2.2 Intermediate models of forcing extensions

2.2.1. Theorem: Let M be an inner model of ZFC and x be a set with $x \subseteq M$. Then there is a unique minimal inner model $M[x]$ of ZFC with $x \in M[x]$ and $M \subseteq M[x]$.

Sketch of the proof: There are $\gamma \in On$ and $f, X \in M$ such that $x \subseteq X$ and $f : \gamma \to X$ is a bijection. Define $a = f^{-1}{}''x$ and $M[x] = \bigcup_{b \in M,\, b \subseteq On} L[a, b]$. Then $M[x]$ is transitive, $On \subseteq M[x]$ and it can be shown that $M[x]$ is a model of ZFC. Since every set of ordinals in M is contained in $M[x]$, we have $M \subseteq M[x]$. Moreover, $a, f \in M[x]$ implies $x \in M[x]$.

Let N be an inner model with $M \subseteq N$ and $x \in N$. ~~Then $b \in N$ for every $b \in M$ with~~ Then $f, x \in N$ implies $a \in N$ and hence $L[a, b] \subseteq N$ for every $b \in M$ with $b \subseteq On$. In particular, we have $M[x] \subseteq N$. $\square$

Goal: Let $V[G]$ be a $\mathbb{P}$-generic extension of V and $x \in V[G]$ with $x \subseteq V$. Then there is a partial order $\mathbb{D}$ and a $\mathbb{D}$-name $\dot{\mathbb{Q}}_{\mathbb{D}}$ for a partial order such that $\mathbb{P}$ and $\mathbb{D} * \dot{\mathbb{Q}}_{\mathbb{D}}$ are forcing equivalent, $V[x] = V[H]$ for some $H \in V[G]$ that is $\mathbb{D}$-generic over V and $V[G]$ is a $\dot{\mathbb{Q}}_{\mathbb{D}}^H$-generic extension of $V[x]$.

VI.

Recall that a complete subalgebra $\mathbb{D}$ of a complete Boolean algebra $\mathbb{B}$ is <u>generated by $X \subseteq \mathbb{B}$</u> if $\mathbb{D}$ is the smallest complete subalgebra of $\mathbb{B}$ containing X. A complete Boolean algebra $\mathbb{B}$ is <u>κ-generated</u> for some cardinal κ if there is an $X \in [\mathbb{B}]^{\leq\kappa}$ that generates $\mathbb{B}$.

<u>2.2.2. Lemma:</u> Let $\mathbb{B}$ be a complete Boolean algebra and $X \subseteq \mathbb{B}$ such that X generates $\mathbb{B}$. If G is $\mathbb{B}$-generic over V, then $V[G] = V[G \cap X]$.

<u>Proof:</u> It suffices to show that $G \in V[G \cap X]$.
Working in V, we inductively construct sequences $\langle X_\alpha \mid \alpha < \Theta \rangle$ and $\langle \bar{X}_\alpha \mid \alpha < \Theta \rangle$ with $\Theta = |\mathbb{B}|^+$ by the following clauses:

- $X_0 = X$
- $\bar{X}_\alpha = \{ -a \mid a \in X_\alpha \}$
- $X_\alpha = \{ \sum Z \mid Z \subseteq \bigcup_{\bar{\alpha} < \alpha} (X_{\bar{\alpha}} \cup \bar{X}_{\bar{\alpha}}) \}$ for $\alpha > 0$

Then $\mathbb{B} = \bigcup_{\alpha < \Theta} X_\alpha$.
Working in $V[G \cap X]$, we inductively construct sequences $\langle G_\alpha \mid \alpha < \Theta \rangle$ and $\langle \bar{G}_\alpha \mid \alpha < \Theta \rangle$ by setting:

- $G_0 = G \cap X$
- $\bar{G}_\alpha = \{ -a \mid a \in X_\alpha \setminus G_\alpha \}$
- $G_\alpha = \{ \sum Z \mid Z \subseteq \bigcup_{\bar{\alpha} < \alpha} (X_{\bar{\alpha}} \cup \bar{X}_{\bar{\alpha}})$ with $Z \cap (G_{\bar{\alpha}} \cup \bar{G}_{\bar{\alpha}}) \neq \emptyset$ for some $\bar{\alpha} < \alpha \}$ for $\alpha > 0$.

