Models of Set Theory II

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Abstract

Martin's Axiom and applications, iterated forcing, forcing Martin's axiom, various types of generic reals, cardinal characteristics, proper forcing.

1 Introduction

The method of *forcing* allows to construct models of set theory with interesting or exotic properties. Further results can be obtained by *transfinite iterations* of this technique. More precisely, iterated forcing defines ordinary generic extensions, which can be analyzed by an increasing well-ordered tower of intermediate models where successor models are ordinary generic extensions of the previous models. Such an analysis is already possible for the Cohen model for $2^{\aleph_0} = \aleph_2$, and we shall indicate some aspects in an introductory chapter. In that model, partially generic filters exist for the standard Cohen forcing $\mathrm{Fn}(\aleph_0, 2, \aleph_0)$. This motivates *forcing axioms* which require the existence of partially generic filters for certain forcings. *Martin's Axiom* MA is a forcing axiom for forcings satisfying the countable antichain condition (ccc). We shall study some consequences of MA and shall then force that axiom by iterated forcing. We shall also study the *Proper Forcing Axiom* PFA for a class of forcings which are *proper*.

Our forcing constructions are mostly directed towards properties of the set $\mathbb{R}$ of real numbers. There are several forcings which adjoin new reals to (ground) models. Different forcings adjoin reals which may be very different with respect to growth behaviour and other aspects. Cardinal characteristics of $\mathbb{R}$ have been introduced to describe such behaviours. They are systematised in CICHON's diagram. Using MA and iterated forcings several constellations of cardinals are realized in CICHON's diagram.

2 Cohen forcing

The most basic forcing construction is the adjunction of a Cohen generic real c to a countable transitive ground model M. The generic extension $M[c]$ is again a countable transitive model of ZFC and it contains the "new" real $c \notin M$. In the previous semester we saw that the adjunction of c has consequences for the set theory within $M[c]$:

Theorem 1. *In the* COHEN *extension $M[c]$ the set $\mathbb{R} \cap M$ of ground model reals has (Lebesgue) measure zero.*

This implies some (relative) consistency results. We may, e.g., assume that M is a model of the axiom of constructibility $V = L$, i.e., $M = L^M$. Since the class L is absolute between transitive models of set theory of the same ordinal height, $L^{M[c]} = L^M = M$. So:

Theorem 2. *Let M be a ground model of* ZFC + $V = L$. *Then the* COHEN *extension $M[c]$ satisfies: the set*

$$\{x \in \mathbb{R} \,|\, x \in L\}$$

of constructible reals has measure zero.

On the other hand, inside a given model of set theory, the set of reals has positive measure, i.e., does not have measure measure.

Exercise 1. Show that the measure zero sets form a proper ideal on $\mathbb{R}$ which is closed under countable unions.

Exercise 2. Show that the following *Cantor set* of reals has cardinality $2^{\aleph_0}$ and measure zero:

$$C = \{x \in \mathbb{R} \,|\, \forall n < \omega\, x(2n) = x(2n+1)\}.$$

So in the model L the set of constructible reals does not have measure zero:

Theorem 3. *The statement "the set of constructible reals has measure zero" is independent of the axioms of* ZFC.

The set of constructible reals in $M[c]$ can be a set of size $\aleph_1$ that has measure zero. This leads to the question whether it is (relatively) consistent that *all* sets of reals of size $\aleph_1$ have measure zero. Of course this necessitates $2^{\aleph_0} > \aleph_1$. It is natural to ask the question about Cohen's canonical model for $2^{\aleph_0} > \aleph_1$.

Consider adjoining λ COHEN reals to a ground model M where $\lambda = \aleph_2^M$. Define λ-fold COHEN forcing $P = (P, \leqslant, 1) \in M$ by $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0)$, $\leqslant = \supseteq$, and $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G: \lambda \times \omega \to 2$ and extract a sequence $(c_\beta | \alpha < \lambda)$ of Cohen reals $c_\beta: \omega \to 2$ from F by:

$$c_\beta(n) = F(\beta, n).$$

Then the generic extension is generated by the sequence of Cohen reals:

$$M[G] = M[(c_\beta | \beta < \lambda)].$$

It is natural to construe $M[G]$ as a limit of the models $M[(c_\beta | \beta < \alpha)]$ when α goes towards λ : Fix $\alpha \leqslant \lambda$. Let $P_\alpha = \mathrm{Fn}(\alpha \times \omega, 2, \aleph_0)$ and $R_\alpha = \mathrm{Fn}((\lambda \setminus \alpha) \times \omega, 2, \aleph_0)$, partially ordered by reverse inclusion. The isomorphisms

$$P \cong P_\alpha \times R_\alpha \text{ and } P_{\alpha+1} \cong P_\alpha \times Q$$

imply that $G_\alpha = G \cap P_\alpha$ is M-generic on P_α and that

$$H_\alpha = \{q \in Q \,|\, \{((\alpha, n), i) \,|\, (n, i) \in q\} \in G_{\alpha+1}\}$$

is $M[G_\alpha]$-generic on Q. Let $M_\alpha = M[G_\alpha]$ be the α-th model in this construction. Then

$$M_{\alpha+1} = M[G_{\alpha+1}] = M[G_\alpha][H_\alpha] = M_\alpha[H_\alpha].$$

It is straightforward to check that $c_\alpha = \bigcup H_\alpha$. So the model $M[G] = M_\lambda$ is obtained by a sequence of models $(M_\alpha \mid \alpha \leqslant \lambda)$ where each successor step is a Cohen extension of the previous step. The whole construction is held together by the "long" generic set G which dictates the sequence of the construction and also the behaviour at limit stages.

Consider a real $x \in M[G]$. Identifying characteristic functions with sets we can view x as a subset of ω. In the previous course we had seen that there is a name $\dot{x} \in M$, $\dot{x}^G = x$ of the form

$$\dot{x} = \{(\check{n}, q) | n < \omega \wedge q \in A_n\},$$

where every A_n is an antichain in P. Since P satisfies the countable chain condition, there is $\alpha < \lambda$ such that $A_n \subseteq P_\alpha$ for every $n < \omega$. Then

$$x = \dot{x}^G = \dot{x}^{(G \cap P_\alpha)} = \dot{x}^{G_\alpha} \in M[G_\alpha]$$

In $M[G]$ consider a set $B = \{x_i \mid i < \aleph_1\}$ of reals of size $\aleph_1$. One can view B as a subset of $\aleph_1^M$. As in the above argument, there is an $\alpha < \lambda$ such that $B \in M_\alpha$. By our previous Lemma, $B \subseteq \mathbb{R} \cap M_\alpha$ has measure zero in the Cohen generic extension $M[c_\alpha]$. So B has measure zero in $M[G]$. The model $M[G]$ establishes:

Theorem 4. *If* ZFC *is consistent then* ZFC + *"every set of reals of size* $\leqslant \aleph_1$ *has Lebesgue measure zero" is consistent.*

Together with models of the Continuum Hypothesis this shows that the statement "every set of reals of size $\leqslant \aleph_1$ has Lebesgue measure zero" is independent of the axioms of ZFC.

One can ask for further properties of Lebesgue measure in connection with the uncountable. Is it consistent that every union of an $\aleph_1$-sequence of measure zero sets has again measure zero?

Exercise 3.

a) Show that in the model $M[G] = M[(c_\beta \mid \beta < \lambda)]$ there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$.

b) Show that $\{c_\beta \mid \beta < \lambda\}$ has measure zero in $M[G]$.

Exercise 4. Define forcing with sets of reals of *positive measure* (i.e., sets which do not have measure zero).

We shall later construct forcing extensions $M[G]$ which are obtained by iterations of forcing notions similar to the above example. We shall require that in the iteration $M_{\alpha+1}$ is a generic extension of M_α by some forcing $Q_\alpha \in M_\alpha = M[G_\alpha]$; the forcing is in general only given by a *name* $\dot{Q}_\alpha \in M$ such that $Q_\alpha = \dot{Q}_\alpha^{G_\alpha}$. To ensure that this is always a partial order we also require that $1_{P_\alpha} \Vdash \dot{Q}_\alpha$ is a partial order. Technical details will be given later.

A principal idea is to let $\dot{Q}_\alpha$ to be some canonical name for a partial order forcing a certain property to hold, like making the set of reals constructed so far a measure zero set. A central concern for such iterations, like for many forcings, is the preservation of cardinals.

