $\bar{\tau} < \tau$, the set $C \cap S \cap (\beta, \alpha)$ contains a closed subset of order-type $\bar{\tau} + 1$. There is a $\bar{\mu} \leq \mu$ and a sequence $\langle \bar{\tau}_\xi \mid \xi < \bar{\mu} \rangle$ such that $\tau = \sum_{\xi < \bar{\mu}} \bar{\tau}_\xi$. Then there is a closed subset c of $C \cap D \cap S$ of order-type $\bar{\mu} + 1$. Let $\langle c_\xi \mid \xi \leq \bar{\mu} \rangle$ be the monotone enumeration of c and pick

a closed subset a_ξ of $C \cap S \cap (c_\xi, c_{\xi+1})$ of order-type $\bar{\alpha}_\xi + 1$ for every $\xi < \bar{\mu}$. This yields a closed subset of $C \cap S$ of order-type $\bar{\alpha}$. $\square$

<u>7.2.3. Corollary:</u> Every stationary subset of ω_1 is fat stationary.

<u>Proof:</u> Let S be a stationary subset of ω_1 and C be a club in ω_1. Assume, towards a contradiction, that there is a $\mu < \omega_1$ such that $C \cap S$ contains no closed subset of order-type $\mu + 1$. By Lemma 7.2.2., we may assume $\mu = \omega$.

<u>Claim:</u> The set $\bar{S} = \{\alpha \in S \mid C \cap S$ is bounded in $\alpha\}$ is not stationary in ω_1.

<u>Proof of the Claim:</u> Otherwise, we would get a regressive function

$$r : \bar{S} \longrightarrow \kappa;\ \alpha \longmapsto \sup(C \cap S \cap \alpha)$$

that is constant with value β on a stationary subset $\tilde{S}$ of S. Pick $\alpha, \bar{\alpha} \in C \cap \tilde{S}$ with ~~β~~ ~~Then $\alpha < \sup(C \cap S)$~~ $\beta < \bar{\alpha} < \alpha$. Then $\beta < \bar{\alpha} < \sup(C \cap S \cap \alpha) = r(\alpha) = \beta$, a contradiction. $\square$ (Claim).

Hence there is a club D in ω_1 such that

$C\cap S\cap \alpha$ is unbounded in α for every $\alpha\in D$. Pick $\alpha\in C\cap D\cap S$. Then $C\cap S\cap(\alpha+1)$ contains a closed subset of order-type $\omega+1$, a contradiction. $\boxed{2}$

Assume that κ is either weakly inacc. or the succ. of a regular cardinal

7.2.4. Proposition: Let $\kappa>\omega_1$ be a regular cardinal and $E\subseteq S^\kappa_\omega$ such that $E\cap\alpha$ is not stationary in α for every $\alpha\in S^\kappa_{>\omega}$. Then $\kappa\setminus E$ is fat stationary.

Proof: Let C be a club in κ and $^{\omega_1}\mu<\kappa$ ~~with $\mathrm{cof}(\mu)>\omega$. Pick $\alpha\in\mathrm{Lim}(C)$ with $\mathrm{cof}(\alpha)=\mu$~~ be a regular cardinal. Pick $\alpha\in\mathrm{Lim}(C)$ with $\mathrm{cof}(\alpha)=\mu$. By assumption, $E\cap\alpha$ is not stationary and there is a club c in α with $c\cap E=\emptyset$. Then $(c\cap C)\cup\{\alpha\}$ is a closed subset of $C\cap(\kappa\setminus E)$ of order-type at least $\mu+1$. By Lemma 7.2.2., the above computations show that $(\kappa\setminus E)$ is fat stationary. $\boxed{3}$

7.2.5. Proposition: Let S be a subset of a regular uncountable cardinal κ such that

$$\mathbb{1}_{\mathbb{P}}\Vdash \text{"}\check{S}\text{ contains a club subset of }\check{\kappa}\text{"}$$

for some $<\kappa$-~~closed~~ distributive partial order $\mathbb{P}$. Then S is fat stationary.

<u>Proof:</u> Let C be a club subset of κ, $\lambda < \kappa$ and G be $\mathbb{P}$-generic over V. In $V[G]$, there is a club D in κ with $D \subseteq S$. Pick $\alpha < \kappa$ with $\operatorname{otp}(C \cap D \cap \alpha) = \lambda$. By our assumption, we have $C \cap D \cap \alpha \in V$ and this set is a closed subset of $C \cap S$ of order-type ~~at least~~ λ. $\square$

Fix a regular uncountable cardinal κ and a fat stationary subset S of κ. Define $\mathbb{C}(S)$ to be the partial order consisting of closed bounded subsets of S ordered by reversed inclusion.

<u>7.2.6. Proposition:</u> If $\kappa = \kappa^{<\kappa}$, then $\mathbb{C}(S)$ has cardinality κ and satisfies the κ^+-chain condition. $\square$

<u>7.2.6. Proposition:</u> If $\alpha < \kappa$, then the D_α set of all conditions c in $\mathbb{C}(S)$ with $\operatorname{otp}(c) \geq \alpha$ is dense in $\mathbb{C}(S)$.

