I.

7. Stationary Reflection

7.1. Introduction

7.1.1. Definition: Let κ be an uncountable regular cardinal, S be a stationary subset of κ and $\lambda \in \kappa \cap \mathrm{Lim}$. We say that *$S$ reflects at λ* if $S \cap \lambda$ is a stationary subset of λ.

7.1.2. Proposition: Let $S \subseteq \kappa$ be a stationary subset of a regular uncountable cardinal κ that reflects at $\lambda \in \kappa \cap \mathrm{Lim}$. If $S \subseteq \mathrm{Lim}$, then $\mathrm{cof}(\lambda) > \omega$.

Proof: Assume, towards a contradiction, that $\mathrm{cof}(\lambda) = \omega$. Then there is a club subset C of λ of order-type ω that consists of successor ordinals. Hence $C \cap S = \emptyset$, a contradiction. $\square$

7.1.3. Proposition: If $\kappa = \mu^+$ is the successor of a regular cardinal μ, then the stationary subset

$$S^\kappa_\mu = \{ \lambda < \kappa \mid \mathrm{cof}(\lambda) = \mu \}$$

does not reflect.

II.

<u>Proof:</u> Assume, towards a contradiction, that S^κ_μ reflects at $\lambda \in \kappa \cap \mathrm{Lim}$. Then there is a club subset C of λ of order-type $\mathrm{cof}(\lambda) \leq \mu$ such that every successor in the monoton enumeration of C is a successor ordinal. By our assumption, there is a $\beta \in C \cap S^\kappa_\mu$ and our construction ensures $\beta \in \mathrm{Lim}$. Then $\mathrm{cof}(\beta) \leq \mathrm{otp}(C \cap \beta) < \mathrm{cof}(\lambda) \leq \mu$, a contradiction. $\square$

<u>7.1.4. Corollary:</u> If κ is an uncountable regular cardinal such that every stationary subset of κ reflects, then κ is either weakly inaccessible or the successor of a singular cardinal. $\square$

<u>7.1.5. Lemma:</u> If κ is a weakly compact cardinal, then every stationary subset of κ reflects.

<u>Proof:</u> Let S be a stationary subset of κ. Then the statement "S is stationary" can be expressed by a Π^1_1-statement in the structure $\langle V_\kappa, \in, S\rangle$. Since κ is Π^1_1-indescribable, there is an inaccessible cardinal $\lambda < \kappa$ such that $S \cap \lambda$ is stationary in λ. $\square$

7.1.6. Lemma: Let κ be an uncountable regular cardinal such that every stationary subset of S^κ_ω reflects. If $\vec{C} = \langle C_\alpha \mid \alpha < \kappa \rangle$ is a $\square(\kappa)$-sequence, then the set

$$S = \{ \alpha \in S^\kappa_\omega \mid otp(C_\alpha) < \alpha \}$$

is not stationary.

Proof: Assume, towards a contradiction, that S is stationary. Define

$$r : S \longrightarrow \kappa ; \ \alpha \longmapsto otp(C_\alpha).$$

Then r is regressive and there is a stationary subset $\bar{S}$ of S such that $r \restriction \bar{S}$ is constant. By our assumptions, there is an $\alpha \in \kappa \cap Lim$ such that $\bar{S} \cap \alpha$ is stationary in α and Proposition 7.1.2. implies $cof(\alpha) > \omega$. Then $Lim(C_\alpha)$ is a club subset of α and there are $\gamma < \beta < \alpha$ such that $\beta, \gamma \in Lim(C_\alpha) \cap \bar{S}$. In this situation, the coherency of $\vec{C}$ implies $C_\beta = C_\alpha \cap \beta$ and $C_\gamma = C_\alpha \cap \gamma = C_\beta \cap \gamma$. Hence C_γ is a proper initial segment of C_β and this implies $otp(C_\gamma) < otp(C_\beta)$. But $\beta, \gamma \in \bar{S}$ and this means $otp(C_\beta) = otp(C_\gamma)$, a contradiction. $\square$

7.1.7. Theorem (Jensen): Assume $V = L$. Let κ be an uncountable regular cardinal that is not weakly compact. Then there is a $\square(\kappa)$-sequence $\langle C_\alpha \mid \alpha < \kappa \rangle$ and a stationary subset $S \subseteq S^\kappa_\omega$ such that $\mathrm{Lim}(C_\alpha) \cap S = \emptyset$ for every $\alpha \in \kappa \cap \mathrm{Lim}$.