3 Forcing axioms

The argument that the set $\mathbb{R}\cap M$ of ground model reals has measure zero in the standard Cohen extension $M[H]=M[c]$ by the Cohen partial order Q rests, like most forcing arguments, on density considerations. For a given $\varepsilon=2^{-i}$, a sequence $I_0, I_1, I_2, \ldots$ of real intervals such that $\sum_{n<\omega}\text{length}(I_n)\leqslant\varepsilon$ is extracted from the Cohen real c. It remains to show that $X\subseteq\bigcup_{n<\omega}I_n$. For $x\in\mathbb{R}\cap M$ a dense set D_x is defined so that $H\cap D_x\neq\emptyset$ implies that $x\in\bigcup_{n<\omega}I_n$. To cover the real x requires a "partially generic filter" which intersects D_x. This approach is captured by the following definition:

Definition 5. *Let $(Q,\leqslant,1_Q)$ be a forcing, $\mathcal{D}$ be any set, and κ a cardinal.*

a) A filter H on Q is $\mathcal{D}$-generic *iff $D\cap G\neq\emptyset$ for every $D\in\mathcal{D}$ which is dense in Q.*

b) The forcing axiom $\mathrm{FA}_\kappa(Q)$ *postulates that there exists a* $\mathcal{D}$-generic *filter on Q for any $\mathcal{D}$ of cardinality $\leqslant\kappa$.*

For any countable $\mathcal{D}$ we obtain the existence of generic filters just like in the case of ground models.

Theorem 6. (Rasiowa-Sikorski) $\mathrm{FA}_{\aleph_0}(Q)$ *holds for any partial order Q.*

Proof. Let $\mathcal{D}$ be countable. Take an enumeration $(D_n|n<\omega)$ of all $D\in\mathcal{D}$ which are dense in Q. Define an ω-sequence $q=q_0\geqslant q_1\geqslant q_2\geqslant\ldots$ recursively, using the axiom of choice:

$$\text{choose } q_{n+1} \text{ such that } q_{n+1}\leqslant q_n \text{ and } q_{n+1}\in D_n.$$

Then $H=\{q\in Q|\exists n<\omega\ q_n\leqslant q\}$ is as desired. □

Exercise 5. Show that $\mathrm{FA}_\kappa(Q)$ holds for any κ-closed partial order Q.

The results of the previous chapter now read as follows:

Theorem 7. *Let $Q=\mathrm{Fn}(\omega,2,\aleph_0)$ be the Cohen partial order and assume $\mathrm{FA}_{\aleph_1}(Q)$. Then every set of reals of cardinality $\leqslant\aleph_1$ has measure zero.*

Theorem 8. *Let $M[G]$ be a generic extension of the ground model M by λ-fold Cohen forcing $P=(P,\leqslant,1)=\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$ where $\lambda=\aleph_2^M$. Then in $M[G]$, $\mathrm{FA}_{\aleph_1}(Q)$ holds.*

Proof. We may assume that every $D\in\mathcal{D}$ is a dense subset of Q. Then $\mathcal{D}$ can be coded as a subset of $\aleph_1^M$. There is $\alpha<\lambda$ such that $\mathcal{D}\in M[G_\alpha]$. The filter H_α corresponding to the α-th Cohen real in the construction is $M[G_\alpha]$-generic on Q. Since $\mathcal{D}\subseteq M[G_\alpha]$, H_α is $\mathcal{D}$-generic on Q. □

So for the Cohen forcing Q we have a strengthening of the Rasiowa-Sikorski Lemma from countable to cardinality $\leqslant\aleph_1$. This is not possible for all forcings:

Lemma 9. *Let $P=\mathrm{Fn}(\aleph_0,\aleph_1,\aleph_0)$ be the canonical forcing for adding a surjection from $\aleph_0$ onto $\aleph_1$. Then $\mathrm{FA}_{\aleph_1}(P)$ is false.*

Proof. For $\alpha < \aleph_1$ define the set

$$D_\alpha = \{p \in P \mid \alpha \in \operatorname{ran}(p)\}$$

which is dense in P. Let $D = \{D_\alpha \mid \alpha < \aleph_1\}$. Assume for a contradiction that H is a $\mathcal{D}$-generic filter on P. Then $\bigcup H$ is a partial function from $\aleph_0$ to $\aleph_1$.

(1) $\bigcup H$ is onto $\aleph_1$.

Proof. Let $\alpha < \aleph_1$. Since H is a $\mathcal{D}$-generic, $H \cap D_\alpha \neq \emptyset$. Take $p \in H \cap D_\alpha$. Then

$$\alpha \in \operatorname{ran}(p) \subseteq \operatorname{ran}\left(\bigcup H\right)$$

qed.

But this is a contradiction since $\aleph_1$ is a cardinal. □

Exercise 6. Show that $\mathrm{FA}_{2^{\aleph_0}}(\mathrm{Fn}(\aleph_0, \aleph_0, \aleph_0))$ is false.

So we cannot have an uncountable generalization of the Rasiowa-Sikorski Lemma for forcings which collapse the cardinal $\aleph_1$. Since countable chain condition (ccc) forcing does not collapse cardinals, this suggests the following axiom:

Definition 10.

a) *Let κ be a cardinal. Then* MARTIN's axiom MA_κ *is the property: for every ccc partial order* $(P, \leqslant, 1_P)$*,* $\mathrm{FA}_\kappa(P)$ *holds.*

b) MARTIN's axiom MA *postulates that* MA_κ *holds for every* $\kappa < 2^{\aleph_0}$.

$\mathrm{MA}_{\aleph_0}$ holds by Theorem 6. Thus the continuum hypothesis $2^{\aleph_0} = \aleph_1$ trivially implies MA. We shall later see by an iterated forcing construction that $2^{\aleph_0} = \aleph_2$ and MA are relatively consistent with ZFC.

4 Consequences of MA+¬CH

4.1 Lebesgue measure

We shall not go into the details of LEBESGUE measure, since we shall only consider measure zero sets. We recall some notions and facts from before. For $s \in {}^{<\omega}2 = \{t | t\colon \operatorname{dom}(t) \to 2 \wedge \operatorname{dom}(t) \in \omega\}$ define the real *interval*

$$I_s = \{x \in \mathbb{R} | s \subseteq x\} \subseteq \mathbb{R}$$

with $\operatorname{length}(I_s) = 2^{-\operatorname{dom}(s)}$. Note that $I_s = I_{s \cup \{(\operatorname{dom}(s),0)\}} \cup I_{s \cup \{(\operatorname{dom}(s),1)\}}$, $\operatorname{length}(\mathbb{R}) = I_\emptyset = 2^{-0} = 1$, and $\operatorname{length}(I_{s \cup \{(\operatorname{dom}(s),0)\}}) = \operatorname{length}(I_{s \cup \{(\operatorname{dom}(s),1)\}}) = \frac{1}{2}\operatorname{length}(I_s)$.

Definition 11. *Let $\varepsilon > 0$. Then a set $X \subseteq \mathbb{R}$ has* measure $<\varepsilon$ *if there exists a sequence $(I_n | n < \omega)$ of intervals in $\mathbb{R}$ such that $X \subseteq \bigcup_{n<\omega} I_n$ and* $\sum_{n<\omega} \operatorname{length}(I_n) \leqslant \varepsilon$. *A set $X \subseteq \mathbb{R}$ has* measure zero *if it has measure $<\varepsilon$ for every $\varepsilon > 0$.*

The measure zero sets form a countably complete ideal on $\mathbb{R}$. It is easy to see that a countable union of measure zero sets is again measure zero. To strengthen this theorem in the context of MA we need some more topological and measure theoretic notions. The (standard) topology on $\mathbb{R}$ is generated by the basic open sets I_s for $s \in {}^{<\omega}2$. Hence every union $\bigcup_{n<\omega} I_n$ of basic open intervals is itself open. The basic open intervals I_s are also compact in the sense of the HEINE-BOREL theorem: every cover of I_s by open sets has a finite subcover.

