Models of Set Theory II

BY PETER KOEPKE

Monday, 14:15-16, Wednesday 12:30-14, Room 1.008, Endenicher Allee 60.

Abstract

Martin's Axiom and applications, iterated forcing, forcing Martin's axiom, adding various types of generic reals, proper forcing.

1 Introduction

The method of *forcing* allows to construct models of set theory with interesting or exotic properties. Further results can be obtained by *transfinite iterations* of this technique. More precisely, iterated forcing defines ordinary generic extensions, which can be analyzed by an increasing well-ordered tower of intermediate models where successor models are ordinary generic extensions of the previous models. Such an analysis is already possible for the Cohen model for $2^{\aleph_0} = \aleph_2$, and we shall indicate some aspects in an introductory chapter. In that model, partially generic filters exist for the standard Cohen forcing $\mathrm{Fn}(\aleph_0, 2, \aleph_0)$. This motivates *forcing axioms* which require the existence of partially generic filters for certain forcings. *Martin's Axiom* MA is a forcing axiom for forcings satisfying the countable antichain condition (ccc). We shall study some consequences of MA and shall then force that axiom by iterated forcing. We shall also study the *Proper Forcing Axiom* PFA for a class of forcings which are *proper*.

Our forcing constructions are mostly directed towards properties of the set $\mathbb{R}$ of real numbers. There are several forcings which adjoin new reals to (ground) models. Different forcings adjoin reals which may be very different with respect to growth behaviour and other aspects. Cardinal characteristics of $\mathbb{R}$ have been introduced to describe such behaviours. They are systematised in CICHON's diagram. Using MA and iterated forcings several constellations of cardinals are realized in CICHON's diagram.

2 Cohen forcing

The most basic forcing construction is the adjunction of a Cohen generic real c to a countable transitive ground model M. The generic extension $M[c]$ is again a countable transitive model of ZFC and it contains the "new" real $c \notin M$. In the previous semester we saw that the adjunction of c has consequences for the set theory within $M[c]$:

Theorem 1. *In the* COHEN *extension $M[c]$ the set $\mathbb{R} \cap M$ of ground model reals has (Lebesgue) measure zero.*

This implies some (relative) consistency results. We may, e.g., assume that M is a model of the axiom of constructibility $V = L$, i.e., $M = L^M$. Since the class L is absolute between transitive models of set theory of the same ordinal height, $L^{M[c]} = L^M = M$. So:

Theorem 2. *Let M be a ground model of* ZFC + $V = L$. *Then the* COHEN *extension $M[c]$ satisfies: the set*

$$\{x \in \mathbb{R} \,|\, x \in L\}$$

of constructible reals has measure zero.

On the other hand, inside a given model of set theory, the set of reals has positive measure, i.e., does not have measure measure.

Exercise 1. Show that the measure zero sets form a proper ideal on $\mathbb{R}$ which is closed under countable unions.

Exercise 2. Show that the following *Cantor set* of reals has cardinality $2^{\aleph_0}$ and measure zero:

$$C = \{x \in \mathbb{R} \,|\, \forall n < \omega x(2n) = x(2n+1)\}.$$

So in the model L the set of constructible reals does not have measure zero:

Theorem 3. *The statement "the set of constructible reals has measure zero" is independent of the axioms of* ZFC.

The set of constructible reals in $M[c]$ can be a set of size $\aleph_1$ that has measure zero. This leads to the question whether it is (relatively) consistent that *all* sets of reals of size $\aleph_1$ have measure zero. Of course this necessitates $2^{\aleph_0} > \aleph_1$. It is natural to ask the question about Cohen's canonical model for $2^{\aleph_0} > \aleph_1$.

Consider adjoining λ COHEN reals to a ground model M where $\lambda = \aleph_2^M$. Define λ-fold COHEN forcing $P = (P, \leqslant, 1) \in M$ by $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, $\leqslant = \supseteq$, and $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G: \lambda \times \omega \to 2$ and extract a sequence $(c_\beta | \alpha < \lambda)$ of Cohen reals $c_\beta: \omega \to 2$ from F by:

$$c_\beta(n) = F(\beta, n).$$

Then the generic extension is generated by the sequence of Cohen reals:

$$M[G] = M[(c_\beta | \beta < \lambda)].$$

It is natural to construe $M[G]$ as a limit of the models $M[(c_\beta | \beta < \alpha)]$ when α goes towards λ: Fix $\alpha \leqslant \lambda$. Let $P_\alpha = \mathrm{Fn}(\alpha \times \omega, 2, \aleph_0)$ and $R_\alpha = \mathrm{Fn}((\lambda \setminus \alpha) \times \omega, 2, \aleph_0)$, partially ordered by reverse inclusion. The isomorphisms

$$P \cong P_\alpha \times R_\alpha \text{ and } P_{\alpha+1} \cong P_\alpha \times Q$$

imply that $G_\alpha = G \cap P_\alpha$ is M-generic on P_α and that

$$H_\alpha = \{q \in Q \mid \{((\alpha, n), i) \mid (n, i) \in q\} \in G_{\alpha+1}\}$$

is $M[G_\alpha]$-generic on Q. Let $M_\alpha = M[G_\alpha]$ be the α-th model in this construction. Then

$$M_{\alpha+1} = M[G_{\alpha+1}] = M[G_\alpha][H_\alpha] = M_\alpha[H_\alpha].$$

It is straightforward to check that $c_\alpha = \bigcup H_\alpha$. So the model $M[G] = M_\lambda$ is obtained by a sequence of models $(M_\alpha \mid \alpha \leqslant \lambda)$ where each successor step is a Cohen extension of the previous step. The whole construction is held together by the "long" generic set G which dictates the sequence of the construction and also the behaviour at limit stages.

Consider a real $x \in M[G]$. Identifying characteristic functions with sets we can view x as a subset of ω. In the previous course we had seen that there is a name $\dot{x} \in M$, $\dot{x}^G = x$ of the form

$$\dot{x} = \{(\check{n}, q) | n < \omega \wedge q \in A_n\},$$

where every A_n is an antichain in P. Since P satisfies the countable chain condition, there is $\alpha < \lambda$ such that $A_n \subseteq P_\alpha$ for every $n < \omega$. Then

$$x = \dot{x}^G = \dot{x}^{(G \cap P_\alpha)} = \dot{x}^{G_\alpha} \in M[G_\alpha]$$

In $M[G]$ consider a set $B = \{x_i \mid i < \aleph_1\}$ of reals of size $\aleph_1$. One can view B as a subset of $\aleph_1^M$. As in the above argument, there is an $\alpha < \lambda$ such that $B \in M_\alpha$. By our previous Lemma, $B \subseteq \mathbb{R} \cap M_\alpha$ has measure zero in the Cohen generic extension $M[c_\alpha]$. So B has measure zero in $M[G]$. The model $M[G]$ establishes:

Theorem 4. *If* ZFC *is consistent then* ZFC + *"every set of reals of size* $\leqslant \aleph_1$ *has Lebesgue measure zero" is consistent.*

Together with models of the Continuum Hypothesis this shows that the statement "every set of reals of size $\leqslant\aleph_1$ has Lebesgue measure zero" is independent of the axioms of ZFC.

