Models of Set Theory II

by Peter Koepke

Monday, 14:15-16, Wednesday 12:30-14, Room 1.008, Endenicher Allee 60.

Abstract

Martin's Axiom and applications, iterated forcing, forcing Martin's axiom, adding various types of generic reals, proper forcing.

1 Introduction

The method of *forcing* allows to construct models of set theory with interesting or exotic properties. Further results can be obtained by *transfinite iterations* of this technique. More precisely, iterated forcing defines ordinary generic extensions, which can be analyzed by an increasing well-ordered tower of intermediate models where successor models are (simple) generic extensions of the previous models. Such an analysis is already possible for the Cohen model for $2^{\aleph_0} = \aleph_2$, and we shall indicate some aspects in an introductory chapter. In that model, partially generic filters exist for the simple Cohen forcing $\mathrm{Fn}(\aleph_0, 2, \aleph_0)$. This motivates *forcing axioms* which require the existence of partially generic filters for certain forcings. *Martin's Axiom* MA is a forcing axiom for forcings satisfying the countable antichain condition (ccc). We shall study some consequences of MA and shall then force that axiom by iterated forcing. We shall also study the *Proper Forcing Axiom* PFA for a class of forcings which are *proper*.

Our forcing constructions are mostly directed towards properties of the set $\mathbb{R}$ of real numbers. There are several forcings which adjoin new reals to (ground) models. Different forcings adjoin reals which may be very different with respect to growth behaviour and other aspects. Cardinal characteristics of $\mathbb{R}$ have been introduced to describe such behaviours. They are systematised in Cichon's diagram. Using MA and iterated forcings several constellations of cardinals are realized in Cichon's diagram.

2 Cohen forcing

The most basic forcing construction is the adjunction of a Cohen generic real c to a countable transitive ground model M. The generic extension $M[c]$ is again a countable transitive model of ZFC and contains the "new" real $c \notin M$. In the previous semester we saw that the adjunction of c has consequences for the set theory within $M[c]$:

Theorem 1. *In the* Cohen *extension $M[c]$ the set $\mathbb{R} \cap M$ of ground model reals has (Lebesgue) measure zero.*

This implies some (relative) consistency results. We may, e.g., assume that M is a model of the axiom of constructibility $V=L$, i.e., $M=L^M$. Since the class L is absolute between transitive models of set theory of the same ordinal height, $L^{M[c]}=L^M=M$. So:

Theorem 2. *Let M be a ground model of* ZFC + $V=L$. *Then the* Cohen *extension $M[c]$ satisfies: the set*

$$\{x\in\mathbb{R}\,|\,x\in L\}$$

of constructible reals has measure zero.

On the other hand, inside a given model of set theory, the set $\mathbb{R}$ has positive measure, i.e., does not have measure measure. So in the model L the set of constructible reals does not have measure zero:

Theorem 3. *The statement "the set of constructible reals has measure zero" is independent of the axioms of* ZFC.

Exercise 1. Show that the measure zero sets form a proper ideal on $\mathbb{R}$ which is closed under countable unions.

Exercise 2. Show that the following *Cantor set* of reals has cardinality $2^{\aleph_0}$ and measure zero:

$$C=\{x\in\mathbb{R}\,|\,\forall n<\omega x(2n)=x(2n+1)\}.$$

The set of constructible reals in $M[c]$ can be a set of size $\aleph_1$ that has measure zero. This poses the question whether it is (relatively) consistent that *all* sets of reals of size $\aleph_1$ have measure zero. Of course this necessitates $2^{\aleph_0}>\aleph_1$. It is natural to ask the question about Cohen's canonical model for $2^{\aleph_0}>\aleph_1$.