Theorem 12. *Assume* MA_κ *and let* $(X_i|i<\kappa)$ *be a family of measure zero sets. Then* $X = \bigcup_{i<\kappa} X_i$ *has measure zero.*

Proof. Fix $\varepsilon > 0$. We show that $X = \bigcup_{i<\kappa} X_i$ has measure $<2\varepsilon$. Let

$$\mathcal{I} = \{(a,b)|a,b \in \mathbb{Q}, a<b\}$$

the countable set of rational intervals $(a,b) = \{c \in \mathbb{R}|a<c<b\}$ in $\mathbb{R}$. The *length* of (a,b) is simply $\mathrm{length}((a,b)) = b-a$. We shall apply MARTIN's axiom to the following forcing $P = (P, \supseteq, \emptyset)$ where

$$P = \{p \subseteq \mathcal{I}|\sum_{I\in p} \mathrm{length}(I) < \varepsilon\}.$$

(1) P is ccc.
Proof. Let $\{p_i|i<\omega_1\} \subseteq P$. For every $i<\omega_1$ there is $n_i<\omega$ such that p_i has measure $<\varepsilon - \frac{1}{n_i}$. By a pigeonhole principle we may assume that all n_i are equal to a common value $n<\omega$. For every p_i we have

$$\sum_{I\in p_i} \mathrm{length}(I) < \varepsilon - \frac{1}{n}.$$

For every $i<\omega_1$ take a finite set $\bar{p}_i \subseteq p_i$ such that

$$\sum_{I\in p_i\setminus\bar{p}_i} \mathrm{length}(I) < \frac{1}{n}.$$

There are only countably many such set $\bar{p}_i$, and again by a pigeonhole argument we may assume that for all $i<\omega_1$

$$\bar{p}_i = \bar{p}$$

takes a fixed value. Now consider $i<j<\omega_1$. Then

$$\begin{aligned}\sum_{I\in p_i\cup p_j} \mathrm{length}(I) &\leqslant \sum_{I\in p_i} \mathrm{length}(I) + \sum_{I\in p_j\setminus\bar{p}} \mathrm{length}(I)\\ &< \varepsilon - \frac{1}{n} + \frac{1}{n}\\ &= \varepsilon\end{aligned}$$

Hence $p_i \cup p_j \in P$ and $p_i \cup p_j \leqslant p_i, p_j$, and so $\{p_i|i<\omega_1\}$ is not an antichain in P. *qed*(1)

For $i<\kappa$ define

$$D_i = \{p \in P|X_i \subseteq \bigcup p\}.$$

(2) D_i is dense in P.
Proof. Let $q \in P$. Take $n < \omega$ such that

$$\sum_{I \in q} \mathrm{length}(I) < \varepsilon - \frac{1}{n}.$$

Since X_i has measure zero, take $r \subseteq \mathcal{I}$ such that $X_i \subseteq \bigcup p$ and $\sum_{I \in r} \mathrm{length}(I) \leqslant \frac{1}{n}$. Then

$$X_i \subseteq \bigcup (q \cup r) \quad \text{and} \quad \sum_{I \in q \cup r} \mathrm{length}(I) \leqslant \sum_{I \in q} \mathrm{length}(I) + \sum_{I \in r} \mathrm{length}(I) < \varepsilon - \frac{1}{n} + \frac{1}{n} = \varepsilon.$$

Hence $p = q \cup r \in P$, $p \supseteq q$, and $p \in D_i$. *qed*(2)

By MA_κ take a filter G on P which is $\{D_i | i < \kappa\}$-generic. Let $U = \bigcup G \subseteq \mathcal{I}$.

(3) $X = \bigcup_{i<\kappa} X_i \subseteq \bigcup_{I \in U} I$.
Proof. Let $i < \kappa$. By the generity of G take $p \in G \cap D_i$. Then

$$X_i \subseteq \bigcup p \subseteq \bigcup U$$

qed(3)

(4) $\sum_{I \in U} \mathrm{length}(I) \leqslant \varepsilon$.
Proof. Assume for a contradiction that $\sum_{I \in U} \mathrm{length}(I) > \varepsilon$. Then take a finite set $\bar{U} \subseteq U$ such that $\sum_{I \in \bar{U}} \mathrm{length}(I) > \varepsilon$. Let $\bar{B} = \{I_0, ..., I_{k-1}\}$. For every $I_j \in \bar{U}$ take $p_j \in G$ such that $I_j \in p_j$. Since all elements of G are compatible within G there is a condition $p \in G$ such that $p \supseteq p_0, ..., p_{k-1}$. Hence $\bar{U} \subseteq p$. But, since $p \in P$, we get a contradiction:

$$\varepsilon < \sum_{I \in \bar{U}} \mathrm{length}(I) \leqslant \sum_{I \in p} \mathrm{length}(I) < \varepsilon.$$

□

Two easy consequences are:

Corollary 13. *Assume* MA_κ *and let* $X \subseteq \mathbb{R}$ *with* $\mathrm{card}(X) \leqslant \kappa$. *Then* X *has measure zero.*

Theorem 14. *Assume* MA. *Then* $2^{\aleph_0}$ *is regular.*

Proof. Assume instead that $\mathbb{R} = \bigcup_{i<\kappa} X_i$ for some $\kappa < 2^{\aleph_0}$, where $\mathrm{card}(X_i) < 2^{\aleph_0}$ for every $i < \kappa$. Every singleton $\{r\}$ has measure zero. By Theorem 12, each X_i has measure zero. Again by Theorem, $\mathbb{R} = \bigcup_{i<\kappa} X_i$ has measure zero. But measure theory (and also intuition) shows that $\mathbb{R}$ does not have measure zero. □

4.2 Almost disjoint forcing

We intend to code subsets of κ by subsets of ω. If such a coding is possible then we shall have

$$2^{\aleph_0} \leqslant 2^\kappa \leqslant 2^{\aleph_0}, \text{ i.e. } 2^\kappa = 2^{\aleph_0}.$$

We shall employ almost disjoint coding.

Definition 15. *A sequence* $(x_i|i\in I)$ *is* almost disjoint *if*

a) x_i *is infinite*

b) $i\neq j<\kappa$ *implies that* $x_i\cap x_j$ *is finite*

Lemma 16. *There is an almost disjoint sequence* $(x_i|i<2^{\aleph_0})$ *of subsets of* ω.

Proof. For $u\in{}^{\omega}2$ let $x_u=\{u\upharpoonright m\,|\,m<\omega\}$. x_u is infinite. Consider $u\neq v$ from ${}^{\omega}2$. Let $n<\omega$ be minimal such that $u\upharpoonright n\neq v\upharpoonright n$. Then

$$x_u\cap x_v=\{u\upharpoonright m\,|\,m<\omega\}\cap\{v\upharpoonright m\,|\,m<\omega\}=\{u\upharpoonright m\,|\,m<n\}$$

is finite. Thus $(x_u|u\in{}^{\omega}2)$ is almost disjoint. Using bijections $\omega\leftrightarrow{}^{<\omega}2$ and $2^{\aleph_0}\leftrightarrow{}^{\omega}2$ one can turn this into an almost disjoint sequence $(x_i|i<2^{\aleph_0})$ of subsets of ω. □

Theorem 17. *Assume* MA_κ. *Then* $2^\kappa=2^{\aleph_0}$.

Proof. By a previous example, $\kappa<2^{\aleph_0}$. By the lemma, fix an almost disjoint sequence $(x_i|i<\kappa)$ of subsets of ω. Define a map $c\colon\mathcal{P}(\omega)\to\mathcal{P}(\kappa)$ by

$$c(x)=\{i<\kappa|x\cap x_i\text{ is infinite}\}.$$

We say that x *codes* $c(x)$. We want to show that every subset of κ can be coded as some $c(x)$. We show this by proving that $c\colon\mathcal{P}(\omega)\to\mathcal{P}(\kappa)$ is surjective.

Let $A\subseteq\kappa$ be given. We use the following forcing $(P,\leqslant,1)$ to code A:

$$P=\{(a,z)|a\subseteq\omega,z\subseteq\kappa,\mathrm{card}(a)<\aleph_0,\mathrm{card}(z)<\aleph_0\},$$

partially ordered by

$$(a',z')\leqslant(a,z)\text{ iff }a'\supseteq a,z'\supseteq z,i\in z\cap(\kappa\setminus A)\to a'\cap x_i=a\cap x_i.$$

The weakest element of P is $1=(\emptyset,\emptyset)$.

The idea of the forcing is to keep the intersection of the first component with x_i fixed, provided $i\notin A$ has entered the second component. This will allow the almost disjoint coding of A by the finite/infinite method.

(1) $(P,\leqslant,1)$ satisfies ccc.

Proof. Conditions $(a,\,y)$ and $(a,\,z)$ with equal first components are compatible, since $(a,y\cup z)\leqslant(a,y)$ and $(a,y\cup z)\leqslant(a,z)$. Incompatibel conditions have different first components. Since there are only countably many first components, an antichain in P can be at most countable. *qed*(1)

The outcome of a forcing construction results from an interplay between the partial order and some dense set arguments. We now define dense sets for our requirements.

