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6.2. Threading square sequences by forcing

6.2.1. Proposition: Let $\lambda \in \mathrm{Lim}$ with $\mathrm{cof}(\lambda) > \omega$ and $\vec{C}$ be a coherent C-sequence of length λ. Then there is at most one thread through $\vec{C}$.

Proof: Let C_0 and C_1 be closed unbounded subsets of λ that thread $\vec{C} = \langle C_\alpha \mid \alpha < \lambda \rangle$ and define $C = C_0 \cap C_1$. Since $\mathrm{cof}(\lambda) > \omega$, C is a club in λ and our assumptions imply that

$$C_0 \cap \alpha = C_\alpha = C_0 \cap \alpha$$

holds for every $\alpha \in \mathrm{Lim}(C) \subseteq \mathrm{Lim}(C_0) \cap \mathrm{Lim}(C_1)$. We can conclude that $C_0 = C_1$. $\square$

6.2.2. Corollary: Let κ be an uncountable regular cardinal and $\vec{C}$ be a $\square(\kappa)$-sequence. If $\mathbb{P}$ is a partial order with

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{“} \vec{C} \text{ is trivial”},$$

then $\mathbb{1}_{\mathbb{P} \times \mathbb{P}} \Vdash$ “$\mathrm{cof}(\check{\kappa}) = \omega$”. In particular, σ-strategically closed forcings do not trivialize $\vec{C}$.

Proof: Let $\dot{C}$ be a $\mathbb{P}$-name for a thread through $\vec{C}$ and $G \times H$ be $(\mathbb{P} \times \mathbb{P})$-generic over V. ~~By 6.2.1, we [illegible] either~~ Assume $\mathrm{cof}(\kappa)^{V[G,H]} > \omega$. Then 6.2.1. implies $\dot{C}^G = \dot{C}^H \in (V[G] \cap V[H]) \setminus V$, a contradiction. $\square$

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Let κ be an uncountable regular cardinal and $\vec{C} = \langle C_\gamma \mid \gamma < \kappa \rangle$ be a $\square(\kappa)$-sequence. Define $\mathbb{T}(\vec{C})$ to be the partial order consisting of closed bounded subsets c of κ such that c is a thread through $\langle C_\gamma \mid \gamma < \sup(c) \rangle$ ordered by reversed inclusion.

If κ is an uncountable regular cardinal, then we let $\dot{\mathbb{T}}$ denote the canonical $\mathbb{S}(\kappa)$-name for the partial order $\mathbb{T}(\bigcup \dot{G})$.

<u>6.2.3. Lemma:</u> The partial order $\mathbb{S}(\kappa) * \dot{\mathbb{T}}$ contains a dense subset of cardinality $\kappa^{<\kappa}$.

<u>Proof:</u> Define

$$D = \{ \langle \vec{C}, \check{c} \rangle \mid \vec{C} \in \mathbb{S},\ c \text{ is a thread through } \vec{C} \}.$$

Pick $\langle \vec{C}_0, \dot{c} \rangle \in \mathbb{S}(\kappa) * \dot{\mathbb{T}}$. By 6.1.14 and 6.1.15, there is a $\vec{C} \in \mathbb{S}$ and $c_0 \subseteq \kappa$ such that $\vec{C} \leq_{\mathbb{S}} \vec{C}_0$, $\vec{C} \Vdash$ "$\dot{c} = \check{c}_0$" and $\vec{C} = \langle C_\gamma \mid \gamma \leq \nu + \omega \rangle$ for some $\nu < \kappa$. Define $c = c_0 \cup \{\sup(c_0)\} \cup (\nu, \nu+\omega)$. Then c is a thread through $\vec{C}$ and $\langle \vec{C}, \check{c} \rangle$ is an element of D that is stronger than $\langle \vec{C}_0, \dot{c} \rangle$.

Let $\lambda \in \kappa \cap \mathrm{Lim}$ and let $\langle \langle \vec{C}_\alpha, \check{c}_\alpha \rangle \mid \alpha < \lambda \rangle$ be a descending sequence of conditions in D with $\vec{C}_\alpha = \langle C^\alpha_\gamma \mid \gamma \leq \nu_\alpha \rangle$ for every $\alpha < \lambda$. Define $\nu = \sup_{\alpha<\lambda} \nu_\alpha$, $C_\nu = \bigcup_{\alpha<\lambda} c_\alpha$ and $C_\gamma = C_\gamma^{\min\{\alpha<\lambda \mid \gamma \leq \nu_\alpha\}}$ for every $\gamma < \nu$. By our assumptions, the

