2.3. The Levy Collapse

Reminder: Given a regular cardinal λ and $S \subseteq On$, the partial order $Col(\lambda, S)$ consists of partial functions p of cardinality less than λ with $dom(p) \subseteq S \times \lambda$ and $p(\alpha, \xi) \in \{0\} \cup \alpha$ for all $(\alpha, \xi) \in dom(p)$ ordered by reversed inclusion.

2.3.1. Lemma: If κ is an uncountable regular cardinal, then $Col(\omega, \kappa)$ satisfies the κ-chain condition. In particular, forcing with $Col(\omega, \kappa)$ preserves cofinalities greater than or equal to κ.

Proof: Let $A_0 \in [Col(\omega, \kappa)]^\kappa$. By the Δ-system Lemma, there is an $A \in [A_0]^\kappa$ and a finite subset r of $\alpha \times \omega$ with $\alpha < \kappa$ such that $r = dom(p) \cap dom(q)$ for all $p, q \in A$ with $p \neq q$. Since there are less than κ-many functions from r to α, there are $p, q \in A$ with $p \upharpoonright r = q \upharpoonright r$ and $p \neq q$. This shows that A_0 is not an antichain. $\square$

2.3.2. Corollary: If κ is an uncountable regular cardinal, then $\mathbb{1}_{Col(\omega,\kappa)} \Vdash$ "$\check{\kappa} = \omega_1$". $\square$

2.3.3. Proposition: If $S = S_0 \sqcup S_1 \subseteq On$, then the canonical map

$$i : Col(\omega, S) \longrightarrow Col(\omega, S_0) \times Col(\omega, S_1)$$
$$p \longmapsto (p \restriction (S_0 \times \omega), p \restriction (S_1 \times \omega))$$

is an isomorphism of partial orders.
In particular, $Col(\omega, S_i)$ is a complete subforcing of $Col(\omega, S)$. $\square$

2.3.4. Lemma: Let κ be a regular uncountable cardinal, $\lambda < \kappa$ and $\dot{x}$ be a $Col(\omega, \kappa)$-name for a subset of λ. Then there is a $\bar{\kappa} < \kappa$ and a $Col(\omega, \bar{\kappa})$-name $\dot{y}$ with $\mathbb{1}_{Col(\omega,\kappa)} \Vdash$ "$\dot{x} = \dot{y}$".

Proof: Let $\dot{y} = \bigcup_{\alpha<\lambda} \{\check{\alpha}\} \times A_\alpha$ be a $Col(\omega, \kappa)$-nice name for $\dot{x}$. Since $Col(\omega, \kappa)$ satisfies the κ-chain condition and κ is regular, there is a $\bar{\kappa} < \kappa$ with $A_\alpha \subseteq Col(\omega, \bar{\kappa})$ for all $\alpha < \lambda$. Then $\dot{y}$ is a $Col(\omega, \bar{\kappa})$-name. $\square$

2.3.5 Corollary: If κ is an inaccessible cardinal, G is $Col(\omega, \kappa)$-generic over V and $x : \lambda \to On$ is a function in $V[G]$ with $\lambda < \kappa$, then κ is inaccessible in $V[x]$. $\square$

2.3.6. Lemma: Let δ be ~~an infinite cardinal~~ limit ordinal and $\mathbb{P}$ be a separative partial order of cardinality at most $|\delta|$ with

$$\mathbb{1}_{\mathbb{P}} \Vdash \exists f : \omega \longrightarrow \delta \; [f \text{ is surjective} \wedge f \notin \check{V}].$$

Then there is an injective dense embedding of a dense subset of $Col(\omega, \{\delta\})$ into $\mathbb{P}$.

Proof: Note that below every condition in $\mathbb{P}$ there is a maximal antichain of size $|\delta|$. Given $n < \omega$, define

$$D_n = \{ p \in Col(\omega, \{\delta\}) \mid dom(p) = \{\delta\} \times n \}$$

Then $D = \bigcup_{n<\omega} D_n$ is a dense subset of $Col(\omega, \{\delta\})$. Let $\dot{f}$ be a $\mathbb{P}$-name for a surjection of ω onto the $\mathbb{P}$-generic filter. We define $e_n : D_n \longrightarrow \mathbb{P}$ by induction:

- $e_0(\emptyset) = \mathbb{1}_{\mathbb{P}}$
- Assume that e_n is already constructed. Fix $p \in D_n$ and fix an injective enumeration $\langle q_\gamma \mid \gamma < \delta \rangle$ of a maximal antichain in $\mathbb{P}$ below $e_n(p)$ consisting of conditions that decide $\dot{f}(\check{n})$. Define

$$e_{n+1}(p \cup \{((\delta, n), \gamma)\}) = q_\gamma.$$

Then $ran(e_{n+1})$ is a maximal antichain in $\mathbb{P}$ for every $n < \omega$ and $e = \bigcup_{n<\omega} e_n : D \longrightarrow \mathbb{P}$ is injective and order-preserving.

Pick $q \in \mathbb{P}$. Since $\mathbb{P}$ is separative, we have $q \Vdash_{\mathbb{P}}$ "$\check{q} \in \dot{G}$" and there is an $r \leq_{\mathbb{P}} q$ and an $n < \omega$ with $r \Vdash_{\mathbb{P}}$ "$\check{q} = \dot{f}(n)$".

By our construction, there is a $p \in D_{n+1}$ such that $l(p_{n+1}) \Vdash_{\mathbb{P}} \dot{r}$. Then $l_{n+1}(p) \Vdash_{\mathbb{P}}$ " $\check{q} = \dot{f}(\check{n}) \in \dot{G}$ " and this implies $l_{n+1}(p) \leq_{\mathbb{P}} q$, because $\mathbb{P}$ is separative. $\square$

2.3.7. Corollary: Let δ be an infinite cardinal and $\mathbb{P}$ be a partial order of cardinality δ. Then the partial orders $Col(\omega, \{\delta\})$ and $\mathbb{P} \times Col(\omega, \{\delta\})$ are forcing equivalent. $\square$

2.3.8. Corollary: Let κ be an uncountable cardinal and $\mathbb{P}$ be a partial order of cardinality less than κ. Then the partial orders $Col(\omega, \kappa)$ and $\mathbb{P} \times Col(\omega, \kappa)$ are forcing equivalent.

Proof: This follows from 2.3.3. and 2.3.7. $\square$

2.3.9. Lemma: Let κ be an uncountable regular cardinal and $\lambda < \kappa$. If G is $Col(\omega, \kappa)$-generic over V and $x : \lambda \to On$ is contained in $V[G]$, then κ is an uncountable regular cardinal in $V[x]$ and $V[G]$ is a $Col(\omega, \kappa)$-generic extension of $V[x]$.

<u>Proof:</u> By 2.3.4., there is $\delta < \kappa$ and $G_0, G_1, G_2 \in V[G]$ such that the following statements hold:

- G_0 is $Col(\omega, \delta)$-gen. $| V$ and $x \in V[G_0]$
- G_1 is $Col(\omega, \{\delta\})$-gen. $| V[G_0]$
- G_2 is $Col(\omega, \kappa \setminus (\delta+1))$-generic $| V$

and $V[G] = V[G_0][G_1][G_2]$.

Since $Col(\omega, \delta)$ satisfies the κ-chain condition in V, κ is an uncountable regular cardinal in $V[G_0]$ and hence also in $V[x]$.

By 2.2.6., there is a partial order $\mathbb{P}$ of cardinality ~~less than~~ $|\delta|$ in $V[x]$ such that $V[G_0]$ is a separative $\mathbb{P}$-generic extension of $V[x]$.

Since 2.3.7 implies that the partial orders $Col(\omega, \{\delta\})$, $\mathbb{P} \times Col(\omega, \{\delta\})$ and $Col(\omega, \delta+1)$ are all forcing equivalent in $V[x]$, we can conclude that $V[G_0][G_1]$ is a $Col(\omega, \delta+1)$-generic extension of $V[x]$ and hence $V[G]$ is a $Col(\omega, \kappa)$-generic extension of $V[x]$.

$\boxtimes$

2.3.10. Definition: A partial order $\mathbb{P}$ is *weakly homogeneous* if for all $p, q \in \mathbb{P}$ there is an automorphism π of $\mathbb{P}$ such that $\pi(p) \parallel_{\mathbb{P}} q$.