Consider adjoining λ Cohen reals to a ground model M where $\lambda=\aleph_2^M$. Define λ-fold Cohen forcing $P=(P,\leqslant,1)\in M$ by $P=\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$, $\leqslant=\supseteq$, and $1=\emptyset$. Let G be M-generic on P. Let $F=\bigcup G\colon\lambda\times\omega\to 2$ and extract a sequence $(c_\beta|\alpha<\lambda)$ of Cohen reals $c_\beta\colon\omega\to 2$ from F by:

$$c_\beta(n)=F(\beta,n).$$

Then the generic extension is generated by the sequence of Cohen reals:

$$M[G]=M[(c_\beta|\beta<\lambda)].$$

It is natural to construe $M[G]$ as a limit of the models $M[(c_\beta|\beta<\alpha)]$ when α goes towards λ : Fix $\alpha\leqslant\lambda$. Let $P_\alpha=\mathrm{Fn}(\alpha\times\omega,2,\aleph_0)$ and $R_\alpha=\mathrm{Fn}((\lambda\setminus\alpha)\times\omega,2,\aleph_0)$, partially ordered by reverse inclusion. The isomorphisms

$$P\cong P_\alpha\times R_\alpha \text{ and } P_{\alpha+1}\cong P_\alpha\times Q$$

imply that $G_\alpha=G\cap P_\alpha$ is M-generic on P_α and that

$$H_\alpha=\{q\in Q\,|\,\{((\alpha,n),i)\,|\,(n,i)\in q\}\in G_{\alpha+1}\}$$

is $M[G_\alpha]$-generic on Q. Let $M_\alpha=M[G_\alpha]$ be the α-th model in this construction. Then

$$M_{\alpha+1}=M[G_{\alpha+1}]=M[G_\alpha][H_\alpha]=M_\alpha[H_\alpha].$$

It is straightforward to check that $c_\alpha = \bigcup H_\alpha$. So the model $M[G] = M_\lambda$ is obtained by a sequence of models $(M_\alpha \mid \alpha \leqslant \lambda)$ where each successor step is a Cohen extension of the previous step. The whole construction is held together by the "long" generic set G which dictates the sequence of the construction and also the behaviour at limit stages.

Consider a real $x \in M[G]$. Identifying characteristic functions with sets we can view x as a subset of ω. In the previous course we had seen that there is a name $\dot{x} \in M$, $\dot{x}^G = x$ of the form

$$\dot{x} = \{(\check{n}, q) | n < \omega \wedge q \in A_n\},$$

where every A_n is an antichain in P. Since P satisfies the countable chain condition, there is $\alpha < \lambda$ such that $A_n \subseteq P_\alpha$ for every $n < \omega$. Then

$$x = \dot{x}^G = \dot{x}^{(G \cap P_\alpha)} = \dot{x}^{G_\alpha} \in M[G_\alpha]$$

In $M[G]$ consider a set $B = \{x_i \mid i < \aleph_1\}$ of reals of size $\aleph_1$. One can view B as a subset of $\aleph_1^M$. Like in the above argument, there is an $\alpha < \lambda$ such that $B \in M_\alpha$. By our previous Lemma, $B \subseteq \mathbb{R} \cap M_\alpha$ has measure zero in the Cohen generic extension $M[c_\alpha]$. So B has measure zero in $M[G]$. The model $M[G]$ establishes:

Theorem 4. *If* ZFC *is consistent then* ZFC + *"every set of reals of size* $\leqslant \aleph_1$ *has Lebesgue measure zero" is consistent.*

Together with models of the Continuum Hypothesis this shows that the statement "every set of reals of size $\leqslant \aleph_1$ has Lebesgue measure zero" is independent of the axioms of ZFC.

One can ask for further properties of Lebesgue measure in connection with the uncountable. Is it consistent that every union of an $\aleph_1$-sequence of measure zero sets has again measure zero?

Exercise 3.

a) Show that in the model $M[G] = M[(c_\beta \mid \beta < \lambda)]$ there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$.

b) Show that $\{c_\beta \mid \beta < \lambda\}$ has measure zero in $M[G]$.

Exercise 4. Define forcing with sets of reals of *positive measure* (i.e., sets which do not have measure zero).