One can ask for further properties of Lebesgue measure in connection with the uncountable. Is it consistent that every union of an $\aleph_1$-sequence of measure zero sets has again measure zero?

Exercise 3.

a) Show that in the model $M[G] = M[(c_\beta \mid \beta < \lambda)]$ there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$.

b) Show that $\{c_\beta \mid \beta < \lambda\}$ has measure zero in $M[G]$.

Exercise 4. Define forcing with sets of reals of *positive measure* (i.e., sets which do not have measure zero).

We shall later construct forcing extensions $M[G]$ which are obtained by iterations of forcing notions similar to the above example. We shall require that in the iteration $M_{\alpha+1}$ is a generic extension of M_α by some forcing $Q_\alpha \in M_\alpha = M[G_\alpha]$; the forcing is in general only given by a *name* $\dot{Q}_\alpha \in M$ such that $Q_\alpha = \dot{Q}_\alpha^{G_\alpha}$. To ensure that this is always a partial order we also require that $1_{P_\alpha} \Vdash \dot{Q}_\alpha$ is a partial order. Technical details will be given later.

A principal idea is to let $\dot{Q}_\alpha$ to be some canonical name for a partial order forcing a certain property to hold, like making the set of reals constructed so far a measure zero set. A central concern for such iterations, like for many forcings, is the preservation of cardinals.

3 Forcing axioms

The argument that the set $\mathbb{R}\cap M$ of ground model reals has measure zero in the standard Cohen extension $M[H]=M[c]$ by the Cohen partial order Q rests, like most forcing arguments, on density considerations. For a given $\varepsilon=2^{-i}$, a sequence $I_0, I_1, I_2, \ldots$ of real intervals such that $\sum_{n<\omega} \text{length}(I_n)\leqslant\varepsilon$ is extracted from the Cohen real c. It remains to show that $X\subseteq\bigcup_{n<\omega} I_n$. For $x\in\mathbb{R}\cap M$ a dense set D_x is defined so that $H\cap D_x\neq\emptyset$ implies that $x\in\bigcup_{n<\omega} I_n$. To cover the real x requires a "partially generic filter" which intersects D_x. This approach is captured by the following definition:

Definition 5. *Let $(Q,\leqslant,1_Q)$ be a forcing, $\mathcal{D}$ be any set, and κ a cardinal.*

a) A filter H on Q is $\mathcal{D}$-generic *iff $D\cap G\neq\emptyset$ for every $D\in\mathcal{D}$ which is dense in Q.*

b) The forcing axiom $\text{FA}_\kappa(Q)$ *postulates that there exists a* $\mathcal{D}$-generic *filter on Q for any $\mathcal{D}$ of cardinality $\leqslant\kappa$.*

For any countable $\mathcal{D}$ we obtain the existence of generic filters just like in the case of ground models.

Theorem 6. (Rasiowa-Sikorski) $\text{FA}_{\aleph_0}(Q)$ *holds for any partial order Q.*

Proof. Let $\mathcal{D}$ be countable. Take an enumeration $(D_n|n<\omega)$ of all $D\in\mathcal{D}$ which are dense in Q. Define an ω-sequence $q=q_0\geqslant q_1\geqslant q_2\geqslant\ldots$ recursively, using the axiom of choice:

$$\text{choose } q_{n+1} \text{ such that } q_{n+1}\leqslant q_n \text{ and } q_{n+1}\in D_n.$$

Then $H=\{q\in Q|\exists n<\omega\ q_n\leqslant q\}$ is as desired. □

Exercise 5. Show that $\text{FA}_\kappa(Q)$ holds for any κ-closed partial order Q.

The results of the previous chapter now read as follows:

Theorem 7. *Let $Q=\text{Fn}(\omega,2,\aleph_0)$ be the Cohen partial order and assume* $\text{FA}_{\aleph_1}(Q)$*. Then every set of reals of cardinality $\leqslant\aleph_1$ has measure zero.*

Theorem 8. *Let $M[G]$ be a generic extension of the ground model M by λ-fold Cohen forcing $P=(P,\leqslant,1)=\text{Fn}(\lambda\times\omega,2,\aleph_0)$ where $\lambda=\aleph_2^M$. Then in $M[G]$,* $\text{FA}_{\aleph_1}(Q)$ *holds.*

Proof. We may assume that every $D\in\mathcal{D}$ is a dense subset of Q. Then $\mathcal{D}$ can be coded as a subset of $\aleph_1^M$. There is $\alpha<\lambda$ such that $\mathcal{D}\in M[G_\alpha]$. The filter H_α corresponding to the α-th Cohen real in the construction is $M[G_\alpha]$-generic on Q. Since $\mathcal{D}\subseteq M[G_\alpha]$, H_α is $\mathcal{D}$-generic on Q. □

So for the Cohen forcing Q we have a strengthening of the Rasiowa-Sikorski Lemma from countable to cardinality $\leqslant\aleph_1$. This is not possible for all forcings:

Lemma 9. *Let $P=\text{Fn}(\aleph_0,\aleph_1,\aleph_0)$ be the canonical forcing for adding a surjection from $\aleph_0$ onto $\aleph_1$. Then* $\text{FA}_{\aleph_1}(P)$ *is false.*

Proof. For $\alpha < \aleph_1$ define the set

$$D_\alpha = \{p \in P \mid \alpha \in \operatorname{ran}(p)\}$$

which is dense in P. Let $D = \{D_\alpha \mid \alpha < \aleph_1\}$. Assume for a contradiction that H is a $\mathcal{D}$-generic filter on P. Then $\bigcup H$ is a partial function from $\aleph_0$ to $\aleph_1$.

(1) $\bigcup H$ is onto $\aleph_1$.

Proof. Let $\alpha < \aleph_1$. Since H is a $\mathcal{D}$-generic, $H \cap D_\alpha \neq \emptyset$. Take $p \in H \cap D_\alpha$. Then

$$\alpha \in \operatorname{ran}(p) \subseteq \operatorname{ran}\left(\bigcup H\right)$$

qed.