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pair $\langle\langle C_\gamma \mid \gamma \leq \nu\rangle, \check{C}_\nu\rangle$ is an element of D that is stronger than $\langle \vec{C}_\alpha, \check{C}_\alpha\rangle$ for every $\alpha < \lambda$. $\square$

<u>6.2.4. Lemma:</u> Let κ be an uncountable regular cardinal and $\mathbb{P}$ be a separative $<\kappa$-closed partial order of cardinality $\lambda \geq \kappa$. If

$$\mathbb{1}_{\mathbb{P}} \Vdash \exists f \, [f : \check{\kappa} \to \check{\lambda} \text{ surjective} \wedge f \notin \check{V}],$$

then there is a dense embedding of a dense subset of $Col(\kappa, \{\lambda\})$ into $\mathbb{P}$.

<u>Proof:</u> ~~We may assume that $\mathbb{P}$ is separative.~~
Our assumptions imply that below every condition in $\mathbb{P}$, there is an antichain of size λ.
By our assumptions, there is a $\mathbb{P}$-name $\dot{g}$ for a surjection of λ onto the $\mathbb{P}$-generic filter. Let

$$D = \{ p \in Col(\kappa, \{\lambda\}) \mid dom(p) = \alpha \times \{\lambda\}, \ \alpha = 0 \vee \alpha = \bar{\alpha}+1\}$$

We inductively define an embedding of D into $\mathbb{P}$. Set $e(\mathbb{1}_{Col(\kappa, \{\lambda\})}) = \mathbb{1}_{\mathbb{P}}$. Now pick $p : \alpha \times \{\lambda\} \to \lambda$ and assume that $e(q)$ is defined for all $q \in D$ with $q \leq p$. Let $\langle r_\gamma \mid \gamma < \lambda\rangle$ enumerate a maximal antichain in the set of all conditions q in $\mathbb{P}$ such that q decides $\dot{f}(\alpha)$ and $q \leq_{\mathbb{P}} e(\bar{p})$ for all $\bar{p} \in D$ with $\bar{p} \leq p$.
Define $e(p \cup \{\langle\langle \alpha, \lambda\rangle, \gamma\rangle\}) = r_\gamma$.
Pick $q \in \mathbb{P}$. Then there is an $r \leq_{\mathbb{P}} q$ with $r \Vdash$ "$\dot{f}(\check{\alpha}) = \check{q}$" for some $\alpha < \kappa$. By construction, the set $\{e(p) \mid p : (\alpha+1) \times \{\lambda\} \to \lambda\}$ is a maximal antichain in $\mathbb{P}$ and there is a $p \in Col(\kappa, \{\lambda\})$ such that $e(p)$ decides $\dot{f}(\alpha)$ and $r \parallel e(p)$. This implies $e(p) \Vdash$ "$\dot{f}(\check{\alpha}) = \check{q}$" and $e(p) \leq_{\mathbb{P}} q$ by separativity. $\square$

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6.2.5 Corollary: Let κ be an uncountable regular cardinal with $\kappa = \kappa^{<\kappa}$. If $\mathbb{P}$ is a nontrivial $<\kappa$-closed partial order of cardinality κ, then $\mathbb{P}$ is forcing equivalent to $Add(\kappa, 1)$. □

6.2.6. Theorem: Let κ be a supercompact cardinal. Then there is a partial order $\mathbb{P}$ and a $\mathbb{P}$-name $\dot{\mathbb{Q}}$ for a partial order such that

$$\mathbb{1}_{\mathbb{P}} \Vdash \text{"}\check{\kappa} \text{ is not weakly compact"}$$

and

$$\mathbb{1}_{\mathbb{P} * \dot{\mathbb{Q}}} \Vdash \text{"}\check{\kappa} \text{ is supercompact"}.$$

Proof: Let $\mathbb{P}_0$ be the partial order that makes the supercompactness of κ indestructible under $<\kappa$-directed closed forcings, $\dot{\mathbb{P}}$ be the canonical $\mathbb{P}_0$-name for $S(\kappa)$ and $\dot{\mathbb{Q}}$ be the canonical $(\mathbb{P}_0 * \dot{\mathbb{P}})$-name for $\mathbb{T}$.

Let $(G * H) * K$ be $((\mathbb{P}_0 * \dot{\mathbb{P}}) * \dot{\mathbb{Q}})$-generic over V. Then $V[G, H]$ is an $S(\kappa)^{V[G]}$-generic extension of $V[G]$ and κ is not weakly compact in $V[G, H]$ by 6.1.7. and 6.1.16. By 6.2.3 and 6.2.5, $V[G, H, K]$ is an $Add(\kappa, 1)^{V[G]}$-generic extension of $V[G]$ and hence κ is supercompact in $V[G, H, K]$. □