<u>Proof:</u> Fix a condition c in $\mathbb{C}(S)$. By our assumptions, the set $S \cap (\max(c), \kappa)$ contains a closed set d of order-type $\alpha+1$. Then $c \cup d$ is a condition in $\mathbb{C}(S)$ that is stronger than c. $\square$

7.2.7. Lemma: Assume that κ is either inaccessible or the successor of a regular cardinal μ with $\mu = \mu^{<\mu}$. Then $\mathbb{C}(S)$ is $<\kappa$-distributive.

Proof: Fix a regular cardinal $\lambda < \kappa$ ~~with~~ [illegible] $\nu^{<\lambda} < \kappa$ [for all $\nu < \kappa$], a $\mathbb{C}(S)$-name $\dot{f}$ for a function from λ to On, $c_0 \in \mathbb{C}(S)$ and a sufficiently large regular cardinal θ. Then there is a continuous ascending sequence $\langle M_\alpha \mid \alpha < \kappa \rangle$ of elementary submodels of $H(\theta)$ of cardinality less than κ and a monotone enumeration $\langle \kappa_\alpha \mid \alpha < \kappa \rangle$ of a club C in κ such that the following statements hold for all $\alpha < \kappa$.

(i) $c_0, \dot{f}, \lambda, S \in M_0$.

(ii) $\kappa_\alpha = M_\alpha \cap \kappa$

(iii) ${}^{<\lambda}M_\alpha \subseteq M_{\alpha+1}$.

By our assumptions, $C \cap S$ contains a closed set c of order-type $\lambda + 1$. Let $\langle \alpha_\gamma \mid \gamma \leq \lambda \rangle$ be the monotone cont. enumeration of all $\alpha < \kappa$ with $\kappa_\alpha \in c$.

We inductively construct a decreasing sequence $\langle c_\gamma \mid \gamma \leq \lambda \rangle$ of conditions in $\mathbb{C}(S)$ such that the following statements hold for all $\gamma \leq \lambda$.

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(a) If $\gamma < \lambda$, then $c_\gamma \in M_{\alpha_\gamma + 1}$ and $\kappa_{\alpha_{\bar{\gamma}}} < \max(c_\gamma)$ for all $\bar{\gamma} < \gamma$.

(b) If $\gamma < \lambda$, then $c_{\gamma+1}$ decides $f(\gamma)$.

(c) If $\gamma \in Lim$, then $c_\gamma = \bigcup_{\bar{\gamma} < \gamma} c_{\bar{\gamma}} \cup \{\kappa_{\alpha_\gamma}\}$.

Assume that $\gamma \leq \lambda$ and a sequence $\langle c_{\bar{\gamma}} \mid \bar{\gamma} < \gamma \rangle$ with the above properties was already constructed.

First, assume $\gamma = \bar{\gamma} + 1$. Then we have $\bar{\gamma} \leq \kappa_{\alpha_{\bar{\gamma}}} < \kappa_{\alpha_\gamma}$ and this implies that M_{α_γ} contains the condition $c_{\bar{\gamma}}$, the dense open subset $D_{\kappa_{\alpha_{\bar{\gamma}}}}$ and the dense open subset $E_{\bar{\gamma}}$ of all conditions deciding $f(\bar{\gamma})$. Hence there is a condition c_γ in $\mathbb{C}(S)$ such that $c_\gamma \leq_{\mathbb{C}(S)} c_{\bar{\gamma}}$ and $c_\gamma \in D_{\kappa_{\alpha_{\bar{\gamma}}}} \cap E_{\bar{\gamma}} \cap M_{\alpha_\gamma}$.

Now, assume that $\gamma \in Lim$ and define $c_\gamma = \{\kappa_{\alpha_\gamma}\} \cup \bigcup_{\bar{\gamma} < \gamma} c_{\bar{\gamma}}$. By construction, c_γ is a closed bounded subset of S. Hence it is a condition in $\mathbb{C}(S)$ that is stronger than $c_{\bar{\gamma}}$ for every $\bar{\gamma} < \gamma$. Since ${}^{<\lambda} M_{\alpha_\gamma} \subseteq M_{\alpha_\gamma + 1}$ and $\kappa_{\alpha_\gamma} < \kappa_{\alpha_\gamma + 1} \subseteq M_{\alpha_\gamma}$, we also have $c_\gamma \in M_{\alpha_\gamma + 1}$.

Our construction ensures that $c_\lambda \leq_{\mathbb{C}(S)} c_0$ and $c_\lambda \Vdash$ "$f \in \check{V}$". This proves the statement of the lemma. ☺

7.2.8. Theorem: Let S be a fat stationary subset of an uncountable regular cardinal κ. Assume that κ is either inaccessible or the successor of a regular cardinal μ with $\mu = \mu^{<\mu}$ and $\kappa = 2^{\mu}$. Then the following statements hold.

(i) Forcing with $\mathbb{C}(S)$ preserves all cofinalities and adds no new functions from $\lambda < \kappa$ to On

(ii) If G is $\mathbb{C}(S)$-generic over V, then S contains a club subset of κ in $V[G]$. $\boxtimes$