But this is a contradiction since $\aleph_1$ is a cardinal. □

Exercise 6. Show that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

So we cannot have an uncountable generalization of the Rasiowa-Sikorski Lemma for forcings which collapse the cardinal $\aleph_1$. Since countable chain condition (ccc) forcing does not collapse cardinals, this suggests the following axiom:

Definition 10.

a) *Let κ be a cardinal. Then* MARTIN's axiom MA_κ *is the property: for every ccc partial order $(P, \leqslant, 1_P)$,* $\mathrm{FA}_\kappa(P)$ *holds.*

b) MARTIN's axiom MA *postulates that* MA_κ *holds for every $\kappa < 2^{\aleph_0}$.*

$\mathrm{MA}_{\aleph_0}$ holds by Theorem 6. Thus the continuum hypothesis $2^{\aleph_0} = \aleph_1$ trivially implies MA. We shall later see by an iterated forcing construction that $2^{\aleph_0} = \aleph_2$ and MA are relatively consistent with ZFC.

4 Consequences of MA+¬CH

4.1 Lebesgue measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t | t\colon \operatorname{dom}(t) \to 2 \wedge \operatorname{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} | s \subseteq x\} \subseteq \mathbb{R}$$

with $\operatorname{length}(I_s) = 2^{-\operatorname{dom}(s)}$. Note that $I_s = I_{s \cup \{(\operatorname{dom}(s), 0)\}} \cup I_{s \cup \{(\operatorname{dom}(s), 1)\}}$, $\operatorname{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\operatorname{length}(I_{s \cup \{(\operatorname{dom}(s), 0)\}}) = \operatorname{length}(I_{s \cup \{(\operatorname{dom}(s), 1)\}}) = \frac{1}{2} \operatorname{length}(I_s)$.

Definition 11. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has* measure $<\varepsilon$ *if there exists a sequence $(I_n | n < \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and $\sum_{n<\omega} \operatorname{length}(I_n) \leqslant \varepsilon$. A set $X \subseteq \mathbb{R}$ has* measure zero *if it has measure $<\varepsilon$ for every $\varepsilon > 0$.*

The measure zero sets form a countably complete ideal on $\mathbb{R}$. It is easy to see that a countable union of measure zero sets is again measure zero. To strengthen this theorem in the context of MA we need some more topological and measure theoretic notions. The (standard) topology on $\mathbb{R}$ is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n<\omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.

Theorem 12. *Assume* MA_κ *and let* $(X_i|i<\kappa)$ *be a family of measure zero sets. Then* $X=\bigcup_{i<\kappa} X_i$ *has measure zero.*

Proof. Fix $\varepsilon>0$. We show that $X=\bigcup_{i<\kappa} X_i$ has measure $<2\varepsilon$. Let

$$\mathcal{I}=\{(a,b)|a,b\in\mathbb{Q},a<b\}$$

the countable set of rational intervals $(a,b)=\{c\in\mathbb{R}|a<c<b\}$ in $\mathbb{R}$. The *length* of (a,b) is simply $\mathrm{length}((a,b))=b-a$. We shall apply MARTIN's axiom to the following forcing $P=(P,\supseteq,\emptyset)$ where

$$P=\{p\subseteq\mathcal{I}|\sum_{I\in p}\mathrm{length}(I)<\varepsilon\}.$$

(1) P is ccc.
Proof. Let $\{p_i|i<\omega_1\}\subseteq P$. For every $i<\omega_1$ there is $n_i<\omega$ such that p_i has measure $<\varepsilon-\frac{1}{n_i}$. By a pigeonhole principle we may assume that all n_i are equal to a common value $n<\omega$. For every p_i we have

$$\sum_{I\in p_i}\mathrm{length}(I)<\varepsilon-\frac{1}{n}.$$

For every $i<\omega_1$ take a finite set $\bar{p}_i\subseteq p_i$ such that

$$\sum_{I\in p_i\setminus\bar{p}_i}\mathrm{length}(I)<\frac{1}{n}.$$

There are only countably many such set $\bar{p}_i$, and again by a pigeonhole argument we may assume that for all $i<\omega_1$

$$\bar{p}_i=\bar{p}$$

takes a fixed value. Now consider $i<j<\omega_1$. Then

$$\begin{aligned}\sum_{I\in p_i\cup p_j}\mathrm{length}(I) &\leqslant \sum_{I\in p_i}\mathrm{length}(I)+\sum_{I\in p_j\setminus\bar{p}}\mathrm{length}(I)\\ &< \varepsilon-\frac{1}{n}+\frac{1}{n}\\ &= \varepsilon\end{aligned}$$

Hence $p_i\cup p_j\in P$ and $p_i\cup p_j\leqslant p_i,p_j$, and so $\{p_i|i<\omega_1\}$ is not an antichain in P. *qed*(1)

For $i<\kappa$ define

$$D_i=\{p\in P|X_i\subseteq\bigcup p\}.$$

(2) D_i is dense in P.
Proof. Let $q \in P$. Take $n < \omega$ such that

$$\sum_{I \in q} \text{length}(I) < \varepsilon - \frac{1}{n}.$$

Since X_i has measure zero, take $r \subseteq \mathcal{I}$ such that $X_i \subseteq \bigcup p$ and $\sum_{I \in r} \text{length}(I) \leqslant \frac{1}{n}$. Then

$$X_i \subseteq \bigcup (q \cup r) \ \text{ and } \sum_{I \in q \cup r} \text{length}(I) \leqslant \sum_{I \in q} \text{length}(I) + \sum_{I \in r} \text{length}(I) < \varepsilon - \frac{1}{n} + \frac{1}{n} = \varepsilon.$$

Hence $p = q \cup r \in P$, $p \supseteq q$, and $p \in D_i$. *qed*(2)

By MA_κ take a filter G on P which is $\{D_i | i < \kappa\}$-generic. Let $U = \bigcup G \subseteq \mathcal{I}$.

(3) $X = \bigcup_{i < \kappa} X_i \subseteq \bigcup_{I \in U} I$.
Proof. Let $i < \kappa$. By the generity of G take $p \in G \cap D_i$. Then

$$X_i \subseteq \bigcup p \subseteq \bigcup U$$

qed(3)

(4) $\sum_{I \in U} \text{length}(I) \leqslant \varepsilon$.
Proof. Assume for a contradiction that $\sum_{I \in U} \text{length}(I) > \varepsilon$. Then take a finite set $\bar{U} \subseteq U$ such that $\sum_{I \in \bar{U}} \text{length}(I) > \varepsilon$. Let $\bar{B} = \{I_0, ..., I_{k-1}\}$. For every $I_j \in \bar{U}$ take $p_j \in G$ such that $I_j \in p_j$. Since all elements of G are compatible within G there is a condition $p \in G$ such that $p \supseteq p_0, ..., p_{k-1}$. Hence $\bar{U} \subseteq p$. But, since $p \in P$, we get a contradiction:

$$\varepsilon < \sum_{I \in \bar{U}} \text{length}(I) \leqslant \sum_{I \in p} \text{length}(I) < \varepsilon.$$

□

Two easy consequences are:

Corollary 13. *Assume* MA_κ *and let* $X \subseteq \mathbb{R}$ *with* $\text{card}(X) \leqslant \kappa$. *Then* X *has measure zero.*

Theorem 14. *Assume* MA. *Then* $2^{\aleph_0}$ *is regular.*

Proof. Assume instead that $\mathbb{R} = \bigcup_{i < \kappa} X_i$ for some $\kappa < 2^{\aleph_0}$, where $\text{card}(X_i) < 2^{\aleph_0}$ for every $i < \kappa$. Every singleton $\{r\}$ has measure zero. By Theorem 12, each X_i has measure zero. Again by Theorem, $\mathbb{R} = \bigcup_{i < \kappa} X_i$ has measure zero. But measure theory (and also intuition) shows that $\mathbb{R}$ does not have measure zero. □

4.2 Almost disjoint forcing

We intend to code subsets of κ by subsets of ω. If such a coding is possible then we shall have

$$2^{\aleph_0} \leqslant 2^\kappa \leqslant 2^{\aleph_0}, \text{ i.e. } 2^\kappa = 2^{\aleph_0}.$$

We shall employ almost disjoint coding.