An easy induction shows $G_\alpha = G \cap X_\alpha$ and $\bar{G}_\alpha = G \cap \bar{X}_\alpha$ for all $\alpha < \Theta$. Hence $G = \bigcup_{\alpha < \Theta} G_\alpha \in V[G \cap X]$. $\square$

2.2.3. Lemma: Let $\mathbb{B}$ be a complete Boolean algebra and κ be a cardinal. If $\dot{x}$ is a $\mathbb{B}^*$-name for a subset of κ, then there is a κ-generated complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ such that $V[\dot{x}^G] = V[G \cap \mathbb{D}]$ holds whenever G is $\mathbb{B}^*$-generic over V.

Proof: Given $\alpha < \kappa$, define
$$b_\alpha = \sum \{ a \in \mathbb{B}^* \mid a \Vdash_{\mathbb{B}^*} \text{"} \check{\alpha} \in \dot{x} \text{"} \}.$$
Let $\mathbb{D}$ denote the complete subalgebra generated by $X = \{ b_\alpha \mid \alpha < \kappa \}$. Set
$$\dot{a} = \{ \langle \check{\alpha}, b_\alpha \rangle \mid \alpha < \kappa,\ b_\alpha \neq 0_{\mathbb{B}} \}.$$
Let G be $\mathbb{B}^*$-generic over V. If $\alpha \in \dot{x}^G$, then $0_{\mathbb{B}} \neq b_\alpha \in G \cap \mathbb{D}$ and hence $\alpha \in \dot{a}^{G \cap \mathbb{D}}$. If $\alpha \in \dot{a}^{G \cap \mathbb{D}}$, then there is an $a \in G$ with $a \Vdash_{\mathbb{B}^*}$ "$\check{\alpha} \in \dot{x}$" and hence $\alpha \in \dot{x}^G$.
This shows $\dot{x}^G = \dot{a}^{G \cap \mathbb{D}}$ and $V[\dot{x}^G] \subseteq V[G \cap \mathbb{D}]$.
By Lemma 2.2.2, we have $V[G \cap \mathbb{D}] = V[G \cap X]$. Since $G \cap X = \{ b_\alpha \mid \alpha \in \dot{x}^G \} \in V[\dot{x}^G]$, we also get $V[G \cap \mathbb{D}] \subseteq V[\dot{x}^G]$. $\square$

2.2.4. Lemma: Let $\mathbb{B}$ be a complete Boolean algebra, G be $\mathbb{B}^*$-generic over V and let M be a transitive model of ZFC with $V \subseteq M \subseteq V[G]$. Then there is a complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ such that $M = V[G \cap \mathbb{D}]$.

Proof: Let $a \in M$ with $a \subseteq On$ and $P(\mathbb{B})^M \in V[a]$. By the above Lemma, it suffices to show $M \subseteq V[a]$.

Let $x \in M$ with $x \subseteq On$. By Lemma 2.2.3, there is a complete subalgebra $\mathbb{D}$ of $\mathbb{B}$ in V with $V[x] = V[G \cap \mathbb{D}]$. Then $G \cap \mathbb{D} \in P(\mathbb{B})^M \subseteq V[a]$ and hence $x \in V[a]$. This shows $M \subseteq V[a]$. $\boxed{\mathbb{B}}$

Let $\mathbb{P}$ be a separative partial order, $i : \mathbb{P} \longrightarrow \mathbb{B}(\mathbb{P})$ be the canonical ~~embedding~~ dense embedding into the complete Boolean algebra of regular open subsets of $\mathbb{P}$ and $\mathbb{D}$ be a complete subalgebra of $\mathbb{B}(\mathbb{P})$. Define $\dot{\mathbb{Q}}_{\mathbb{D}}$ to be the canonical $\mathbb{D}^*$-name with the property that whenever H is $\mathbb{D}^*$-generic over V, then $\dot{\mathbb{Q}}^H_{\mathbb{D}}$ is a partial order with domain

~~$\{p \in \mathbb{P} \mid \forall q \in \mathbb{P} \, [i(q) \in H \Rightarrow p \parallel_{\mathbb{P}} q]\}$ ordered by $\leq_{\mathbb{P}}$.~~

$$\{p \in \mathbb{P} \mid \forall q \in H \;\; i(p) \parallel_{\mathbb{B}(\mathbb{P})^*} q\}$$

ordered by $\leq_{\mathbb{P}}$.