For $i<\kappa$ let $D_i=\{(a,z)\in P|i\in z\}$. D_i is obviously dense in P. For $i\in A$ and $n\in\omega$ let $D_{i,n}=\{(a,z)\in P|\exists m>n\colon m\in a\cap x_i\}$.

(2) If $i\in A$ and $n\in\omega$ then $D_{i,n}$ is dense in P.

Proof. Consider $(a,z)\in P$. For $j\in z$, $j\neq i$ is the intersection $x_i\cap x_j$ finite. Take some $m\in x_i$, $m>n$ such that $m\notin x_i\cap x_j$ for $j\in z$, $j\neq i$. Then

$$(a\cup\{m\},z)\leqslant(a,z)\text{ and }(a\cup\{m\},z)\in D_{i,n}.$$

qed(2)

By MA_κ take a filter G on P which is generic for the dense sets in

$$\{D_i | i < \kappa\} \cup \{D_{i,n} | i \in A, n \in \omega\}.$$

Let

$$x = \bigcup \{a | (a, y) \in G\} \subseteq \omega.$$

(3) Let $i \in A$. Then $x \cap x_i$ is infinite.
Proof. Let $n < \omega$. By genericity take $(a, y) \in G \cap D_{i,n}$. By the definition of $D_{i,n}$ take $m > n$ such that $m \in a \cap x_i$. Then $m \in x \cap x_i$, and so $x \cap x_i$ is cofinal in ω. *qed*(3)
(4) Let $i \in \kappa \setminus A$. Then $x \cap x_i$ is finite.
Proof. By genericity take $(a, y) \in G \cap D_i$. Then $i \in y$. We show that $x \cap x_i \subseteq a \cap x_i$. Consider $n \in x \cap x_i$. Take $(b, z) \in G$ such that $n \in b$. By the filter properties of G take $(a', y') \in P$ such that $(a', y') \leqslant (a, y)$ and $(a', y') \leqslant (b, z)$. Then $n \in a'$, and by the definition of $\leqslant$, $a' \cap x_i = a \cap x_i$. Thus $n \in a \cap x_i$. *qed*(4)

So

$$c(x) = \{i < \kappa | x \cap x_i \text{ is infinite}\} = A \in \mathrm{range}(c).$$

□

4.3 Baire category

Lebesgue measure defines an ideal of "small" sets, namely the ideal of measure zero sets: arbitrary subsets of measure zero sets are measure zero, and, under MA, every union of less than $2^{\aleph_0}$ measure zero sets is again measure zero.

We now look at another ideal of small sets, namely the ideal of subsets X of $\mathbb{R}$ which are nowhere dense in $\mathbb{R}$: every nonempty open interval in $\mathbb{R}$ has a nonempty open subinterval which is disjoint from X. The union of all such subintervals is open, dense in $\mathbb{R}$, and disjoint from X.

The BAIRE category theorem says that the intersection of countably many dense open sets of reals in dense in $\mathbb{R}$. We can strengthen this to:

Theorem 18. *Assume* MA_κ. *Then the intersection of* κ *many dense open sets of reals is dense in* $\mathbb{R}$.

Proof. Consider a sequence $(O_i | i < \kappa)$ of dense open subsets of $\mathbb{R}$. We use standard COHEN forcing $P = \mathrm{Fn}(\omega, 2, \aleph_0)$ for the density argument. Since P is countable it trivially has the ccc. For $i < \kappa$ define $D_i = \{p \in P | \forall x \in \mathbb{R}\, (x \supseteq p \to x \in O_i)\}$. This means that the interval determined by p lies within O_i. The density of D_i follows readily since O_i is open dense. For $n < \omega$ let $D_n = \{p \in P | n \in \mathrm{dom}(p)\}$. Obviously, D_n is also dense in P. By MA_κ let $G \subseteq P$ be $\{D_i | i < \kappa\}$-$\{D_n | n < \kappa\}$generic. Let $x = \bigcup G$. $p \in G \cap D_n$ implies that $n \in \mathrm{dom}(p) \subseteq \mathrm{dom}(x)$. So $x\colon \omega \to 2$ is a real number. □

Since $\mathrm{MA}_{\aleph_0}$ is always true in ZFC, we get the BAIRE category theorem:

Theorem 19. *The intersection of countably many dense open sets of reals is dense in* $\mathbb{R}$.

This says that dense open sets (of reals) have a largeness property, and correspondingly complements of dense open sets are small.

Definition 20. *A set* $A \subseteq \mathbb{R}$ *is* nowhere dense *if there is a dense open set* $O \subseteq \mathbb{R}$ *such that* $A \cap O = \emptyset$. *A set* $A \subseteq \mathbb{R}$ *is* meager *or* of 1st category *if it is a union of countably many nowhere dense sets.*

Proposition 21.

a) *A singleton set* $\{x\} \subseteq \mathbb{R}$ *is nowhere dense since* $\mathbb{R} \setminus \{x\}$ *is dense open in* $\mathbb{R}$.

b) *A countable set* C *is meager.*

c) *A set* $A \subseteq \mathbb{R}$ *is meager iff there are open dense sets* $(O_n | n < \omega)$ *such that* $A \cap \bigcap_{n<\omega} O_n = \emptyset$.

d) $\mathbb{R}$ *is not meager. Sets which are not meager are said to be of* 2nd category.

Proof. *c*) Let $A = \bigcup_{n<\omega} A_n$ be meager where each A_n is nowhere dense. For each n choose O_n dense open in $\mathbb{R}$ such that $A_n \cap O_n = \emptyset$. Then

$$(\bigcup_{n<\omega} A_n) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

Conversely assume that $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ where each O_n is dense open. $(A \setminus O_n) \cap O_n = \emptyset$, and so by definition, every $A_n = A \setminus O_n$ is nowhere dense. Obviously

$$\bigcup_{n<\omega} A_n \subseteq A.$$

For the converse consider $x \in A$. The property $A \cap (\bigcap_{n<\omega} O_n) = \emptyset$ implies that we may take $n < \omega$ such that $x \notin O_n$. Hence $x \in A \setminus O_n = A_n$. So $A = \bigcup_{n<\omega} A_n$ is meager.

d) If $\mathbb{R}$ were meager then there would be open dense sets $(O_n | n < \omega)$ such that $\mathbb{R} \cap \bigcap_{n<\omega} O_n = \emptyset$. But by Theorem 19,

$$\mathbb{R} \cap \bigcap_{n<\omega} O_n = \bigcap_{n<\omega} O_n \neq \emptyset,$$

contradiction. □

We would now like to show as in the case of measure that a union of $<2^{\aleph_0}$ small sets in the sense of category is again small if MARTIN's axiom holds.

Theorem 22. *Assume* MA_κ . *Let* $(A_i | i < \kappa)$ *be a family of meager sets. Then* $A = \bigcup_{i<\kappa} A_i$ *is meager.*

Proof. Obviously it suffices to consider the case where each A_i is nowhere dense. We shall use MA_κ to find dense open sets $(O_n | n < \omega)$ such that

$$(\bigcup_{i<\kappa} A_i) \cap (\bigcap_{n<\omega} O_n) = A \cap (\bigcap_{n<\omega} O_n) = \emptyset.$$

The forcing will consist of approximations to a family $(O_n|n<\omega)$ of open dense sets which makes this equality true.

The forcing conditions will consist of finitely many finite approximations to the O_n . Moreover there will be for every n a finite collection of $i<\kappa$ such that an approximation to the equation holds for those i. We shall see that by appropriate density considerations the full equality may be satisfied.

For ccc-reasons, much like in the argument of measure-zero sets, we only consider approximations to the O_n by finitely many *rational* intervals. Let

$$\mathcal{I}=\{(a,b)|a,b\in\mathbb{Q},a<b\}$$

the countable set of rational open intervals $(a,b)=\{c\in\mathbb{R}|a<c<b\}$ in $\mathbb{R}$. Now let

$P=\{(r,\ s)|r\colon \omega\to[\mathcal{I}]^{<\omega},\ s\colon \omega\to[\kappa]^{<\omega},\ \{n<\omega|r(n)\neq\emptyset\}$ is finite, $\{n<\omega|s(n)\neq\emptyset\}$ is finite, $\forall n<\omega\forall i\in s(n)\ A_i\cap\bigcup r(n)=\emptyset\}$.