7.1.8. Corollary (Jensen): Assume $V = L$. Given an uncountable regular cardinal κ, the following statements are equivalent.

(i) κ is weakly compact.

(ii) Every stationary subset of κ reflects.

(iii) Every stationary subset of S^κ_ω reflects.

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) follow from Lemma 7.1.5.

Assume κ is not weakly compact. Let $\langle C_\alpha \mid \alpha < \kappa \rangle$ and $S \subseteq S^\kappa_\omega$ be the objects obtained by an application of Theorem 7.1.7. Then there is a sequence $\vec{D} = \langle D_\alpha \mid \alpha < \kappa \rangle$ such that $C_\alpha = D_\alpha$ for all $\alpha \in \kappa \setminus S$ and D_α is a cofinal subset of α of order-type ω for every $\alpha \in S$. Then $\vec{D}$ is a $\square(\kappa)$-sequence and the set $\{\alpha \in S^\kappa_\omega \mid \mathrm{otp}(D_\alpha) < \alpha\}$ is stationary. By Lemma 7.1.6., this implies that there is a stationary subset of S^κ_ω that does not reflect. $\boxtimes$

7.1.9. Definition: Let κ be an infinite cardinal. A coherent C-sequence $\langle C_\alpha \mid \alpha < \kappa^+ \rangle$ is a *$\square_\kappa$-sequence* if $\mathrm{otp}(C_\alpha) \leq \kappa$ holds for every $\alpha < \kappa^+$.

7.1.10. Proposition: Let κ be an infinite cardinal. Then every $\square_\kappa$-sequence is a $\square(\kappa^+)$-sequence.

Proof: Assume that there is a $\square_\kappa$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$ and a club subset C of κ^+ that threads $\vec{C}$. Pick $\alpha \in \mathrm{Lim}(C)$ with $\mathrm{otp}(C \cap \alpha) > \kappa$. Then $\kappa < \mathrm{otp}(C \cap \alpha) = \mathrm{otp}(C_\alpha) \leq \kappa$, a contradiction. $\square$

7.1.11. Proposition: Let κ be an uncountable regular cardinal such that every stationary subset of S^κ_ω reflects. Then there is no $\square_\kappa$-sequence.

Proof: Assume, towards a contradiction, that there is a $\square_\kappa$-sequence $\vec{C} = \langle C_\alpha \mid \alpha < \kappa^+ \rangle$. Then $\vec{C}$ is a $\square(\kappa^+)$-sequence and $\mathrm{otp}(C_\alpha) < \alpha$ for every $\alpha \in S^{\kappa^+}_\omega \setminus \kappa$. By Lemma 7.1.6., this yields a contradiction. $\square$

7.1.12. Definition: An inaccessible cardinal κ is a Mahlo cardinal if the set of regular cardinals smaller than κ form a stationary subset of κ.

7.1.13. Theorem (Jensen): If κ is an infinite cardinal such that κ^+ is not Mahlo in L, then there is a $\square_\kappa$-sequence.

7.1.14. Corollary (Jensen): If κ is an uncountable cardinal such that every stationary subset of $S^{\kappa^+}_\omega$ reflects, then κ^+ is a Mahlo cardinal in L. $\square$

We will later prove the following theorem that shows that the above corollary yields the correct consistency strength for stationary reflection for stationary subsets of $S^{\kappa^+}_\omega$.

7.1.15. Theorem (Harrington-Shelah): Let κ be an uncountable regular cardinal and $\delta \geq \kappa$ be a Mahlo cardinal. Then there is a partial order $\mathbb{P}$ with the following properties.

(i) Forcing with $\mathbb{P}$ preserves cofinalities $\leq \kappa$ and $\geq \delta$.

(ii) If $\kappa < \delta$, then $\mathbb{1}_{\mathbb{P}} \Vdash$ "$\check{\delta} = \check{\kappa}^+$".

(iii) If G is $\mathbb{R}$-generic over V, then every stationary subset of S^δ_ω reflects in $V[G]$.

7.1.16. Corollary: Let κ be an uncountable regular cardinal and $\delta > \kappa$ be a Mahlo cardinal that is not weakly compact in L. Then there is a generic extension $V[G]$ of V such that there is a $\square(\kappa^+)$-sequence in $V[G]$ and there is no $\square_\kappa$-sequence in $V[G]$. $\square$