Definition 15. *A sequence* $(x_i|i\in I)$ *is* almost disjoint *if*

a) x_i *is infinite*

b) $i\neq j<\kappa$ *implies that* $x_i\cap x_j$ *is finite*

Lemma 16. *There is an almost disjoint sequence* $(x_i|i<2^{\aleph_0})$ *of subsets of* ω.

Proof. For $u\in{}^\omega 2$ let $x_u=\{u\restriction m\,|\,m<\omega\}$. x_u is infinite. Consider $u\neq v$ from ${}^\omega 2$. Let $n<\omega$ be minimal such that $u\restriction n\neq v\restriction n$. Then

$$x_u\cap x_v=\{u\restriction m\,|\,m<\omega\}\cap\{v\restriction m\,|\,m<\omega\}=\{u\restriction m\,|\,m<n\}$$

is finite. Thus $(x_u|u\in{}^\omega 2)$ is almost disjoint. Using bijections $\omega\leftrightarrow{}^{<\omega}2$ and $2^{\aleph_0}\leftrightarrow{}^\omega 2$ one can turn this into an almost disjoint sequence $(x_i|i<2^{\aleph_0})$ of subsets of ω. □

Theorem 17. *Assume* MA_κ. *Then* $2^\kappa=2^{\aleph_0}$.

Proof. By a previous example, $\kappa<2^{\aleph_0}$. By the lemma, fix an almost disjoint sequence $(x_i|i<\kappa)$ of subsets of ω. Define a map $c\colon\mathcal{P}(\omega)\to\mathcal{P}(\kappa)$ by

$$c(x)=\{i<\kappa|x\cap x_i\text{ is infinite}\}.$$

We say that x *codes* $c(x)$. We want to show that every subset of κ can be coded as some $c(x)$. We show this by proving that $c\colon\mathcal{P}(\omega)\to\mathcal{P}(\kappa)$ is surjective.

Let $A\subseteq\kappa$ be given. We use the following forcing $(P,\leqslant,1)$ to code A:

$$P=\{(a,z)|a\subseteq\omega,z\subseteq\kappa,\mathrm{card}(a)<\aleph_0,\mathrm{card}(z)<\aleph_0\},$$

partially ordered by

$$(a',z')\leqslant(a,z)\text{ iff }a'\supseteq a,z'\supseteq z,i\in z\cap(\kappa\setminus A)\to a'\cap x_i=a\cap x_i.$$

The weakest element of P is $1=(\emptyset,\emptyset)$.

The idea of the forcing is to keep the intersection of the first component with x_i fixed, provided $i\notin A$ has entered the second component. This will allow the almost disjoint coding of A by the finite/infinite method.

(1) $(P,\leqslant,1)$ satisfies ccc.

Proof. Conditions $(a,\,y)$ and $(a,\,z)$ with equal first components are compatible, since $(a,y\cup z)\leqslant(a,y)$ and $(a,y\cup z)\leqslant(a,z)$. Incompatibel conditions have different first components. Since there are only countably many first components, an antichain in P can be at most countable. *qed*(1)

The outcome of a forcing construction results from an interplay between the partial order and some dense set arguments. We now define dense sets for our requirements.

For $i<\kappa$ let $D_i=\{(a,z)\in P|i\in z\}$. D_i is obviously dense in P. For $i\in A$ and $n\in\omega$ let $D_{i,n}=\{(a,z)\in P|\exists m>n\colon m\in a\cap x_i\}$.

(2) If $i\in A$ and $n\in\omega$ then $D_{i,n}$ is dense in P.

Proof. Consider $(a,z)\in P$. For $j\in z$, $j\neq i$ is the intersection $x_i\cap x_j$ finite. Take some $m\in x_i$, $m>n$ such that $m\notin x_i\cap x_j$ for $j\in z$, $j\neq i$. Then

$$(a\cup\{m\},z)\leqslant(a,z)\text{ and }(a\cup\{m\},z)\in D_{i,n}.$$

qed(2)

By MA_κ take a filter G on P which is generic for the dense sets in

$$\{D_i | i < \kappa\} \cup \{D_{i,n} | i \in A, n \in \omega\}.$$

Let

$$x = \bigcup \{a | (a, y) \in G\} \subseteq \omega.$$

(3) Let $i \in A$. Then $x \cap x_i$ is infinite.
Proof. Let $n < \omega$. By genericity take $(a, y) \in G \cap D_{i,n}$. By the definition of $D_{i,n}$ take $m > n$ such that $m \in a \cap x_i$. Then $m \in x \cap x_i$, and so $x \cap x_i$ is cofinal in ω. *qed*(3)
(4) Let $i \in \kappa \setminus A$. Then $x \cap x_i$ is finite.
Proof. By genericity take $(a, y) \in G \cap D_i$. Then $i \in y$. We show that $x \cap x_i \subseteq a \cap x_i$. Consider $n \in x \cap x_i$. Take $(b, z) \in G$ such that $n \in b$. By the filter properties of G take $(a', y') \in P$ such that $(a', y') \leqslant (a, y)$ and $(a', y') \leqslant (b, z)$. Then $n \in a'$, and by the definition of $\leqslant$, $a' \cap x_i = a \cap x_i$. Thus $n \in a \cap x_i$. *qed*(4)

So

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\} = A \in \mathrm{range}(c).$$

□

4.3 Baire category

Lebesgue measure defines an ideal of “small” sets, namely the ideal of measure zero sets: arbitrary subsets of measure zero sets are measure zero, and, under MA, every union of less than $2^{\aleph_0}$ measure zero sets is again measure zero.

We now look at another ideal of small sets, namely the ideal of subsets X of $\mathbb{R}$ which are nowhere dense in $\mathbb{R}$: every nonempty open interval in $\mathbb{R}$ has a nonempty open subinterval which is disjoint from X. The union of all such subintervals is open, dense in $\mathbb{R}$, and disjoint from X.