Define

$$(r',s')\leqslant(r,s)\ \text{ iff }\ \forall n<\omega\,(r'(n)\supseteq r(n)\wedge s'(n)\supseteq s(n)).$$

(1) $(P,\leqslant)$ satisfies the countable chain condition.
Proof. Consider $(r,\ s)$ and $(r,\ s')$ in P having the same first component. Then define s'': $\omega\to[\kappa]^{<\omega}$ by $s''(n)=s(n)\cup s'(n)$. It is easy to check that $(r,s'')\in P$, and also $(r,s'')\leqslant(r,s)$ and $(r,s'')\leqslant(r,s')$. So (r,s) and (r,s') are compatible in P.

An antichain in P must consist of conditions whose first components are pairwise distinct. Since there are only countably many first components, an antichain in P is at most countable. *qed*(1)

For each $n<\omega$ the following dense sets ensures the density of the O_n in $\mathbb{R}$: for $I\in\mathcal{I}$ let

$$D_{n,I}=\{(r',s')|\exists J\in r'(n)\ J\subseteq I\}.$$

(2) $D_{n,I}$ is dense in P.
Proof. Let $(r,\ s)\in P$. Let $s(n)=\{i_0,\ldots,i_{k-1}\}$. Since $A_{i_0},\ldots,A_{i_{k-1}}$ are nowhere dense one can go find intervals $I\supseteq I_{i_0}\supseteq\ldots\supseteq I_{k-1}=J$ in $\mathcal{I}$ such that $A_{i_l}\cap I_{i_l}=\emptyset$. Define $r'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $r'\restriction(\omega\setminus\{n\})=r\restriction(\omega\setminus\{n\})$ and $r'(n)=r(n)\cup\{J\}$. Then $(r',s)\in P$, $(r',s)\leqslant(r,s)$, and $(r',s)\in D_{n,I}$. *qed*(2)

We also need that every $i<\kappa$ is considered by some O_n. Define

$$D_i=\{(r',s')|\exists n<\omega\ i\in s'(n)\}.$$

(3) D_i is dense in P.
Proof. Let $(r,\ s)\in P$. Take $n<\omega$ such that $r(n)=\emptyset$. Define $s'\colon\omega\to[\mathcal{I}]^{<\omega}$ by $s'\restriction(\omega\setminus\{n\})=s\restriction(\omega\setminus\{n\})$ and $s'(n)=s(n)\cup\{i\}$. Then $(r,s')\in P$, $(r,s')\leqslant(r,s)$, and $(r,s')\in D_i$. *qed*(3)

By MA_κ we can take a filter G on P which is generic for

$$\{D_{n,I}|n<\omega,I\in\mathcal{I}\}\cup\{D_i|i<\kappa\}.$$

For $n<\omega$ define

$$O_n=\bigcup\bigcup\{r(n)|(r,s)\in G\}.$$

(4) O_n is open, since it is a union of open intervals.
(5) O_n is dense in $\mathbb{R}$.
Proof. Let $I\in\mathcal{I}$. By genericity take $(r',s')\in G\cap D_{n,I}$. Take $J\in r'(n)$ such that $J\subseteq I$. Then

$$\emptyset\neq J\subseteq\bigcup r'(n)\subseteq\bigcup\bigcup\{r(n)|(r,s)\in G\}=O_n.$$

qed(5)
(6) Let $i<\kappa$. Then $A_i\cap\bigcap_{n<\omega}O_n=\emptyset$.
Proof. By genericity take $(r',s')\in G\cap D_i$. Take $n<\omega$ such that $i\in s'(n)$. We show that $A_i\cap O_n=\emptyset$. Assume not, and let $x\in A_i\cap O_n$. Take $(r,s)\in G$ and $I\in r(n)$ such that $x\in I$. Since G is a filter, take $(r'',s'')\in P$ such that $(r'',s'')\leqslant(r,s)$ and $(r'',s'')\leqslant(r',s')$. Then $I\in r''(n)$, $i\in s''(n)$, and

$$x\in A_i\cap I\subseteq A_i\cap\bigcup r''(n)\neq\emptyset.$$

The last inequality contradicts the definition of P. *qed*(6)

By (6), $\bigcup_{i<\kappa}A_i\cap\bigcap_{n<\omega}O_n=\emptyset$, and so $\bigcup_{i<\kappa}A_i$ is meager. □

5 Iterated forcing

MARTIN's axiom postulates that for every ccc partial order $(P,\leqslant,1_P)$ and $\mathcal{D}$ with $\mathrm{card}(\mathcal{D})<2^{\aleph_0}$ there is a $\mathcal{D}$-generic filter G on P. Syntactically this axiom has a $\forall\exists$-form: $\forall P\forall\mathcal{D}\exists G\ldots$. $\forall\exists$-properties are often realised through chain constructions: build a chain

$$M=M_0\subseteq M_1\subseteq\ldots\subseteq M_\alpha\subseteq\ldots\subseteq M_\beta\subseteq\ldots$$

of models such that for any $P,\mathcal{D}\in M_\alpha$ there is some $\beta\geqslant\alpha$ such that M_β contains a generic G as required. Then the "union" or limit of the chain should contain appropriate G's for all P's and $\mathcal{D}$'s.

Such chain constructions are wellknown from algebra. To satisfy closure under square roots ($\forall x\exists y$: $yy=x$) one can e.g. start with a countable field M_0 and along a chain $M_0\subseteq M_1\subseteq M_2\subseteq\ldots$ adjoin square roots for all elements of M_n. Then $\bigcup_{n<\omega}M_n$ satisfies the closure property.

In set theory there is a difficulty that unions of models of set theory usually do not satisfy the theory ZF: assume that $M_0\subseteq M_1\subseteq M_2\subseteq\ldots$ is an ascending chain of transitive models of ZF such that $(M_{n+1}\setminus M_n)\cap\mathcal{P}(\omega)\neq\emptyset$ for all $n<\omega$. Let $M_\omega=\bigcup_{n<\omega}M_n$. Then $\mathcal{P}(\omega)\cap M_\omega\notin M_\omega$. Indeed, if one had $\mathcal{P}(\omega)\cap M_\omega\in M_\omega$ then $\mathcal{P}(\omega)\cap M_\omega\in M_n$ for some $n<\omega$ and $\mathcal{P}(\omega)\cap M_{n+1}\in M_n$ contradicts the initial assumption. So a "limit" model of models of ZF has to be more complicated, and it will itself be constructed by some limit forcing which is called *iterated forcing*.

Exercise 7. Check which axioms of set theory hold in $M_\omega=\bigcup_{n<\omega}M_n$ where $(M_n)_{n<\omega}$ is an ascending sequence of transitive models of ZF(C).

Since we want to obtain the limit by forcing over a ground model M the construction must be visible in the ground model. This means that the sequence of forcings to be employed to pass from M_α to $M_{\alpha+1}$ has to exist as a sequence $(\dot{Q}_\beta|\beta<\kappa)$ of names in the ground model. The initial sequence $(\dot{Q}_\beta|\beta<\alpha)$ already determines a forcing P_α and $\dot{Q}_\alpha$ is intended to be a P_α-name. If G_α is M-generic over P_α then furthermore $Q_\alpha=(\dot{Q}_\alpha)^{G_\alpha}$ is intended to be a forcing in the model $M_\alpha=M[G_\alpha]$, and $M_{\alpha+1}$ is a generic extension of M_α by forcing with Q_α. The following iteration theorem says that any sequence $(\dot{Q}_\beta|\beta<\kappa)\in M$ gives rise to an iteration of forcing extensions. In applications the sequence has to be chosen carefully to ensure that some $\forall\exists$-property holds in the final model M_κ. Without loss of generality we only consider forcings Q_α whose maximal element is $\emptyset$.

