Models of Set Theory II - Winter 2013

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Problem 29 (6 Points)**.** Suppose that $(P, \leq_P)$ and $(Q, \leq_Q)$ are transitive reflexive relations (also called *weak partial orders*, *quasiorders*, or *preorders*). A *Galois-Tukey reduction from P to Q* is a pair of functions $\phi\colon P \to Q$, $\phi^*\colon Q \to P$ such that

$$\forall p \in P\ \forall q \in Q\ (\phi(p) \leq_Q q \to p \leq_P \phi^*(q)).$$

Suppose that $\mathcal{I}$ and $\mathcal{J}$ are ideals such that there is a Galois-Tukey reduction from $(\mathcal{I}, \subseteq)$ to $(\mathcal{J}, \subseteq)$. Prove the following statements.

(a) $\mathrm{add}(\mathcal{I}) \geq \mathrm{add}(\mathcal{J})$.

(b) $\mathrm{cof}(\mathcal{I}) \leq \mathrm{cof}(\mathcal{J})$.

(c) Suppose that D is a cofinal subset of P and and that E is a cofinal subset of Q. If there is a Galois-Tukey reduction from $(D, \leq_P \cap (D \times D))$ to $(E, \leq_Q \cap (E \times E))$, then there is a Galois-Tukey reduction from $(P, \leq_P)$ to $(Q, \leq_Q)$.

Problem 30 (8 Points)**.** If $f \in {}^\omega\omega$, let $f'(n) = \max\{f(i) \mid i \leq n\} + 1$ and $\phi(f) = \{x \in {}^\omega\omega \mid x \leq^* f'\}$. Now suppose that $F_0 \subseteq F_1 \subseteq ...$ is a sequence of nowhere dense closed subsets of ${}^\omega\omega$ and $F = \bigcup_{i\in\omega} F_i$. We define $(k_n, s_n)_{n\in\omega}$ by induction. Let $k_0 = 0$. If k_n is defined, choose $s_n \in {}^{<\omega}\omega$ such that

$$\forall t \in {}^{\leq k_n}k_n\ \forall i \leq n\ N_{t^\frown s_n} \cap F_i = \emptyset.$$

Let

$$k_{n+1} = k_n + |s_n| + \max(\mathrm{range}(s_n)) + 1.$$

and

$$\phi^*(F) = \max(\mathrm{range}(s_n)).$$

(a) Show that $\phi(f)$ is a meager subset of ${}^\omega\omega$ for all $f \in {}^\omega\omega$.

(b) Suppose that $f \in {}^\omega\omega$ and $f \not\leq^* \phi^*(F)$. Suppose that $Z \subseteq \omega$ is infinite with $f(n) > \phi^*(F)(n)$ for all $n \in Z$ and that $(z_i)_{i\in\omega}$ is the order preserving enumeration of Z. Let

$$x = 0^\frown 0^\frown ... 0^\frown s_{z_0}^\frown 0^\frown ... 0^\frown s_{z_1} ...$$

such that s_{z_i} begins in place $k_{z_i} + 1$ for all $i \in \omega$. Show that $x \in \phi(f) \setminus F$.

(c) Show that $\mathrm{add}(\mathcal{M}) \leq \mathfrak{b}$.

Problem 31 (6 Points)**.** A forcing $(P, \leq_P, 1_P)$ is called *σ-linked* if there is a function $f\colon P \to \omega$ such that $p, q \in P$ are compatible if $f(p) = f(q)$.

Let μ denote the unique measure on the Borel subsets of ${}^{\omega}2$ with $\mu(N_t) = 2^{-|t|}$ for all $t \in {}^{<\omega}2$. Let P denote the weak partial order consisting of the Borel subset B of ${}^{\omega}2$ with $\mu(B) > 0$, ordered by reverse inclusion. Let

$$C_t = \{B \subseteq {}^{\omega}2 \mid B \text{ is Borel and } \mu(N_t \setminus B) < \frac{1}{2}\mu(N_t)\}.$$

for $t \in {}^{<\omega}2$. Every Borel subset of ${}^{\omega}2$ with positive measure is an element of C_t for some $t \in {}^{<\omega}2$ by the Lebesgue density theorem.

(a) Show all $A, B \in C_t$ are compatible for any $t \in {}^{<\omega}2$. Conclude that P is σ-linked.

(b) Suppose that κ is a cardinal and P_α is a σ-linked forcing for all $\alpha < \kappa$. Show that the finite support product $\prod_{\alpha<\kappa} P_\alpha$ is c.c.c.

Please hand in your solutions on Monday, December 16 before the lecture.