The BAIRE category theorem says that the intersection of countably many dense open sets of reals in dense in $\mathbb{R}$. We can strengthen this to:

Theorem 18. *Assume* MA_κ. *Then the intersection of* κ *many dense open sets of reals is dense in* $\mathbb{R}$.

Proof. Consider a sequence $(O_i | i < \kappa)$ of dense open subsets of $\mathbb{R}$. We use standard COHEN forcing $P = \mathrm{Fn}(\omega, 2, \aleph_0)$ for the density argument. Since P is countable it trivially has the ccc. For $i < \kappa$ define $D_i = \{p \in P | \forall x \in \mathbb{R}\, (x \supseteq p \rightarrow x \in O_i)\}$. This means that the interval determined by p lies within O_i. The density of D_i follows readily since O_i is open dense. For $n < \omega$ let $D_n = \{p \in P | n \in \mathrm{dom}(p)\}$. Obviously, D_n is also dense in P. By MA_κ let $G \subseteq P$ be $\{D_i | i < \kappa\}$-$\{D_n | n < \kappa\}$generic. Let $x = \bigcup G$. $p \in G \cap D_n$ implies that $n \in \mathrm{dom}(p) \subseteq \mathrm{dom}(x)$. So $x: \omega \rightarrow 2$ is a real number. □

Since $\mathrm{MA}_{\aleph_0}$ is always true in ZFC, we get the BAIRE category theorem:

Theorem 19. *The intersection of countably many dense open sets of reals is dense in* $\mathbb{R}$.

This says that dense open sets (of reals) have a largeness property, and correspondingly complements of dense open sets are small.

Definition 20. *A set $A \subseteq \mathbb{R}$ is* nowhere dense *if there is a dense open set $O \subseteq \mathbb{R}$ such that $A \cap O = \emptyset$. A set $A \subseteq \mathbb{R}$ is* meager *or* of 1st category *if it is a union of countably many nowhere dense sets.*

Proposition 21.

a) A singleton set $\{x\} \subseteq \mathbb{R}$ is nowhere dense since $\mathbb{R} \setminus \{x\}$ is dense open in $\mathbb{R}$.

b) A countable set C is meager.

c) A set $A \subseteq \mathbb{R}$ is meager iff there are open dense sets $(O_n | n < \omega)$ such that $A \cap \bigcap_{n<\omega} O_n = \emptyset$.

d) $\mathbb{R}$ is not meager. Sets which are not meager are said to be of 2nd category.

Proof. *c*) Let $A = \bigcup_{n<\omega} A_n$ be meager where each A_n is nowhere dense. For each n choose O_n dense open in $\mathbb{R}$ such that $A_n \cap O_n = \emptyset$. Then

$$(\bigcup_{n<\omega} A_n) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

Conversely assume that $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ where each O_n is dense open. $(A \setminus O_n) \cap O_n = \emptyset$, and so by definition, every $A_n = A \setminus O_n$ is nowhere dense. Obviously

$$\bigcup_{n<\omega} A_n \subseteq A.$$

For the converse consider $x \in A$. The property $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ implies that we may take $n < \omega$ such that $x \notin O_n$. Hence $x \in A \setminus O_n = A_n$. So $A = \bigcup_{n<\omega} A_n$ is meager.

d) If $\mathbb{R}$ were meager then there would be open dense sets $(O_n | n < \omega)$ such that $\mathbb{R} \cap \bigcap_{n<\omega} O_n = \emptyset$. But by Theorem 19,

$$\mathbb{R} \cap \bigcap_{n<\omega} O_n = \bigcap_{n<\omega} O_n \neq \emptyset,$$

contradiction. □

We would now like to show as in the case of measure that a union of $< 2^{\aleph_0}$ small sets in the sense of category is again small if MARTIN's axiom holds.

Theorem 22. *Assume* MA_κ *. Let $(A_i | i < \kappa)$ be a family of meager sets. Then $A = \bigcup_{i<\kappa} A_i$ is meager.*

Proof. Obviously it suffices to consider the case where each A_i is nowhere dense. We shall use MA_κ to find dense open sets $(O_n | n < \omega)$ such that

$$(\bigcup_{i<\kappa} A_i) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

The forcing will consist of approximations to a family $(O_n|n<\omega)$ of open dense sets which makes this equality true.

The forcing conditions will consist of finitely many finite approximations to the O_n . Moreover there will be for every n a finite collection of $i<\kappa$ such that an approximation to the equation holds for those i. We shall see that by appropriate density considerations the full equality may be satisfied.

For ccc-reasons, much like in the argument of measure-zero sets, we only consider approximations to the O_n by finitely many *rational* intervals. Let

$$\mathcal{I}=\{(a,b)|a,b\in\mathbb{Q},a<b\}$$

the countable set of rational open intervals $(a,b)=\{c\in\mathbb{R}|a<c<b\}$ in $\mathbb{R}$. Now let

$P=\{(r,\,s)|r\colon\omega\to[\mathcal{I}]^{<\omega},\ s\colon\omega\to[\kappa]^{<\omega},\ \{n<\omega|r(n)\neq\emptyset\}$ is finite, $\{n<\omega|s(n)\neq\emptyset\}$ is finite, $\forall n<\omega\forall i\in s(n)\ A_i\cap\bigcup r(n)=\emptyset\}$.

Define

$$(r',s')\leqslant(r,s)\ \text{ iff }\ \forall n<\omega\,(r'(n)\supseteq r(n)\wedge s'(n)\supseteq s(n)).$$

(1) $(P,\leqslant)$ satisfies the countable chain condition.
Proof. Consider $(r,\,s)$ and $(r,\,s')$ in P having the same first component. Then define s'': $\omega\to[\kappa]^{<\omega}$ by $s''(n)=s(n)\cup s'(n)$. It is easy to check that $(r,s'')\in P$, and also $(r,s'')\leqslant(r,s)$ and $(r,s'')\leqslant(r,s')$. So (r,s) and (r,s') are compatible in P.