We shall later construct several forcing extensions $M[G]$ which are obtained by iterations of forcing notions similar to the above example. We shall require that in the iteration $M_{\alpha+1}$ is a generic extension of M_α by some forcing $Q_\alpha \in M_\alpha = M[G_\alpha]$; the forcing is in general only given by a *name* $\dot{Q}_\alpha \in M$ such that $Q_\alpha = \dot{Q}_\alpha^{G_\alpha}$. To ensure that this is always a partial order we also require that $1_{P_\alpha} \Vdash \dot{Q}_\alpha$ is a partial order. Technical details will be given later.

A principal idea is to let $\dot{Q}_\alpha$ to be some canonical name for a partial order forcing a certain property to hold, like making the set of reals constructed so far a measure zero set. A central concern for such iterations, like for many forcings, is the preservation of cardinals.

3 Forcing axioms

The argument that the set $\mathbb{R}\cap M$ of ground model reals has measure zero in the standard Cohen extension $M[H]=M[c]$ by the Cohen partial order Q rests, like most forcing arguments, on density considerations. For a given $\varepsilon = 2^{-i}$, a sequence $I_0, I_1, I_2, \ldots$ of real intervals such that $\sum_{n<\omega} \text{length}(I_n) \leqslant \varepsilon$ is extracted from the Cohen real c. It remains to show that $X \subseteq \bigcup_{n<\omega} I_n$. For $x \in \mathbb{R}\cap M$ a dense set D_x is defined so that $H \cap D_x \neq \emptyset$ implies that $x \in \bigcup_{n<\omega} I_n$. To cover the real x requires a "partially generic filter" which intersects D_x. This approach is captured by the following definition:

Definition 5. *Let $(Q, \leqslant, 1_Q)$ be a forcing, $\mathcal{D}$ be any set, and κ a cardinal.*

a) A filter H on Q is $\mathcal{D}$-generic *iff $D \cap G \neq \emptyset$ for every $D \in \mathcal{D}$ which is dense in Q.*

b) The forcing axiom $\text{FA}_\kappa(Q)$ *postulates that there exists a* $\mathcal{D}$-generic *filter on Q for any $\mathcal{D}$ of cardinality $\leqslant\kappa$.*

For any countable $\mathcal{D}$ we obtain the existence of generic filters just like in the case of ground models.

Theorem 6. (Rasiowa-Sikorski) $\text{FA}_{\aleph_0}(Q)$ *holds for any partial order Q.*

Proof. Let $\mathcal{D}$ be countable. Take an enumeration $(D_n | n < \omega)$ of all $D \in \mathcal{D}$ which are dense in Q. Define an ω-sequence $q = q_0 \geqslant q_1 \geqslant q_2 \geqslant \ldots$ recursively, using the axiom of choice:

$$\text{choose } q_{n+1} \text{ such that } q_{n+1} \leqslant q_n \text{ and } q_{n+1} \in D_n .$$

Then $H = \{q \in Q | \exists n < \omega \; q_n \leqslant q\}$ is as desired. □

Exercise 5. Show that $\text{FA}_\kappa(Q)$ holds for any κ-closed partial order Q.

The results of the previous chapter now read as follows:

Theorem 7. *Let $Q = \text{Fn}(\omega, 2, \aleph_0)$ be the Cohen partial order and assume* $\text{FA}_{\aleph_1}(Q)$. *Then every set of reals of cardinality $\leqslant\aleph_1$ has measure zero.*

Theorem 8. *Let $M[G]$ be a generic extension of the ground model M by λ-fold Cohen forcing $P = (P, \leqslant, 1) = \text{Fn}(\lambda \times \omega, 2, \aleph_0)$ where $\lambda = \aleph_2^M$. Then in $M[G]$, $\text{FA}_{\aleph_1}(Q)$ holds.*