2.3.11 Proposition: If $\mathbb{P}$ is a weakly homogeneous partial order, then

$$\mathbb{1}_{\mathbb{P}} \Vdash \varphi(\check{x}_0, \dots, \check{x}_{n-1})$$
$$\Longleftrightarrow \exists p \in \mathbb{P} \quad p \Vdash_{\mathbb{P}} \varphi(\check{x}_0, \dots, \check{x}_{n-1})$$

for every formula $\varphi(v_0, \dots, v_{n-1})$ and all $x_0, \dots, x_{n-1}$.

Proof: Let $p \Vdash_{\mathbb{P}} \varphi(\check{x}_0, \dots, \check{x}_{n-1})$ and assume that there is a $q \in \mathbb{P}$ with $q \Vdash \neg\varphi(\check{x}_0, \dots, \check{x}_{n-1})$. Then there is a $\pi \in \mathrm{Aut}(\mathbb{P})$ and an $r \in \mathbb{P}$ with $r \leq_{\mathbb{P}} \pi(p), q$ and hence
$r \Vdash$ "$\varphi(\check{x}_0, \dots, \check{x}_{n-1}) \wedge \neg\varphi(\check{x}_0, \dots, \check{x}_{n-1})$",
a contradiction. $\boxtimes$

2.3.12. Proposition: If λ is a regular cardinal and $S \subseteq On$, then $\mathrm{Col}(\lambda, S)$ is weakly homogeneous.

Proof: Fix $p, q \in \mathrm{Col}(\lambda, S)$. Then there is $f \in \mathrm{Sym}(\lambda)$ with $\langle \beta, f(\xi)\rangle \notin \mathrm{dom}(q)$ for all $\langle \alpha, \xi\rangle \in \mathrm{dom}(p)$ and $\beta \in S$. There is $\pi \in \mathrm{Aut}(\mathbb{P})$ with $\mathrm{dom}(\pi(r)) = \{\langle \alpha, f(\xi)\rangle \mid \langle \alpha, \xi\rangle \in \mathrm{dom}(p)\}$ and $\pi(r)\langle \alpha, f(\xi)\rangle = r(\alpha, \xi)$ for all $r \in \mathrm{Col}(\lambda, S)$ and $(\alpha, \xi) \in \mathrm{dom}(r)$. Then $\mathrm{dom}(\pi(p)) \cap \mathrm{dom}(q) = \emptyset$ and hence $\pi(p) \parallel_{\mathrm{Col}(\lambda, S)} q$. $\boxtimes$

2.3.13. Definition: Let $A \subseteq {}^\omega\omega$ and M be an inner model of ZFC. We say that A is <u>Solovay over M</u> if there is a formula $\varphi(u, v_0, \dots, v_{n-1})$ and $y_0, \dots, y_{n-1} \in M$ such that
$$A = \{x \in {}^\omega\omega \mid M[x] \models \varphi(x, y_0, \dots, y_{n-1})\}$$

2.3.14 Theorem: Let κ be an inaccessible cardinal, G be $\mathrm{Col}(\omega, \kappa)$-generic over V and $A \subseteq {}^\omega\omega$ with $A \in \mathrm{HOD}({}^\omega\mathrm{On})^{V[G]}$. Then there is an inner model M of ZFC such that A is Solovay over M in $V[G]$ and $(2^{\aleph_0})^M$ is countable in $V[G]$.

<u>Proof:</u> By definition, there is a formula $\varphi(u, v)$ and $y \in ({}^\omega\mathrm{On})^{V[G]}$ such that
$$A = \{x \in ({}^\omega\omega)^{V[G]} \mid V[G] \models \varphi(x, y)\}$$
Set $M = V[y]$. By 2.3.5 and 2.3.9, $(2^{\aleph_0})^M$ is countable in $V[G]$. Given $x \in ({}^\omega\omega)^{V[G]}$, pick $H_x \in V[G]$ $\mathrm{Col}(\omega, \kappa)$-gen. $/ M[x]$ with $V[G] = M[x][H_x]$. By 2.3.11 and 2.3.12, we have
$$x \in A \iff V[G] \models \varphi(x, y) \iff \exists p \in H_x \; M[x] \models (p \Vdash \varphi(\check{x}, \check{y}))$$
$$\iff M[x] \models (\mathbb{1}_{\mathrm{Col}(\omega,\kappa)} \Vdash \varphi(\check{x}, \check{y})).$$
This yields a formula that witnesses that A is Solovay over M in $V[G]$.

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