An antichain in P must consist of conditions whose first components are pairwise distinct. Since there are only countably many first components, an antichain in P is at most countable. *qed*(1)

For each $n<\omega$ the following dense sets ensures the density of the O_n in $\mathbb{R}$: for $I\in\mathcal{I}$ let

$$D_{n,I}=\{(r',s')|\exists J\in r'(n)\ J\subseteq I\}.$$

(2) $D_{n,I}$ is dense in P.
Proof. Let $(r,\,s)\in P$. Let $s(n)=\{i_0,\,...,\,i_{k-1}\}$. Since $A_{i_0},\,...,\,A_{i_{k-1}}$ are nowhere dense one can go find intervals $I\supseteq I_{i_0}\supseteq...\supseteq I_{k-1}=J$ in $\mathcal{I}$ such that $A_{i_l}\cap I_{i_l}=\emptyset$. Define $r'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $r'\restriction(\omega\setminus\{n\})=r\restriction(\omega\setminus\{n\})$ and $r'(n)=r(n)\cup\{J\}$. Then $(r',s)\in P$, $(r',s)\leqslant(r,s)$, and $(r',s)\in D_{n,I}$. *qed*(2)

We also need that every $i<\kappa$ is considered by some O_n. Define

$$D_i=\{(r',s')|\exists n<\omega\ i\in s'(n)\}.$$

(3) D_i is dense in P.
Proof. Let $(r,\,s)\in P$. Take $n<\omega$ such that $r(n)=\emptyset$. Define $s'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $s'\restriction(\omega\setminus\{n\})=s\restriction(\omega\setminus\{n\})$ and $s'(n)=s(n)\cup\{i\}$. Then $(r,s')\in P$, $(r,s')\leqslant(r,s)$, and $(r,s')\in D_i$. *qed*(3)

By MA_κ we can take a filter G on P which is generic for

$$\{D_{n,I}|n<\omega,I\in\mathcal{I}\}\cup\{D_i|i<\kappa\}.$$

For $n<\omega$ define

$$O_n=\bigcup\bigcup\{r(n)|(r,s)\in G\}.$$

(4) O_n is open, since it is a union of open intervals.
(5) O_n is dense in $\mathbb{R}$.
Proof. Let $I\in\mathcal{I}$. By genericity take $(r', s')\in G\cap D_{n,I}$. Take $J\in r'(n)$ such that $J\subseteq I$. Then

$$\emptyset\neq J\subseteq\bigcup r'(n)\subseteq\bigcup\bigcup\{r(n)|(r,s)\in G\}=O_n.$$

qed(5)
(6) Let $i<\kappa$. Then $A_i\cap\bigcap_{n<\omega}O_n=\emptyset$.
Proof. By genericity take $(r', s')\in G\cap D_i$. Take $n<\omega$ such that $i\in s'(n)$. We show that $A_i\cap O_n=\emptyset$. Assume not, and let $x\in A_i\cap O_n$. Take $(r,s)\in G$ and $I\in r(n)$ such that $x\in I$. Since G is a filter, take $(r'', s'')\in P$ such that $(r'', s'')\leqslant(r, s)$ and $(r'', s'')\leqslant(r', s')$. Then $I\in r''(n)$, $i\in s''(n)$, and

$$x\in A_i\cap I\subseteq A_i\cap\bigcup r''(n)\neq\emptyset.$$

The last inequality contradicts the definition of P. *qed*(6)

By (6), $\bigcup_{i<\kappa}A_i\cap\bigcap_{n<\omega}O_n=\emptyset$, and so $\bigcup_{i<\kappa}A_i$ is meager. □

5 Iterated forcing

MARTIN's axiom postulates that for every ccc partial order $(P, \leqslant, 1_P)$ and $\mathcal{D}$ with $\mathrm{card}(\mathcal{D})<2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter G on P. Syntactically this axiom has a $\forall\exists$-form: $\forall P\forall\mathcal{D}\exists G\ldots$. $\forall\exists$-properties are often realised through chain constructions: build a chain

$$M=M_0\subseteq M_1\subseteq\ldots\subseteq M_\alpha\subseteq\ldots\subseteq M_\beta\subseteq\ldots$$

of models such that for any P, $\mathcal{D}\in M_\alpha$ there is some $\beta\geqslant\alpha$ such that M_β contains a generic G as required. Then the "union" or limit of the chain should contain appropriate G's for all P's and $\mathcal{D}$'s.

Such chain constructions are wellknown from algebra. To satisfy closure under square roots ($\forall x\exists y$: $yy=x$) one can e.g. start with a countable field M_0 and along a chain $M_0\subseteq M_1\subseteq M_2\subseteq\ldots$ adjoin square roots for all elements of M_n. Then $\bigcup_{n<\omega}M_n$ satisfies the closure property.

In set theory there is a difficulty that unions of models of set theory usually do not satisfy the theory ZF: assume that $M_0\subseteq M_1\subseteq M_2\subseteq\ldots$ is an ascending chain of transitive models of ZF such that $(M_{n+1}\setminus M_n)\cap\mathcal{P}(\omega)\neq\emptyset$ for all $n<\omega$. Let $M_\omega=\bigcup_{n<\omega}M_n$. Then $\mathcal{P}(\omega)\cap M_\omega\notin M_\omega$. Indeed, if one had $\mathcal{P}(\omega)\cap M_\omega\in M_\omega$ then $\mathcal{P}(\omega)\cap M_\omega\in M_n$ for some $n<\omega$ and $\mathcal{P}(\omega)\cap M_{n+1}\in M_n$ contradicts the initial assumption. So a "limit" model of models of ZF has to be more complicated, and it will itself be constructed by some limit forcing which is called *iterated forcing*.

Exercise 7. Check which axioms of set theory hold in $M_\omega=\bigcup_{n<\omega}M_n$ where $(M_n)_{n<\omega}$ is an ascending sequence of transitive models of ZF(C).

Since we want to obtain the limit by forcing over a ground model M the construction must be visible in the ground model. This means that the sequence of forcings to be employed to pass from M_α to $M_{\alpha+1}$ has to exist as a sequence $(\dot{Q}_\beta|\beta<\kappa)$ of names in the ground model. The initial sequence $(\dot{Q}_\beta|\beta<\alpha)$ already determines a forcing P_α and $\dot{Q}_\alpha$ is intended to be a P_α-name. If G_α is M-generic over P_α then furthermore $Q_\alpha=(\dot{Q}_\alpha)^{G_\alpha}$ is intended to be a forcing in the model $M_\alpha=M[G_\alpha]$, and $M_{\alpha+1}$ is a generic extension of M_α by forcing with Q_α. The following iteration theorem says that any sequence $(\dot{Q}_\beta|\beta<\kappa)\in M$ gives rise to an iteration of forcing extensions. In applications the sequence has to be chosen carefully to ensure that some $\forall\exists$-property holds in the final model M_κ. Without loss of generality we only consider forcings Q_α whose maximal element is $\emptyset$.

