2.4. Borel Codes

Recall that the Baire space is the topological space consisting of the set ${}^\omega\omega$ of all functions $x: \omega \to \omega$ equipped with the topology whose basic open sets are of the form

$$N_s = \{x \in {}^\omega\omega \mid s \subseteq x\},$$

where s is an element of the set ${}^{<\omega}\omega$ of all functions $t: n \to \omega$ with $n < \omega$.

2.4.1. Definition: (i): We say that $T \subseteq ({}^{<\omega}\omega)^{k+1}$ is a subtree of $({}^{<\omega}\omega)^{k+1}$ if $lh(t_0) = \ldots = lh(t_k)$ and $(t_0 \restriction n, \ldots, t_k \restriction n) \in T$ for all $(t_0, \ldots, t_k) \in T$ and $n \leq lh(t_0)$.

(ii): Given a subtree T of $({}^{<\omega}\omega)^{k+1}$, we define

$$[T] = \{(x_0, \ldots, x_k) \in ({}^\omega\omega)^{k+1} \mid \forall n < \omega \ (x_0 \restriction n, \ldots, x_k \restriction n) \in T\}$$

to be the set of cofinal branches through T.

(iii): We say that a subtree T of $({}^{<\omega}\omega)^{k+1}$ is well-founded if $[T] = \emptyset$.

2.4.2 Proposition (ZF^-): A subtree T of $({}^{<\omega}\omega)^{k+1}$ is well-founded if and only if $(T, \supsetneq)$ is a well-founded relation.

$\square$

Remark: Note that a subset A of $({}^{\omega}\omega)^{k+1}$ is closed if and only if there is a subtree T of $({}^{<\omega}\omega)^{k+1}$ with $A = [T]$.

2.4.3. Lemma: Let T be a subtree of $({}^{<\omega}\omega)^{k+1}$ and M be a transitive model of ZF^- with $T \in M$. Then the statement "T is well-founded" is absolute between M and V.

Proof: Assume that $[T]^M = \emptyset$. By 2.4.2, $(T, \supseteq)$ is a well-founded relation in M and there is a rank-function $r: T \to On$ with

$$r(t_0, \dots, t_k) = lub\{r(s_0, \dots, s_k) \mid (s_0, \dots, s_k) \in T, \ s_i \supseteq t_i \text{ for all } i \leq k\}$$

in M. Then r witnesses that $(T, \supseteq)$ is a well-founded relation in V and we can conclude that $[T]^V = \emptyset$. Since we also have $[T]^M \subseteq [T]^V$, this shows the statement of the Lemma. $\boxtimes$

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2.4.4 Definition: Let $\mathcal{T} = (X, \tau)$ be a topological space and $A \subseteq X$.

(i) We say that A is a G_δ-subset of $\mathcal{T}$ if A is equal to the intersection of countably many open sets in $\mathcal{T}$.

(ii) We say that A is an F_σ-subset of $\mathcal{T}$ if $X \setminus A$ is a G_δ-subset of $\mathcal{T}$.

Fix a recursive enumeration $\langle s_i \mid i < \omega \rangle$ of ${}^{<\omega}\omega$ such that $lh(s_i) \leq i$, $s_i \subseteq s_j \rightarrow i \leq j$ and $i \neq j \rightarrow s_i \neq s_j$ holds for all $i, j < \omega$.

2.4.5 Definition: (i) We say that $x \in {}^{\omega}\omega$ is a code for...

- ... an open subset if $x(0) = 0$.
- ... a closed subset if $x(0) = 1$.
- ... a G_δ-subset if $x(0) > 1$.

(ii) Given $c \in {}^{\omega}\omega$, we define

- $A_c = \bigcup \{ N_{s_i} \mid c(i+1) = 1 \}$ if $c(0) = 0$.
- $A_c = {}^{\omega}\omega \setminus \bigcup \{ N_{s_i} \mid c(i+1) = 1 \}$ if $c(0) = 1$.
- $A_c = \bigcap_{n<\omega} \bigcup \{ N_{s_i} \mid c(2^n \cdot 3^{i+1}) = 1 \}$ if $c(0) > 1$.

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2.4.6. Lemma: Let M be a transitive model of ZF^- and $c, d, e, x \in ({}^\omega\omega)^M$. Then the following statements are absolute between M and V.