Theorem 23. *Let M be a ground model, and let $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)\in M$ with the property that $\forall\beta<\kappa\colon\emptyset\in\operatorname{dom}(\dot{Q}_\beta)$. Then there is a uniquely determined sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)\in M$ such that*

a) $(P_\alpha,\leqslant_\alpha,1_\alpha)$ is a partial order which consists of α-sequences;

b) $P_0=\{\emptyset\}$, $\leqslant_0=\{(\emptyset,\emptyset)\}$, $1_0=\emptyset$;

c) If $\lambda\leqslant\kappa$ is a limit ordinal then the forcing P_λ is defined by:

$$\begin{array}{rcl} P_\lambda &=& \{p\colon\lambda\to V\,|\,(\forall\gamma<\lambda\colon p\upharpoonright\gamma\in P_\gamma)\wedge\exists\gamma<\lambda\forall\beta\in[\gamma,\lambda)\ p(\beta)=\emptyset)\}\\ p\leqslant_\lambda q & \mathit{iff} & \forall\gamma<\lambda\colon p\upharpoonright\gamma\leqslant_\gamma q\upharpoonright\gamma\\ 1_\lambda &=& (\emptyset|\gamma<\lambda)\end{array}$$

d) If $\alpha<\kappa$ and $1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:

$$\begin{array}{rcl} P_{\alpha+1} &=& \{p\colon\alpha+1\to V\,|\,p\upharpoonright\alpha\in P_\alpha\wedge p(\alpha)\in\operatorname{dom}(\dot{Q}_\alpha)\wedge p\upharpoonright\alpha\Vdash_{P_\alpha}p(\alpha)\in\dot{Q}_\alpha\}\\ p\leqslant_{\alpha+1}q & \mathit{iff} & p\upharpoonright\alpha\leqslant_\alpha q\upharpoonright\alpha\wedge p\upharpoonright\alpha\Vdash_{P_\alpha}p(\alpha)\dot{\leqslant}_\alpha q(\alpha)\\ 1_{\alpha+1} &=& (\emptyset|\gamma<\alpha+1)\end{array}$$

e) If $\alpha<\kappa$ and not *$1_\alpha\Vdash_{P_\alpha}$ "$(\dot{Q}_\alpha,\dot{\leqslant}_\alpha,\emptyset)$ is a forcing", then the forcing $P_{\alpha+1}$ is defined by:*

$$\begin{array}{rcl} P_{\alpha+1} &=& \{p\colon\alpha+1\to V\,|\,p\upharpoonright\alpha\in P_\alpha\wedge p(\alpha)=\emptyset\}\\ p\leqslant_{\alpha+1}q & \mathit{iff} & p\upharpoonright\alpha\leqslant_\alpha q\upharpoonright\alpha\\ 1_{\alpha+1} &=& (\emptyset|\gamma<\alpha+1)\end{array}$$

$((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$, and in particular P_κ are called the (*finite support*) *iteration* of the sequence $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$.

Proof. To justify the above recursive definition of the sequence $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ it suffices to show recursively that every P_α is a forcing.

Obviously, P_0 is a trivial one-element forcing.

Consider a limit $\lambda\leqslant\kappa$ and assume that P_γ is a forcing for $\gamma<\alpha$. We have to show that the relation $\leqslant_\lambda$ is transitive with maximal element 1_λ. Consider $p\leqslant_\lambda q\leqslant_\lambda r$. Then $\forall\gamma<\lambda\colon p\upharpoonright\gamma\leqslant_\gamma q\upharpoonright\gamma$ and $\forall\gamma<\lambda\colon q\upharpoonright\gamma\leqslant_\gamma r\upharpoonright\gamma$. Since all $\leqslant_\gamma$ with $\gamma<\lambda$ are transitive relations, $\forall\gamma<\lambda\colon p\upharpoonright\gamma\leqslant_\gamma r\upharpoonright\gamma$ and so $p\leqslant_\lambda r$. Now consider $p\in P_\lambda$. Then $\forall\gamma<\lambda\colon p\upharpoonright\gamma\in P_\gamma$. By the inductive assumption, $\forall\gamma<\lambda\colon p\upharpoonright\gamma\leqslant_\gamma 1_\gamma=1_\lambda\upharpoonright\gamma$ and so $p\leqslant_\lambda 1_\lambda$.

For the successor step assume that $\alpha<\kappa$ and that P_α is a forcing.

Case 1. $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p\restriction\alpha \leqslant_\alpha q\restriction\alpha \wedge p\restriction\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$ and $q\restriction\alpha \leqslant_\alpha r\restriction\alpha \wedge q\restriction\alpha \Vdash_{P_\alpha} q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$. By the transitivity of $\leqslant_\alpha$: $p\restriction\alpha\leqslant_\alpha r\restriction\alpha$. Moreover $p\restriction\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha q(\alpha)$, $p\restriction\alpha \Vdash_{P_\alpha} q(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and $p\restriction\alpha \Vdash_{P_\alpha}$ "$\dot{\leqslant}_\alpha$ is transitive". This implies $p\restriction\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha r(\alpha)$ and together that $p\leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\restriction\alpha\in P_\alpha \wedge p\restriction\alpha \Vdash_{P_\alpha} p(\alpha)\in\dot{Q}_\alpha$. Then $p\restriction\alpha \leqslant_\alpha 1_\alpha = 1_{\alpha+1}\restriction\alpha$. Moreover $p\restriction\alpha \Vdash_{P_\alpha}$ "$\emptyset$ is maximal in $\dot{\leqslant}_\alpha$" implies that $p\restriction\alpha \Vdash_{P_\alpha} p(\alpha)\dot{\leqslant}_\alpha\emptyset = 1_{\alpha+1}(\alpha)$. Hence $p\leqslant_{\alpha+1} 1_{\alpha+1}$.

Case 2. It is not the case that $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing.

For the transitivity of $\leqslant_{\alpha+1}$ consider $p \leqslant_{\alpha+1} q \leqslant_{\alpha+1} r$. Then $p\restriction\alpha \leqslant_\alpha q\restriction\alpha$ and $q\restriction\alpha\leqslant_\alpha r\restriction\alpha$. By the transitivity of $\leqslant_\alpha$: $p\restriction\alpha\leqslant_\alpha r\restriction\alpha$ and so $p\leqslant_{\alpha+1} r$.

For the maximality of $1_{\alpha+1}$ consider $p\in P_{\alpha+1}$. Then $p\restriction\alpha\in P_\alpha$. By induction, $p\restriction\alpha\leqslant_\alpha 1_\alpha$ and so $p\leqslant_{\alpha+1} 1_{\alpha+1}$. $\square$

The term "finite support iteration" is justified by the following

Lemma 24. *In the above situation let $p\in P_\kappa$. Then*

$$\operatorname{supp}(p)=\{\alpha<\kappa \mid p(\alpha)\neq\emptyset\}$$

is finite.

Proof. Prove by induction on $\alpha\leqslant\kappa$ that supp(p) is finite for every $q\in P_\alpha$. The crucial property is the definition of P_λ at limit λ in the above iteration theorem. $\square$

Let us fix a ground model M and the iteration $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta<\kappa)\in M$ and $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha\leqslant\kappa)\in M$ as above. Let G_κ be M-generic for P_κ. We analyse the generic extension $M_\kappa = M[G_\kappa]$ by an ascending chain

$$M=M_0\subseteq M_1=M[G_1]=M_0[H_0]\subseteq M_2=M[G_2]=M_1[H_1]\subseteq\ldots\subseteq M_\alpha=M[G_\alpha]\subseteq\ldots\subseteq M_\kappa$$

of generic extensions.

Let us first note some relations within the tower $(P_\alpha)_{\alpha\leqslant\kappa}$ of forcings.

Lemma 25.

a) *Let $\alpha\leqslant\kappa$ and $p,q\in P_\alpha$.*
Then $p\leqslant_\alpha q$ iff $\forall\gamma\in\operatorname{supp}(p)\cup\operatorname{supp}(q)$: $p\restriction\gamma\Vdash_{P_\gamma} p(\gamma)\dot{\leqslant}_\gamma q(\gamma)$.

b) *Let $\alpha\leqslant\beta\leqslant\kappa$ and $p\in P_\beta$. Then $p\restriction\alpha\in P_\alpha$.*

c) *Let $\alpha\leqslant\beta\leqslant\kappa$ and $p\leqslant_\beta q$. Then $p\restriction\alpha\leqslant_\alpha q\restriction\alpha$.*

d) *Let $\alpha\leqslant\beta\leqslant\kappa$, $q\in P_\beta$, $\bar{p}\leqslant_\alpha q\restriction\alpha$. Then $\bar{p}\cup(q(\gamma)|\alpha\leqslant\gamma<\beta)\in P_\beta$ and $\bar{p}\cup(q(\gamma)|\alpha\leqslant\gamma<\beta)\leqslant_\beta q$.*

Proof. *a*) By a straightforward induction on $\alpha\leqslant\kappa$. Now *b*) $-$ *d*) follow immediately. $\square$

For $\alpha \leqslant \kappa$ define $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$.
(1) G_α is M-generic for P_α.
Proof. By (a), $G_\alpha \subseteq P_\alpha$. Consider $p \upharpoonright \alpha$, $q \upharpoonright \alpha \in G_\alpha$ with p, $q \in G_\kappa$. Take $r \in G_\kappa$ such that $r \leqslant_\kappa p, q$. By (a), $r \upharpoonright \alpha \leqslant_\alpha p \upharpoonright \alpha, q \upharpoonright \alpha$. Thus all elements of G_α are compatible in P_α.

