Models of Set Theory II - Winter 2013

Prof. Dr. Peter Koepke, Dr. Philipp Schlicht — Problem sheet 2

Let μ denote Lebesgue measure on $\mathbb{R}$. A set $A \subseteq \mathbb{R}$ has *strong measure 0* if for every sequence $(\epsilon_n)_{n\in\omega}$ with $\epsilon_n > 0$ for all n, there is a sequence $(U_n)_{n\in\omega}$ of open intervals of $\mathbb{R}$ with $A \subseteq \bigcup_{n\in\omega} U_n$ and $\mu(U_n) \leq \epsilon_n$.

For subsets of the Cantor space ${}^{\omega}2$, we define *strong measure* 0 analogously, where open intervals are replaced by basic open sets N_t, and $\mu(N_t) := 2^{-|t|}$.

Problem 5 (8 Points)**.**

(a) Suppose that κ is an infinite cardinal. Prove from MA_κ that every set $A \subseteq \mathbb{R}$ of size $\leq \kappa$ has strong measure 0.

(b) Suppose that $T \subseteq {}^{<\omega}2$ is such that $(T, \subseteq)$ is isomorphic to $({}^{<\omega}2, \subseteq)$. Show that

$$A := [T] = \{x \in {}^{\omega}2 \mid \forall n \in \omega\ (x \upharpoonright n \in T)\}$$

does not have strong measure 0.

Consider the relation $\leq^*$ defined in Problem 4(c). A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is *unbounded* if $\mathcal{F}$ has no upper bound $g\colon \omega \to \omega$ with respect to $\leq^*$. The *bounding number* $\mathfrak{b}$ is defined as the least size of an unbounded family $\mathcal{F} \subseteq {}^{\omega}\omega$.

Problem 6 (5 Points)**.** Consider Hechler forcing from Problem 33, Models of Set Theory I (available on our website at teaching: lectures in past semesters). Suppose that κ is an infinite cardinal. Prove from MA_κ that $\mathfrak{b} \geq \kappa$.

A family $\mathcal{F} \subseteq {}^{\omega}\omega$ is *dominating* if for every $g\colon \omega \to \omega$, there is a function $f \in \mathcal{F}$ with $g \leq^* f$. The *dominating number* $\mathfrak{d}$ is defined as the least size of a dominating family $\mathcal{F} \subseteq {}^{\omega}\omega$.

Let $\mathcal{M}({}^{\omega}\omega)$ denote the set of meager subsets of ${}^{\omega}\omega$. Let $\mathrm{cov}(\mathcal{M}({}^{\omega}\omega))$ denote the least size of a family of meager subsets of ${}^{\omega}\omega$ whose union is ${}^{\omega}\omega$. Let $\mathrm{non}(\mathcal{M}({}^{\omega}\omega))$ denote the least size of a nonmeager subset of ${}^{\omega}\omega$.

Problem 7 (9 Points)**.** Prove the following inequalities.

(a) $\mathfrak{b} \leq cof(\mathfrak{d})$.

(b) $\mathfrak{b} \leq \mathrm{non}(\mathcal{M}({}^{\omega}\omega))$.

(c) $\mathrm{cov}(\mathcal{M}({}^{\omega}\omega)) \leq \mathfrak{d}$.

A partial order $(\mathbb{P}, \leq)$ is *Knaster* if every uncountable subset of $\mathbb{P}$ has a uncountable subsets consisting of pairwise compatible elements. The next problem shows from MA_{ω_1} that c.c.c. implies Knaster.

Problem 8 (6 Points)**.** Suppose that $\mathbb{P} = (\mathbb{P}, \leq)$ is a c.c.c. partial order and that $X \subseteq \mathbb{P}$ is uncountable.

(a) Show that there is some $p_0 \in X$ such that every $p \leq p_0$ is compatible with uncountably many elements of X.

(b) Prove from MA_{ω_1} that there is a filter $G \subseteq \mathbb{P}$ such that $G \cap X$ is uncountable.

Please hand in your solutions on Monday, November 04 before the lecture.