(i) $x \in A_c$ (ii) $A_c = \emptyset$ (iii) $A_c \subseteq A_d$

(iv) $A_c \subseteq {}^\omega\omega \setminus A_d$ (v) $A_c \cap A_d = A_e$

Proof: (i) If $c(0) = 0$, then

$$x \in A_c \iff \exists i < \omega\, [c(i+1) = 1 \wedge s_i \subseteq x]$$

If $c(0) = 1$, then

$$x \in A_c \iff \forall i < \omega\, [c(i+1) = 1 \rightarrow s_i \not\subseteq x]$$

If $c(0) > 1$, then

$$x \in A_c \iff \forall n < \omega\, \exists i < \omega\, [c(2^n \cdot 3^{i+1}) = 1 \wedge s_i \subseteq x]$$

This shows that the statement is absolute.

(ii) If $c(0) = 0$, then $A_c = \emptyset \iff \forall i < \omega\ c(i+1) = 0$.

Now, assume $c(0) = 1$ and define

$$T = \{t \in {}^{<\omega}\omega \mid \forall i < lh(t)\, [c(i+1) = 1 \rightarrow s_i \not\subseteq t]\} \in M.$$

Then T is a subtree of ${}^{<\omega}\omega$ and

$$A_c^M = \emptyset \iff [T]^M = \emptyset \iff [T]^V = \emptyset \iff A_c^V = \emptyset.$$

Finally, assume $c(0) > 1$ and define

$$T = \{\langle t, u\rangle \in {}^{<\omega}\omega \times {}^{<\omega}\omega \mid lh(t) = lh(u) \wedge \forall n < lh(t)\, [c(2^n \cdot 3^{u(n)+1}) = 1 \wedge (u(n) < lh(t) \rightarrow s_{u(n)} \subseteq t)]\}$$

Then $T \in M$ is a subtree of ${}^{<\omega}\omega \times {}^{<\omega}\omega$ and

$$A_c^M = \emptyset \iff [T]^M = \emptyset \iff [T]^V = \emptyset \iff A_c^V = \emptyset.$$

The statements (iii) - (v) follow in the same way.

$\square$

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Remember that we use λ to denote the unique measure on $({}^\omega\omega, \mathcal{B})$ with

$$\lambda(N_s) = \prod_{i<\operatorname{lh}(s)} 2^{-(s(i)+1)}$$

for all $\emptyset \neq s \in {}^{<\omega}\omega$.

2.4.7. Proposition (ZF + (DC)): Let $A \subseteq {}^\omega\omega$ be λ-measurable.

(i): Given $\varepsilon > 0$, there is a closed subset C of ${}^\omega\omega$ and an open subset U of ${}^\omega\omega$ such that $C \subseteq A \subseteq U$ and $\lambda(U) - \lambda(C) < \varepsilon$.

(ii): There is an F_σ-set F and a G_δ-set G such that $F \subseteq A \subseteq G$ and $\lambda(F) = \lambda(G)$.

$\boxtimes$

2.4.8. Lemma: If M is a transitive model of ZF + (DC) and $c \in ({}^\omega\omega)^M$, then

$$\lambda(A_c) = (\lambda(A_c))^M.$$

Proof: (i) Assume $c(0) = 0$. Then there is a $d \in ({}^\omega\omega)^M$ such that $d(1) = 0$ and

$$A_c^M = A_d^M = \dot{\bigcup}\{N_{s(i)}^M \mid d(i+1) = 1\}.$$

Then $A_c^V = A_d^V = \dot{\bigcup}\{N_{s(i)}^V \mid d(i+1) = 1$

$$(\lambda(A_c))^M = (\lambda(A_d))^M = \sum_{i<\omega,\, d(i+1)=1} \Big(\prod_{n<\operatorname{lh}(s_i)} 2^{-(s_i(n)+1)}\Big)$$

$= (\lambda(A_d))^V = (\lambda(A_c))^V$.

(ii) Assume $c(0) = 1$. Then there is a $d \in ({}^{\omega}\omega)^M$ with $d(0) = 0$ and $A_c^M = ({}^{\omega}\omega \setminus A_d)^M$. This implies $A_c^V = ({}^{\omega}\omega \setminus A_d)^V$ and $(\lambda(A_c^M))^M = 1 - (\lambda(A_d))^M = 1 - \lambda(A_d) = \lambda(A_c)$.

(iii) Assume $c(0) > 1$. In M, there is a sequence $\langle c_m \in {}^{\omega}\omega \mid m < \omega \rangle$ such that $c_m = 0$ and

(*) $A_{c_m} = \bigcap_{n<m} \bigcup \{ N_{s(i)} \mid c(2^n \cdot 3^{i+1}) = 1 \}$

for all $m < \omega$. Then (*) also holds in V and we get

$$(\lambda(A_c))^M = \inf_{m<\omega} (\lambda(A_{c_m}))^M = \inf_{m<\omega} \lambda(A_{c_m}) = \lambda(A_c).$$

$\square$