2. The Solovay Model

2.1. Introduction

Let $\mathcal{B}_{\mathbb{R}}$ denote the σ-algebra of Borel sets of real numbers (i.e. Dedekind cuts of rational numbers).

2.1.1. Theorem (Lebesgue, 1901): There is a unique translation-invariant measure $\lambda_{\mathbb{R}}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$ with $\lambda([0,1]) = 1$.

2.1.2. Definition: Let $(X, \mathcal{A}, \mu)$ be a measure space. We say that $Y \subseteq X$ is *μ-measurable* if there are $B, N \in \mathcal{A}$ with $B \Delta Y \subseteq N$ and $\mu(N) = 0$.

2.1.3. Theorem (Vitali, 1905): There is a subset of $\mathbb{R}$ that is not $\lambda_{\mathbb{R}}$-measurable.

Proof: Define an equivalence relation $\sim$ on $\mathbb{R}$ by setting:

$$x \sim y :\Longleftrightarrow x - y \in \mathbb{Q}.$$

Then there is a $T \subseteq [0,1]$ such that ~~the~~ T contains exactly one element of every $\sim$-equivalence class. Assume, towards a contradiction, that $B, N \in \mathcal{B}_{\mathbb{R}}$ witness that T is $\lambda_{\mathbb{R}}$-measurable.

Then $\lambda_{\mathbb{R}}(B) > 0$, because otherwise

$$\lambda_{\mathbb{R}}(\mathbb{R}) = \lambda_{\mathbb{R}}\Big(\bigcup_{q\in\mathbb{Q}} T+q\Big) \leq \lambda_{\mathbb{R}}\Big(\bigcup_{q\in\mathbb{Q}}((B\cup N)+q)\Big)$$

$$\leq \sum_{q\in\mathbb{Q}} \lambda_{\mathbb{R}}((B\cup N)+q) = \sum_{q\in\mathbb{Q}} \lambda(B\cup N) = 0.$$

But this implies

$$2 = \lambda_{\mathbb{R}}([0,2]) \geq \lambda_{\mathbb{R}}\Big(\dot{\bigcup_{q\in\mathbb{Q}\cap[0,1]}}(B\setminus N + q)\Big)$$

$$= \sum_{q\in\mathbb{Q}\cap[0,1]} \lambda_{\mathbb{R}}((B\setminus N)+q) = \sum_{q\in\mathbb{Q}\cap[0,1]} \lambda_{\mathbb{R}}(B\setminus N) = \sum_{q\in\mathbb{Q}\cap[0,1]} \lambda_{\mathbb{R}}(B) = +\infty,$$

a contradiction. $\square$

The above argument uses the axiom of choice.

<u>Naive Question:</u> Is (AC) necessary to construct a subset of $\mathbb{R}$ that is not $\lambda_{\mathbb{R}}$-measurable?

Since the development of measure theory requires some amount of choice, we have to clarify the above question.

<u>2.1.4. Definition:</u> The <u>Principle of Dependent Choice</u> is the statement

(DC) $\forall X \neq \emptyset \ \forall R \subseteq X \times X$
$\quad [\forall x\in X \ \exists y\in X \ \langle x,y\rangle \in R$
$\quad\quad \longrightarrow \exists f: \omega \longrightarrow X \ \forall n<\omega \ \langle f(n), f(n+1)\rangle \in R].$

2.15. Proposition (ZF): (DC) implies (AC_ω).

Fact: (DC) suffices for the development of measure theory and the construction of $\lambda_{\mathbb{R}}$.

Question: Con(ZF + (DC) + "all subsets of $\mathbb{R}$ are $\lambda_{\mathbb{R}}$-measurable")?

2.16. Theorem (Solovay, 1964):

Con(ZFC + "there is an inaccessible cardinal")
$\longrightarrow$ Con(ZF + (DC) + "all subsets of $\mathbb{R}$ are $\lambda_{\mathbb{R}}$-measurable").

Remark: Shelah showed that these theories are equiconsistent.

2.1.7. Definition: Let λ be a regular cardinal and $S \subseteq On$. Define $Col(\lambda, S)$ to be the partial order whose conditions are partial functions p of cardinality less than λ such that $dom(p) \subseteq S \times \lambda$ and $p(\alpha, \xi) \in \{0\} \cup \alpha$ for all $\langle \alpha, \xi \rangle \in dom(p)$ ordered by reversed inclusion.

2.1.8. Definition: We let $HOD({}^\omega On)$ denote the class of all sets that are hereditarily definable over the class ${}^\omega On$, i.e. the class of all x such that for every $y \in tc(\{x\})$ there is a formula $\varphi(u, v_0, \ldots, v_{n-1})$ and $z_0, \ldots, z_{n-1} \in {}^\omega On$ such that

$$y = \{a \mid \varphi(a, z_0, \ldots, z_{n-1})\}.$$

Then $\mathrm{HOD}({}^{\omega}\mathrm{On})$ is an inner model of ZF containing all the reals.

2.1.9 Theorem (Solovay, 1964): Let κ be an inaccessible cardinal. If G is $\mathrm{Col}(\omega, \kappa)$-generic over V, then*) $\mathrm{HOD}({}^{\omega}\mathrm{On})^{V[G]}$ is a model of ZF + (DC) + "all subsets of $\mathbb{R}$ are $\lambda_{\mathbb{R}}$-measurable".

2.1.10 Definition: The Baire space is the topological space consisting of the set ${}^{\omega}\omega$ of all functions from ω to ω equipped with the topology whose basic open sets are of the form

$$N_s = \{ x \in {}^{\omega}\omega \mid s \subseteq x \}$$

for some $s : n \to \omega$ with $n < \omega$.

We let $\mathcal{B}$ denote the σ-algebra of Borel subsets of ${}^{\omega}\omega$.

2.1.11 Theorem (ZF + (DC)): There is a unique measure λ on $({}^{\omega}\omega, \mathcal{B})$ such that

$$\lambda(N_s) = \prod_{i < lh(s)} 2^{-(s(i)+1)}$$

holds for every $s : n \to \omega$ with $0 < n < \omega$.

2.1.12. Theorem (ZF + (DC)): The measure spaces $(\mathbb{R}, \mathcal{B}_{\mathbb{R}}, \lambda_{\mathbb{R}})$ and $({}^{\omega}\omega, \mathcal{B}, \lambda)$ are isomorphic.

*) every subset of $\mathbb{R}$ in $\mathrm{HOD}({}^{\omega}\mathrm{On})^{V[G]}$ is $\lambda_{\mathbb{R}}$-measurable in $V[G]$ and ...