Theorem 23. *Let M be a ground model, and let $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)\in M$ with the property that $\forall\beta<\kappa\colon\emptyset\in\operatorname{dom}(\dot{Q}_\beta)$. Then there is a uniquely determined sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)\in M$ such that*

a) $(P_\alpha,\leqslant_\alpha,1_\alpha)$ *is a partial order which consists of α-sequences;*

b) $P_0=\{\emptyset\}$, $\leqslant_0=\{(\emptyset,\emptyset)\}$, $1_0=\emptyset$;

c) If $\lambda\leqslant\kappa$ is a limit ordinal then the forcing P_λ is defined by:

$$\begin{aligned} P_\lambda &= \{p\colon\lambda\to V\,|\,(\forall\gamma<\lambda\colon p\restriction\gamma\in P_\gamma)\wedge\exists\gamma<\lambda\forall\beta\in[\gamma,\lambda)\ p(\beta)=\emptyset)\}\\ p\leqslant_\lambda q &\text{ iff } \forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma q\restriction\gamma\\ 1_\lambda &= (\emptyset|\gamma<\lambda)\end{aligned}$$

d) If $\alpha<\kappa$ and $1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{aligned} P_{\alpha+1} &= \{p\colon\alpha+1\to V\,|\,p\restriction\alpha\in P_\alpha\wedge p(\alpha)\in\operatorname{dom}(\dot{Q}_\alpha)\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\in\dot{Q}_\alpha\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\wedge p\restriction\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

e) If $\alpha<\kappa$ and not *$1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:*

$$\begin{aligned} P_{\alpha+1} &= \{p\colon\alpha+1\to V\,|\,p\restriction\alpha\in P_\alpha\wedge p(\alpha)=\emptyset\}\\ p\leqslant_{\alpha+1}q &\text{ iff } p\restriction\alpha\leqslant_\alpha q\restriction\alpha\\ 1_{\alpha+1} &= (\emptyset|\gamma<\alpha+1)\end{aligned}$$

$((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$, and in particular P_κ are called the (*finite support*) *iteration* of the sequence $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$.

Proof. To justify the above recursive definition of the sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ it suffices to show recursively that every P_α is a forcing.

Obviously, P_0 is a trivial one-element forcing.

Consider a limit $\lambda\leqslant\kappa$ and assume that P_γ is a forcing for $\gamma<\alpha$. We have to show that the relation $\leqslant_\lambda$ is transitive with maximal element 1_λ. Consider $p\leqslant_\lambda q\leqslant_\lambda r$. Then $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma q\restriction\gamma$ and $\forall\gamma<\lambda\colon q\restriction\gamma\leqslant_\gamma r\restriction\gamma$. Since all $\leqslant_\gamma$ with $\gamma<\lambda$ are transitive relations, $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma r\restriction\gamma$ and so $p\leqslant_\lambda r$. Now consider $p\in P_\lambda$. Then $\forall\gamma<\lambda\colon p\restriction\gamma\in P_\gamma$. By the inductive assumption, $\forall\gamma<\lambda\colon p\restriction\gamma\leqslant_\gamma 1_\gamma=1_\lambda\restriction\gamma$ and so $p\leqslant_\lambda 1_\lambda$.

For the successor step assume that $\alpha<\kappa$ and that P_α is a forcing.

Case 1. $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p\upharpoonright\alpha \leqslant_\alpha q\upharpoonright\alpha \wedge p\upharpoonright\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$ and $q\upharpoonright\alpha \leqslant_\alpha r\upharpoonright\alpha \wedge q\upharpoonright\alpha \Vdash_{P_\alpha} q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$. By the transitivity of $\leqslant_\alpha$: $p\upharpoonright\alpha \leqslant_\alpha r\upharpoonright\alpha$. Moreover $p\upharpoonright\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$, $p\upharpoonright\alpha \Vdash_{P_\alpha} q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and $p\upharpoonright\alpha \Vdash_{P_\alpha}$ "$\dot{\leqslant}_\alpha$ is transitive". This implies $p\upharpoonright\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and together that $p \leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\upharpoonright\alpha \in P_\alpha \wedge p\upharpoonright\alpha \Vdash_{P_\alpha} p(\alpha)\in \dot{Q}_\alpha$. Then $p\upharpoonright\alpha \leqslant_\alpha 1_\alpha = 1_{\alpha+1}\upharpoonright\alpha$. Moreover $p\upharpoonright\alpha \Vdash_{P_\alpha}$ "$\emptyset$ is maximal in $\dot{\leqslant}_\alpha$" implies that $p\upharpoonright\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha \emptyset = 1_{\alpha+1}(\alpha)$. Hence $p \leqslant_{\alpha+1} 1_{\alpha+1}$.

Case 2. It is not the case that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p\upharpoonright\alpha \leqslant_\alpha q\upharpoonright\alpha$ and $q\upharpoonright\alpha \leqslant_\alpha r\upharpoonright\alpha$. By the transitivity of $\leqslant_\alpha$: $p\upharpoonright\alpha \leqslant_\alpha r\upharpoonright\alpha$ and so $p\leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\upharpoonright\alpha \in P_\alpha$. By induction, $p\upharpoonright\alpha \leqslant_\alpha 1_\alpha$ and so $p\leqslant_{\alpha+1} 1_{\alpha+1}$. □

The term "finite support iteration" is justified by the following

Lemma 24. *In the above situation let $p\in P_\kappa$. Then*

$$\operatorname{supp}(p) = \{\alpha<\kappa \mid p(\alpha)\neq\emptyset\}$$

is finite.

Proof. Prove by induction on $\alpha\leqslant\kappa$ that $\operatorname{supp}(p)$ is finite for every $q\in P_\alpha$. The crucial property is the definition of P_λ at limit λ in the above iteration theorem. □

Let us fix a ground model M and the iteration $((\dot{Q}_\beta, \dot{\leqslant}_\beta)\mid\beta<\kappa)\in M$ and $((P_\alpha, \leqslant_\alpha, 1_\alpha)\mid\alpha\leqslant\kappa)\in M$ as above. Let G_κ be M-generic for P_κ. We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M = M_0 \subseteq M_1 = M[G_1] = M_0[H_0] \subseteq M_2 = M[G_2] = M_1[H_1] \subseteq \ldots \subseteq M_\alpha = M[G_\alpha] \subseteq \ldots \subseteq M_\kappa$$

of generic extensions.

Let us first note some relations within the tower $(P_\alpha)_{\alpha\leqslant\kappa}$ of forcings.

Lemma 25.

a) *Let $\alpha\leqslant\kappa$ and $p,q\in P_\alpha$.*
Then $p\leqslant_\alpha q$ iff $\forall\gamma\in\operatorname{supp}(p)\cup\operatorname{supp}(q)$: $p\upharpoonright\gamma \Vdash_{P_\gamma} p(\gamma)\dot{\leqslant}_\gamma q(\gamma)$.

b) *Let $\alpha\leqslant\beta\leqslant\kappa$ and $p\in P_\beta$. Then $p\upharpoonright\alpha\in P_\alpha$.*

c) *Let $\alpha\leqslant\beta\leqslant\kappa$ and $p\leqslant_\beta q$. Then $p\upharpoonright\alpha \leqslant_\alpha q\upharpoonright\alpha$.*

d) *Let $\alpha\leqslant\beta\leqslant\kappa$, $q\in P_\beta$, $\bar{p}\leqslant_\alpha q\upharpoonright\alpha$. Then $\bar{p}\cup(q(\gamma)\mid\alpha\leqslant\gamma<\beta)\in P_\beta$ and $\bar{p}\cup(q(\gamma)\mid\alpha\leqslant\gamma<\beta)\leqslant_\beta q$.*

Proof. *a)* By a straightforward induction on $\alpha\leqslant\kappa$. Now *b)* − *d)* follow immediately. □

For $\alpha \leqslant \kappa$ define $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$.

