Models of Set Theory II - Winter 2013

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht Problem sheet 5

Problem 16 (6 Points)**.** Suppose that κ is an infinite regular cardinal, $(P, \leq_P, 1_P)$ is a κ-distributive partial order, and $\dot{Q}$, $\dot{\leq}_Q$, $\dot{1}_Q$ are P-names such that

$$1_P \Vdash_P \text{"}(\dot{Q}, \dot{\leq}_Q, \dot{1}_Q) \text{ is a } \kappa - \text{distributive partial order"}.$$

Show that $P * \dot{Q}$ is κ-distributive.

Problem 17 (4 Points)**.** Suppose that κ is an infinite cardinal.

(a) Show that $H_{\kappa^+} \prec_{\Sigma_1} V$. (*Hint: form the transitive collapse of a witness for the Σ_1 statement together with the transitive closure of the parameters*)

(b) Show that $H_{\omega_1} \not\prec_{\Sigma_2} V$.

Problem 18 (8 Points)**.** Suppose that $(B, \leq, \wedge, \vee, 0, 1)$ is a complete Boolean algebra and $S \subseteq B^* := B \setminus \{0\}$.

(a) Show that $p \wedge \bigvee S = \bigvee \{p \wedge s \mid s \in S\}$ (*hint: show that $s = (p \wedge s) \vee (\neg p \wedge s)$ for each $s \in S$*).

(b) Show that the following conditions are equivalent:

   (i) S is predense below p (i.e. every $q \leq p$ is compatible with some $s \in S$),

   (ii) q is compatible with $\bigvee S$ for all $q \leq p$,

   (iii) $p \leq \bigvee S$.

(c) Now suppose that M is a ground model and that $(B, \leq, \wedge, \vee, 0, 1)$ is a complete Boolean algebra in M. Suppose that φ is a formula and $\sigma \in M^B$. Let $q := [\![\varphi(\sigma)]\!] := \bigvee \{p \mid p \Vdash^M_{B^*} \varphi(\sigma)\}$. Show that

$$q \Vdash^M_{B^*} \varphi(\sigma)$$

(in particular, $\{[\![\varphi(\sigma)]\!], [\![\neg\varphi(\sigma)]\!]\}$ is a maximal antichain in B^*).

Problem 19 (4 Points)**.** Suppose that M is a ground model. Suppose that in M, $(P_n, \leq_n, 1_{P_n})_{n \leq \omega}$ is a finite support iteration of $(\dot{Q}_n, \dot{\leq}_{Q_m}, \dot{1}_{Q_n})_{n<\omega}$ such that

(i) $P_0 = Add(\omega_1, 1)$ and

(ii) $1_{P_n} \Vdash_{P_n} \text{"}\dot{Q}_n = Add(\omega_1, 1)\text{"}$ for all $n < \omega$.

If G is P_ω-generic over M, show that there is a Cohen real over M in $M[G]$.

Problem 20 (Extra problem, 6 Points)**.** Suppose that M is a ground model. Suppose that in M, $(P_n, \leq_n, 1_{P_n})_{n \leq \omega}$ is a finite support iteration of $(\dot{Q}_n, \dot{\leq}_{Q_m}, \dot{1}_{Q_n})_{n<\omega}$ such that for all $n < \omega$, $1_{P_n} \Vdash_{P_n} \dot{Q}_n$ is nonatomic.

Suppose that G is P_ω-generic over M. Show that there is a Cohen real over M in $M[G]$ (*Hint: We can assume that there is an ordinal λ such that for all $n < \omega$, $1_{P_n} \Vdash_{P_n} \text{"}dom(\dot{Q}_n) \subseteq \lambda\text{"}$*).

Please hand in your solutions on Monday, November 25 before the lecture.