Consider $p \upharpoonright \alpha \in G_\alpha$ with $p \in G_\kappa$ and $\bar{q} \in P_\alpha$ with $p \upharpoonright \alpha \leqslant_\alpha \bar{q}$. By (a),

$$q = \bar{q} \cup (\emptyset | \alpha \leqslant \gamma < \kappa)$$

is an element of P_κ and $p \leqslant_\kappa q$. Since G_κ is a filter, $q \in G_\kappa$, and so $\bar{q} = q \upharpoonright \alpha \in G_\alpha$. Thus G_α is upwards closed.

For the genericity consider a set $\bar{D} \in M$ which is dense in P_α. We claim that the set

$$D = \{d \in P_\kappa \,|\, d \upharpoonright \alpha \in \bar{D}\} \in M$$

is dense in P_κ: let $p \in P_\kappa$. Then $p \upharpoonright \alpha \in P_\alpha$. Take $\bar{d} \in \bar{D}$ such that $\bar{d} \leqslant_\alpha p \upharpoonright \alpha$. By (c,d),

$$d = \bar{d} \cup (p(\gamma) | \alpha \leqslant \gamma < \kappa) \in P_\kappa$$

and $d \leqslant_\kappa p$.

By the genericity of G_κ take $p \in D \cap G_\kappa$. Then $p \upharpoonright \alpha \in \bar{D} \cap G_\alpha \neq \emptyset$. *qed*(1)

So $M_\alpha = M[G_\alpha]$ is a welldefined generic extension of M by G_α.
(2) Let $\alpha < \beta \leqslant \kappa$. Then $G_\alpha \in M[G_\beta]$ and $M[G_\alpha] \subseteq M[G_\beta]$.
Proof. $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{(p \upharpoonright \beta) \upharpoonright \alpha \,|\, p \in G_\kappa\} = \{q \upharpoonright \alpha \,|\, q \in G_\beta\} \in M[G_\beta]$. *qed*(2)

For $\alpha < \kappa$ define

$$Q_\alpha = (Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = \begin{cases} (\dot{Q}_\alpha^{G_\alpha}, \dot{\leqslant}_\alpha^{G_\alpha}, \emptyset), \text{ if } 1_\alpha \Vdash_{P_\alpha} \text{“}(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) \text{ is a forcing”} \\ (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset), \text{ else} \end{cases}$$

Then $Q_\alpha \in M_\alpha = M[G_\alpha]$ is a forcing. For $\alpha < \kappa$ define

$$H_\alpha = \{p(\alpha)^{G_\alpha} | p \in G_\kappa\}.$$

(3) H_α is M_α-generic for Q_α.
Proof. If it is not the case that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”, then $(Q_\alpha, \leqslant^{Q_\alpha}, \emptyset) = (\{\emptyset\}, \{(\emptyset, \emptyset)\}, \emptyset)$ and $H_\alpha = \{\emptyset\}$ is trivially M_α-generic. So assume that $1_\alpha \Vdash_{P_\alpha}$ “$(\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset)$ is a forcing”.
(a) $H_\alpha \subseteq Q_\alpha$.
Proof. Let $p \in G_\kappa$. Then $p \upharpoonright \alpha + 1 \in P_{\alpha+1}$ and so $p \upharpoonright \alpha \Vdash_{P_\alpha} p(\alpha) \in \dot{Q}_\alpha$. Since $p \upharpoonright \alpha \in G_\alpha$ we have that $p(\alpha)^{G_\alpha} \in \dot{Q}_\alpha^{G_\alpha} = Q_\alpha$. *qed*(a)
(b) H_α is a filter.
Proof. Let $p(\alpha)^{G_\alpha} \in H_\alpha$ and $p(\alpha)^{G_\alpha} \leqslant^{Q_\alpha} r \in Q_\alpha$.

(e) Let $\bar{D} \in M_\alpha$ be dense in Q_α. Then $\bar{D} \cap H_\alpha \neq \emptyset$.
Proof. Take $\dot{D} \in M$ such that $\bar{D} = \dot{D}^{G_\alpha}$. Take $p \in G_\kappa$ such that

$$p \upharpoonright \alpha \Vdash_{P_\alpha} \dot{D} \text{ is dense in } \dot{Q}_\alpha.$$

Define

$$D = \{d \in P_\kappa \mid d \restriction \alpha \Vdash d(\alpha) \in \dot{D}\} \in M.$$

We show that D is dense in P_κ below p. Let $q \leqslant_\kappa p$. Then $q \restriction \alpha \leqslant_\alpha p \restriction \alpha$ and $q \restriction \alpha \Vdash q(\alpha) \dot{\leqslant}_\alpha p(\alpha)$. Hence $q \restriction \alpha \Vdash_{P_\alpha} \dot{D}$ is dense in $\dot{Q}_\alpha$ and there is $\bar{d} \leqslant_\alpha q \restriction \alpha$ and some $d(\alpha) \in \mathrm{dom}(\dot{Q}_\alpha)$ such that

$$\bar{d} \Vdash_{P_\alpha} (d(\alpha) \dot{\leqslant}_\alpha q(\alpha) \wedge d(\alpha) \in \dot{D}).$$

Define

$$d = \bar{d} \cup \{(\alpha, d(\alpha))\} \cup (q(\gamma) | \alpha < \gamma < \kappa).$$

Then $d \in P_\kappa$, $d \leqslant_\kappa q$, and $d \in D$.

By the genericity of G_κ take $d \in D \cap G_\kappa$. Then $d(\alpha)^{G_\alpha} \in H_\alpha$, $d \restriction \alpha \in G_\alpha$, and $d(\alpha)^{G_\alpha} \in (\dot{D})^{G_\alpha} = \bar{D}$. Thus $H_\alpha \cap \bar{D} \neq \emptyset$.

(4) $M_{\alpha+1} = M_\alpha[H_\alpha]$.

Proof. $\supseteq$ is straightforward. For the other direction, if suffices to show that $G_{\alpha+1} \in M_\alpha[H_\alpha]$, and indeed we show that

$$G_{\alpha+1} = \{q \in P_{\alpha+1} \mid q \restriction \alpha \in G_\alpha \wedge q(\alpha)^{G_\alpha} \in H_\alpha\}.$$

Let $q \in G_{\alpha+1}$. Take $p \in G_\kappa$ such that $p \restriction \alpha + 1 = q$. Then $q \restriction \alpha = p \restriction \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} = p(\alpha)^{G_\alpha} \in H_\alpha$. For the converse consider $q \in P_{\alpha+1}$ such that $q \restriction \alpha \in G_\alpha$ and $q(\alpha)^{G_\alpha} \in H_\alpha$. Take $p_1, p_2 \in G_\kappa$ such that $q \restriction \alpha = p_1 \restriction \alpha$ and $q(\alpha)^{G_\alpha} = p_2(\alpha)^{G_\alpha}$. Take $p \in G_\kappa$ such that $p \leqslant_\kappa p_1, p_2$. We also may assume that $p \restriction \alpha \Vdash q(\alpha) = p_2(\alpha)$. $p \restriction \alpha \leqslant_\alpha p_1 \restriction \alpha = q \restriction \alpha$ and $p \restriction \alpha \Vdash_{P_\alpha} p(\alpha) \dot{\leqslant}_\alpha p_2(\alpha) = q(\alpha)$. Hence $p \restriction \alpha + 1 \leqslant_{\alpha+1} q$. Since $p \restriction \alpha + 1 \in G_{\alpha+1}$ and since $G_{\alpha+1}$ is upward closed, we get $q \in G_{\alpha+1}$.

5.1 Embeddings

In the above construction, $M[G_\alpha] \subseteq M[G_\beta]$ canonically. This corresponds to canonical transformations of names used in the construction of $M[G_\alpha]$ into names used to construct $M[G_\beta]$. Such transformation of names is important for the construction and analysis of interations. We first reduce our "name spaces" from all of M to more specific P-names.

Definition 26. *Let P be a forcing. Define recursively: $\dot{x}$ is a P-name if every element of $\dot{x}$ is an ordered pair $(\dot{y}, p)$ where $\dot{y}$ is a P-name and $p \in P$. Let V^P be the class or name space of all P-names.*

The generic interpretation of an arbitrary name only depends on ordered pairs whose second component is in P. This is observation leads to

Lemma 27. *Let P be a forcing. Define $\tau: V \to V^P$ recursively by*

$$\tau(\dot{x}) = \{(\tau(\dot{y}), p) | (\dot{y}, p) \in \dot{x}\}.$$

Then $\tau(\dot{x})$ is a P-name and

$$1_P \Vdash \dot{x} = \tau(\dot{x}).$$

I.e., $\dot{x}^G = (\tau(\dot{x}))^G$ for every generic filter on P.