2.2.5. Lemma: (i) The map

$$L: \mathbb{P} \longrightarrow \mathbb{D}^*;\ p \longmapsto \sum\{a \in \mathbb{D}^* \mid a \Vdash_{\mathbb{D}^*} \text{"}\check{p} \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}\}$$

is well-defined and order-preserving
(ii) The map

$$d: \mathbb{P} \longrightarrow \mathbb{D}^* * \dot{\mathbb{Q}}_{\mathbb{D}};\ p \longmapsto (L(p), \check{p})$$

is a well-defined dense embedding.

Proof: Fix $p \in \mathbb{P}$. If G is $\mathbb{P}$-generic over V and $H = i[G]$, then $p \in \dot{\mathbb{Q}}_{\mathbb{D}}^{H \cap \mathbb{D}}$ and there is a $q \in H \cap \mathbb{D}$ with $q \Vdash_{\mathbb{D}^*} \text{"}\check{p} \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}$. This shows that $\sum\{a \in \mathbb{D}^* \mid a \Vdash_{\mathbb{D}^*} \text{"}\check{p} \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}\} \neq 0_{\mathbb{D}^*}$.

If $p_0 \leq_{\mathbb{P}} p_1$ and $q \in \mathbb{D}^*$ with $q \Vdash_{\mathbb{D}^*} \text{"}\check{p}_0 \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}$, then $q \Vdash_{\mathbb{D}^*} \text{"}\check{p}_1 \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}$ and we can conclude that $L(p_0) \leq_{\mathbb{D}^*} L(p_1)$.

Pick $(q_0, \dot{p}) \in \mathbb{D}^* * \dot{\mathbb{Q}}_{\mathbb{D}}$. Then there is $q \leq_{\mathbb{D}^*} q_0$ and $p \in \mathbb{P}$ with $q \Vdash_{\mathbb{D}^*} \text{"}\check{p} = \dot{p}\text{"}$ and hence $(q, \check{p}) \leq_{\mathbb{D}^* * \dot{\mathbb{Q}}_{\mathbb{D}}} (q_0, \dot{p})$. Since $q \Vdash \text{"}\check{p} \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}$ we have $i(p) \parallel_{\mathbb{B}(\mathbb{P})^*} q$ and there is a $p_* \leq_{\mathbb{P}} p$ with $i(p_*) \leq_{\mathbb{B}(\mathbb{P})^*} q$.
Assume, towards a contradiction, that $L(p_*) \not\leq_{\mathbb{D}^*} q$. Then there is an $r \leq_{\mathbb{D}^*} L(p_*)$ with $q \perp_{\mathbb{D}^*} r$. But $r \Vdash_{\mathbb{D}^*} \text{"}\check{p}_* \in \dot{\mathbb{Q}}_{\mathbb{D}}\text{"}$ and hence $i(p_*) \parallel_{\mathbb{B}(\mathbb{P})^*} r$. Since $i(p_*) \leq_{\mathbb{B}(\mathbb{P})^*} q$, this yields $q \parallel_{\mathbb{D}} r$, a contradiction.
We can conclude $d(p_*) \leq_{\mathbb{D}^* * \dot{\mathbb{Q}}_{\mathbb{D}}} (q_0, \dot{p})$. ☐

2.2.6. Corollary: Let $\mathbb{P}$ be a partial order of cardinality κ in V, G be $\mathbb{P}$ generic over V and M be a transitive model of ZFC with $V \subseteq M \subseteq V[G]$. Then there is a partial order $\mathbb{Q}$ of cardinality at most κ in M such that $V[G]$ is a $\mathbb{Q}$-generic extension of M.

Proof: By forming the separative quotient, we may assume that $\mathbb{P}$ is separative. ~~By~~ Let H denote the filter on $\mathbb{B}(\mathbb{P})^*$ induced by G. By Lemma 2.2.4, there is a complete subalgebra $\mathbb{D}$ of $\mathbb{B}(\mathbb{P})$ in V such that $M = V[H \cap \mathbb{D}]$. In this situation, Lemma 2.2.5 shows that there is $K \in V[G]$ that is $\dot{\mathbb{Q}}_{\mathbb{D}}^{H \cap \mathbb{D}}$-generic over M with $V[G] = M[K]$. By definition, $\dot{\mathbb{Q}}_{\mathbb{D}}^{H \cap \mathbb{D}}$ has cardinality at most κ in M.