Reconstructing ω-categorical structures from their clones

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Joint work with Manuel Bodirsky and András Pongrácz

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Outline

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- **Part I:** Reconstructing structures from their automorphism groups and polymorphism clones

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- **Part II:** The topology of algebras

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- **Part III:** Topological reconstruction notions for clones

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- **Part IV:** Negative results

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- **Part V:** Positive results

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- **Part II:** The topology of algebras
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- **Part IV:** Negative results
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Part I

Reconstructing structures from their automorphism groups and polymorphism clones

Reconstructing structures up to first-order . . .

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countable

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countable, ω-categorical

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Aut()

Aut() $\to$

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Theorem (Ryll-Nardzewski)

Let Δ, Γ be ω-categorical structures on the same domain. Then $\mathrm{Aut}(\Delta) = \mathrm{Aut}(\Gamma)$ iff Δ, Γ are first-order interdefinable.

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- Can we reconstruct the topological structure of closed oligomorphic permutation groups from their algebraic structure?

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Observe: $\mathrm{Pol}(\Delta) \supseteq \mathrm{End}(\Delta) \supseteq \mathrm{Aut}(\Delta)$.

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Theorem (Bodirsky + Nešetřil '03)

Let Δ, Γ be ω-categorical structures on the same domain. Then:
Pol(Δ) $=$ Pol(Γ) iff Δ, Γ are primitive positive interdefinable.

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Applications in theoretical computer science.

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Fact: When there is a pp interpretation of Δ in Γ, then there is a polynomial-time reduction from CSP(Δ) to CSP(Γ).

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Part II

The topology of algebras

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Structural conclusions about *finite* $\mathfrak{A}$ from variety of $\mathfrak{A}$
(i.e., from abstract clone $\mathrm{Clo}(\mathfrak{A})$).

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Theorem (Birkhoff 1935)

Let $\mathfrak{A}$, $\mathfrak{B}$ be finite.

$\mathfrak{B}$ is in $\mathsf{HSP}^{\mathrm{fin}}(\mathfrak{A}) \leftrightarrow$
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Theorem ('Topological Birkhoff'; Bodirsky + MP '12)

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Part III

Reconstruction notions

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Similarly for closed subgroups of $\mathbf{S}_\infty$ and closed submonoids of $\mathbf{O}^{(1)}$.

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Observation. Automatic homeomorphicity implies reconstruction.

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Observation. Automatic homeomorphicity implies reconstruction.

Fact. For groups automatic continuity implies automatic homeomorphicity.

Comparing the notions

Definition

- **C** has reconstruction iff $\mathbf{C} \cong \mathbf{D}$ implies $\mathbf{C} \cong^T \mathbf{D}$ for all closed subclones **D** of **O**;
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Unclear for monoids and clones.

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- the random *k*-hypergraphs
  the Henson digraphs

  (Barbina+MacPherson '07).

Part IV

Negative results

Automatic continuity for monoids / clones

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Important in constraint satisfaction:
"main reason" for NP-hardness of the CSP of a structure.

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Theorem (Bodirsky + MP + Pongrácz '13)

There exists an oligomorphic closed subclone of **O** with a discontinuous homomorphism to the projection clone **1**.

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Moreover, this clone also has a continuous homomorphism to **1**.

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Theorem (Bodirsky + MP + Pongrácz '13)

There exists an oligomorphic closed submonoid $\mathbf{M}$ of $\mathbf{O}^{(1)}$ and $\xi\colon \mathbf{M} \to \mathbf{M}$ such that:

- ξ is an isomorphism;
- ξ fixes the invertibles of $\mathbf{M}$ pointwise;
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In particular $\mathbf{M}$ does not have automatic homeomorphicity.

Theorem (Evans + Hewitt '90)

There exists an oligomorphic closed subgroup $\mathbf{G}$ of $\mathbf{S}_\infty$ which does not have reconstruction.

Reconstruction

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Problem

Find an oligomorphic closed subclone of **O** without reconstruction.

Part V

Positive results

Automatic continuity via Birkhoff's theorem

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The only possibly discontinuous step is an infinite product.

Theorem (Bodirsky + MP + Pongrácz '13)

Any closed subclone of **O** containing $\mathbf{O}^{(1)}$ has automatic continuity and automatic homeomorphicity.

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- Show that the closure of $\mathbf{G}_C$ has reconstruction;
- show that the monoid $\mathbf{C}^{(1)}$ of unary functions of **C** has reconstruction;

Automatic homeomorphicity via groups

Let $\mathbf{C}$ be a closed subclone of $\mathbf{O}$
whose group $\mathbf{G}_C$ of invertibles has reconstruction.

- Show that the closure of $\mathbf{G}_C$ has reconstruction;
- show that the monoid $\mathbf{C}^{(1)}$ of unary functions of $\mathbf{C}$ has reconstruction;
- then show that $\mathbf{C}$ has reconstruction.

Today's reconstruction theorem

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Theorem

The polymorphism clone of the random graph
has automatic homeomorphicity.

Open problems

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- Which oligomorphic closed subclones of **O** have automatic homeomorphicity?

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- Which oligomorphic closed subclones of $\mathbf{O}$ have automatic homeomorphicity?
- Which topological clones are closed subclones of $\mathbf{O}$?
- Is there an oligomorphic closed subclone of $\mathbf{O}$ which does not have reconstruction?

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- Which oligomorphic closed subclones of **O** have automatic homeomorphicity?
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- Which oligomorphic closed subclones of **O** have automatic homeomorphicity?
- Which topological clones are closed subclones of **O**?
- Is there an oligomorphic closed subclone of **O** which does not have reconstruction?
- Is there an oligomorphic closed subclone of **O** which has a homomorphism to the projection clone **1**, but no continuous one?
- Is there a model of ZF where all homomorphisms from oligomorphic closed subclones of **O** to the projection clone **1** are continuous?

Thank you!