Proof. We may assume that every $D \in \mathcal{D}$ is a dense subset of Q. Then $\mathcal{D}$ can be coded as a subset of $\aleph_1^M$. There is $\alpha < \lambda$ such that $\mathcal{D} \in M[G_\alpha]$. The filter H_α corresponding to the α-th Cohen real in the construction is $M[G_\alpha]$-generic on Q. Since $\mathcal{D} \subseteq M[G_\alpha]$, H_α is $\mathcal{D}$-generic on Q. □

So for the Cohen forcing Q we have a strengthening of the Rasiowa-Sikorski Lemma from countable to cardinality $\leqslant\aleph_1$. This is not possible for all forcings:

Lemma 9. *Let $P = \text{Fn}(\aleph_0, \aleph_1, \aleph_0)$ be the canonical forcing for adding a surjection from $\aleph_0$ onto $\aleph_1$. Then $\text{FA}_{\aleph_1}(P)$ is false.*

Proof. For $\alpha < \aleph_1$ define the set

$$D_\alpha = \{p \in P \mid \alpha \in \operatorname{ran}(p)\}$$

which is dense in P. Let $D = \{D_\alpha \mid \alpha < \aleph_1\}$. Assume for a contradiction that H is a $\mathcal{D}$-generic filter on P. Then $\bigcup H$ is a partial function from $\aleph_0$ to $\aleph_1$.

(1) $\bigcup H$ is onto $\aleph_1$.

Proof. Let $\alpha < \aleph_1$. Since H is a $\mathcal{D}$-generic, $H \cap D_\alpha \neq \emptyset$. Take $p \in H \cap D_\alpha$. Then

$$\alpha \in \operatorname{ran}(p) \subseteq \operatorname{ran}\left(\bigcup H\right)$$

qed.

But this is a contradiction since $\aleph_1$ is a cardinal. □

Exercise 6. Show that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

So we cannot have an uncountable generalization of the Rasiowa-Sikorski Lemma for forcings which collapse the cardinal $\aleph_1$. Since countable chain condition (ccc) forcing does not collapse cardinals, this suggests the following axiom:

Definition 10.

a) *Let κ be a cardinal. Then* MARTIN's axiom MA_κ *is the property: for every ccc partial order $(P, \leqslant, 1_P)$, $\mathrm{FA}_\kappa(P)$ holds.*

b) MARTIN's axiom MA *postulates that* MA_κ *holds for every* $\kappa < 2^{\aleph_0}$.

$\mathrm{MA}_{\aleph_0}$ holds by Theorem 6. Thus the continuum hypothesis $2^{\aleph_0} = \aleph_1$ trivially implies MA. We shall later see by an iterated forcing construction that $2^{\aleph_0} = \aleph_2$ and MA are relatively consistent with ZFC.

4 Consequences of MA+¬CH

4.1 Lebesgue measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t | t\colon \mathrm{dom}(t) \to 2 \wedge \mathrm{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} | s \subseteq x\} \subseteq \mathbb{R}$$

with $\mathrm{length}(I_s) = 2^{-\mathrm{dom}(s)}$. Note that $I_s = I_{s \cup \{(\mathrm{dom}(s), 0)\}} \cup I_{s \cup \{(\mathrm{dom}(s), 1)\}}$, $\mathrm{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\mathrm{length}(I_{s \cup \{(\mathrm{dom}(s), 0)\}}) = \mathrm{length}(I_{s \cup \{(\mathrm{dom}(s), 1)\}}) = \frac{1}{2} \mathrm{length}(I_s)$.

Definition 11. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has* measure $< \varepsilon$ *if there exists a sequence $(I_n | n < \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and $\sum_{n<\omega} \mathrm{length}(I_n) \leqslant \varepsilon$. A set $X \subseteq \mathbb{R}$ has* measure zero *if it has measure $< \varepsilon$ for every $\varepsilon > 0$.*

The measure zero sets form a countably complete ideal on $\mathbb{R}$. It is easy to see that a countable union of measure zero sets is again measure zero. To strengthen this theorem in the context of MA we need some more topological and measure theoretic notions. The (standard) topology on $\mathbb{R}$ is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n<\omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.