(1) G_α is M-generic for P_α.

Proof. By (a), $G_\alpha \subseteq P_\alpha$. Consider $p \upharpoonright \alpha$, $q \upharpoonright \alpha \in G_\alpha$ with $p, q \in G_\kappa$. Take $r \in G_\kappa$ such that $r \leqslant_\kappa p, q$. By (a), $r \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha, q \upharpoonright \alpha$. Thus all elements of G_α are compatible in P_α.

Consider $p \upharpoonright \alpha \in G_\alpha$ with $p \in G_\kappa$ and $\bar{q} \in P_\alpha$ with $p \upharpoonright \alpha \leqslant_\alpha \bar{q}$. By (a),

$$q = \bar{q} \cup (\emptyset | \alpha \leqslant \gamma < \kappa)$$

is an element of P_κ and $p \leqslant_\kappa q$. Since G_κ is a filter, $q \in G_\kappa$, and so $\bar{q} = q \upharpoonright \alpha \in G_\alpha$. Thus G_α is upwards closed.

For the genericity consider a set $\bar{D} \in M$ which is dense in P_α. We claim that the set

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \in \bar{D}\} \in M$$

is dense in P_κ: let $p \in P_\kappa$. Then $p \upharpoonright \alpha \in P_\alpha$. Take $\bar{d} \in \bar{D}$ such that $\bar{d} \leqslant_\alpha p \upharpoonright \alpha$. By (c,d),

$$d = \bar{d} \cup (p(\gamma) | \alpha \leqslant \gamma < \kappa) \in P_\kappa$$

and $d \leqslant_\kappa p$.

By the genericity of G_κ take $p \in D \cap G_\kappa$. Then $p \upharpoonright \alpha \in \bar{D} \cap G_\alpha \neq \emptyset$. *qed*(1)

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α.

(2) Let $\alpha < \beta \leqslant \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.

Proof. $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{(p \upharpoonright \beta) \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{q \upharpoonright \alpha \,|\, q \in G_\beta\} \in M[G_\beta]$. *qed*(2)

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leqslant}_\alpha^{G_\alpha}, \emptyset), \text{ if } 1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) \text{ is a forcing”} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), \text{ else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} | p \in G_\kappa\}.$$

(3) H_α is M_α-generic for Q_α.

Proof. If it is not the case that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”, then $(Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset)$ and $H_\alpha = \{\emptyset\}$ is trivially M_α-generic. So assume that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”.

(a) $H_\alpha \subseteq Q_\alpha$.

Proof. Let $p \in G_\kappa$. Then $p \upharpoonright \alpha + 1 \in P_{\alpha+1}$ and so $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Since $p \upharpoonright \alpha \in G_\alpha$ we have that $p(\alpha)^{G_\alpha} \in \dot{Q}_\alpha^{G_\alpha} = Q_\alpha$. *qed*(a)

(b) H_α is a filter.

Proof. Let $p(\alpha)^{G_\alpha} \in H_\alpha$ and $p(\alpha)^{G_\alpha} \leqslant^{Q_\alpha} r \in Q_\alpha$.

(e) Let $\bar{D} \in M_\alpha$ be dense in Q_α. Then $\bar{D} \cap H_\alpha \neq \emptyset$.

Proof. Take $\dot{D} \in M$ such that $\bar{D} = \dot{D}^{G_\alpha}$. Take $p \in G_\kappa$ such that

$$p \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D} \text{ is dense in } \dot{Q}_\alpha.$$

Define

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \Vdash d(\alpha) \in \dot{D}\} \in M.$$

We show that D is dense in P_κ below p. Let $q \leqslant_\kappa p$. Then $q \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha$ and $q \upharpoonright \alpha \Vdash q(\alpha) \dot{\leqslant}_\alpha p(\alpha)$. Hence $q \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D}$ is dense in $\dot{Q}_\alpha$ and there is $\bar{d} \leqslant_\alpha q \upharpoonright \alpha$ and some $d(\alpha) \in \mathrm{dom}(\dot{Q}_\alpha)$ such that

$$\bar{d} \Vdash_{P_\alpha} (d(\alpha) \dot{\leqslant}_\alpha q(\alpha) \wedge d(\alpha) \in \dot{D}).$$

Define

$$d = \bar{d} \cup \{(\alpha, d(\alpha))\} \cup (q(\gamma) | \alpha < \gamma < \kappa).$$

Then $d \in P_\kappa$, $d \leqslant_\kappa q$, and $d \in D$.

By the genericity of G_κ take $d \in D \cap G_\kappa$. Then $d(\alpha)^{G_\alpha} \in H_\alpha$, $d \upharpoonright \alpha \in G_\alpha$, and $d(\alpha)^{G_\alpha} \in (\dot{D})^{G_\alpha} = \bar{D}$. Thus $H_\alpha \cap \bar{D} \neq \emptyset$.

(4) $M_{\alpha+1} = M_\alpha[H_\alpha]$.

Proof. $\supseteq$ is straightforward. For the other direction, if suffices to show that $G_{\alpha+1} \in M_\alpha[H_\alpha]$, and indeed we show that

$$G_{\alpha+1} = \{q \in P_{\alpha+1} \,|\, q \upharpoonright \alpha \in G_\alpha \wedge q(\alpha)^{G_\alpha} \in H_\alpha\}.$$

Let $q \in G_{\alpha+1}$. Take $p \in G_\kappa$ such that $p \upharpoonright \alpha + 1 = q$. Then $q \upharpoonright \alpha = p \upharpoonright \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} = p(\alpha)^{G_\alpha} \in H_\alpha$. For the converse consider $q \in P_{\alpha+1}$ such that $q \upharpoonright \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} \in H_\alpha$. Take $p_1, p_2 \in G_\kappa$ such that $q \upharpoonright \alpha = p_1 \upharpoonright \alpha$ and $q(\alpha)^{G_\alpha} = p_2(\alpha)^{G_\alpha}$. Take $p \in G_\kappa$ such that $p \leqslant_\kappa p_1, p_2$. We also may assume that $p \upharpoonright \alpha \Vdash q(\alpha) = p_2(\alpha)$. $p \upharpoonright \alpha \leqslant_\alpha p_1 \upharpoonright \alpha = q \upharpoonright \alpha$ and $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha p_2(\alpha) = q(\alpha)$. Hence $p \upharpoonright \alpha + 1 \leqslant_{\alpha+1} q$. Since $p \upharpoonright \alpha + 1 \in G_{\alpha+1}$ and since $G_{\alpha+1}$ is upward closed, we get $q \in G_{\alpha+1}$.