Let $\pi: P \rightarrow Q$ be an orderpreserving embedding of forcings. This induces an embedding of name spaces $\pi^*: V^P \rightarrow V^Q$ which is defined recursively:

$$\pi^*(\dot{x}) = \{(\pi^*(\dot{y}), \pi(p)) | (\dot{y}, p) \in \dot{x}\}.$$

One can study such embeddings in general. They satisfy "universal properties", sometimes relying on structural properties of the embedding π.

Exercise 8. Examine, how generic filters are mapped by π and its inverse and how this induces embeddings of generic extensions. Formulate sufficient properties for the original map π.

We restrict our considerations to embeddings connected to iterated forcing. So let $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa)$ be a finite support iteration of the sequence $((\dot{Q}_\alpha, \dot{\leqslant}_\alpha)|\alpha < \kappa)$. In view of the previous lemma we also require in the iteration that every $\dot{Q}_\alpha$ is a P_α-name.

There are canonical maps between the P_α's. For $\alpha \leqslant \beta \leqslant \kappa$ define $\pi_{\alpha\beta}: P_\alpha \rightarrow P_\beta$ by

$$\pi_{\alpha\beta}(p) = p \cup (\emptyset \mid \alpha \leqslant \gamma < \beta).$$

Also define $\pi_{\beta\alpha}: P_\beta \rightarrow P_\alpha$ by $\pi_{\beta\alpha}(q) = q \upharpoonright \alpha$. $\pi_{\beta\alpha}$ is a left inverse of $\pi_{\alpha\beta}$:

$$\pi_{\beta\alpha} \circ \pi_{\alpha\beta} = \mathrm{id}_{P_\alpha}.$$

Let the previous constructions take place within a ground model M. Let G_κ be M-generic for P_κ and let $M_\alpha = M[G_\alpha]$ for $\alpha \leqslant \kappa$ be the associated tower of extensions. Let $\alpha \leqslant \beta \leqslant \kappa$. The inclusion $M[G_\alpha] \subseteq M[G_\beta]$ corresponds to the following

Lemma 28. *Let $\dot{x} \in M^{P_\alpha}$ be a P_α-name and $\ddot{x} = \pi^*_{\alpha\beta}(\dot{x}) \in M^{P_\beta}$ its "lift" to P_β. Then*

$$\dot{x}^{G_\alpha} = \ddot{x}^{G_\beta}.$$

Proof. By induction on $\dot{x}$:

$$\begin{aligned}
\ddot{x}^{G_\beta} &= \{\ddot{y}^{G_\beta} \mid \exists q \in G_\beta \, (\ddot{y}, q) \in \ddot{x}\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists q \, (q \in G_\beta \wedge (\ddot{y}, q) \in \ddot{x})\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists q \, (q \in G_\beta \wedge \exists (\dot{y}, p) \in \dot{x} \, ((\pi^*_{\alpha\beta}(\dot{y}), \pi_{\alpha\beta}(p)) \in \ddot{x} \wedge \ddot{y} = \pi^*_{\alpha\beta}(\dot{y}) \wedge q = \pi_{\alpha\beta}(p)))\} \\
&= \{\ddot{y}^{G_\beta} \mid \exists p \in G_\alpha \, \exists (\dot{y}, p) \in \dot{x} \; \ddot{y} = \pi^*_{\alpha\beta}(\dot{y})\} \\
&= \{\pi^*_{\alpha\beta}(\dot{y})^{G_\beta} \mid \exists p \in G_\alpha \, (\dot{y}, p) \in \dot{x}\} \\
&= \{\dot{y}^{G_\alpha} \mid \exists p \in G_\alpha \, (\dot{y}, p) \in \dot{x}\} \\
&= \dot{x}^{G_\alpha}
\end{aligned}$$

□

In the intended applications of iterated forcing we shall usually be confronted at "time" α with several tasks which have to be dealt with "one by one" along the ordinal axis κ: there will be, e.g., two distinct partial orders $R, S \in M[G_\alpha]$ for which we want to adjoin generic filters. These have P_α-names $\dot{R}, \dot{S} \in M^{P_\alpha}$. In the iteration we may set $\dot{Q}_\alpha = \dot{R}$, but then we have to deal with $\dot{S}$ at some later "time" β. This will be possible by lifting $\dot{S}$ to a P_β-name: set $\dot{Q}_\beta = \pi^*_{\alpha\beta}(\dot{S})$. In the construction some "bookkeeping mechanism" will ensure that eventually all tasks will be looked after.

5.2 Two-step iterations

Definition 29. *Consider a forcing* $(P, \leqslant_P, 0)$ *and names* $\dot{Q}$, $\dot{\leqslant}$ *such that*

$$1_P \Vdash (\dot{Q}, \dot{\leqslant}, 0) \text{ is a forcing.}$$

and $0 \in \mathrm{dom}(\dot{Q})$. *Then the* two-step iteration $(P*\dot{Q}, \preccurlyeq, 1)$ *is defined by:*

$$\begin{array}{rcl} P*\dot{Q} & = & \{(p, \dot{q}) | p \in P \wedge \dot{q} \in \mathrm{dom}(\dot{Q}) \wedge p \Vdash_P \dot{q} \in \dot{Q}\} \\ (p', \dot{q}') \preccurlyeq (p, \dot{q}) & \textit{iff} & p' \leqslant_P p \wedge p' \Vdash_P \dot{q}' \dot{\leqslant} \dot{q}' \\ 1 & = & (0, 0) \end{array}$$

The two-step iteration can be construed as an iteration of a sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < 2)$ of length 2: Let $\dot{Q}_0 = \check{P}$, $\dot{\leqslant}_0 = \check{\leqslant}_P$ where the canonical names $\check{P}$ and $\check{\leqslant}_P$ are formed with respect to the trivial forcing $P_0 = \{\emptyset\}$, $\leqslant_0 = \{(\emptyset, \emptyset)\}$, $1_0 = \emptyset$. Then $(P, \leqslant_P, 0)$ is canonically isomorphic to the induced forcing $(P_1, \leqslant_1, 1_1)$ by the map $h: p \longmapsto \check{p}$. We may assume that $\dot{Q}$ is a P-names in the restricted sense that for every ordered pair $(a, p) \in \mathrm{TC}\big(\dot{Q}\big)$ $p \in P$. Then define a corresponding P_1-name $\dot{Q}_1$ by replacing recursively each $(a, p) \in \mathrm{TC}\big(\dot{Q}\big)$ by $(..., h(p))$. Similarly for $\dot{\leqslant}_1$.

One can check that the iterated forcing $(P_2, \leqslant_2, 1_2)$ defined from $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < 2)$ is canonically isomorphic to $\big(P*\dot{Q}, \preccurlyeq, 1\big)$.

Such identifications using subtle but canonical isomorphisms occur often in the theory of iterated forcing.

Exercise 9. If G is M-generic for $P*\dot{Q} \in M$ where M is a ground model define

$$\begin{array}{rcl} G_0 & = & \big\{p \in P \mid \exists \dot{q} \in \mathrm{dom}\big(\dot{Q}\big): (p, \dot{q}) \in G\big\} \\ G_1 & = & \{\dot{q}^{G_0} \mid \exists p \in P: (p, \dot{q}) \in G\} \end{array}$$

Show that G_0 is M-generic for P and that G_1 is M-generic for $\dot{Q}^{G_0}$.

Conversely let G_0 be M-generic for P and G_1 M-generic for $\dot{Q}^{G_0}$. Show that

$$G = \{(p, \dot{q}) \mid p \in G_0, \dot{q}^{G_0} \in G_1\}$$

is M-generic for $P*\dot{Q}$.

5.3 Products of partial orders

A special case of a finite support iteration is a finite support *product*. So let M be a ground model, and let $((Q_\beta, \leqslant_\beta) | \beta < \kappa) \in M$ be a sequence of forcings such that $\emptyset$ is a maximal element of every Q_β. Define the *finite support product* $\prod_{\beta<\kappa} Q_\beta$ as the following forcing:

$$\begin{array}{rcl} \displaystyle\prod_{\beta<\kappa} Q_\beta & = & \{p: \kappa \to V \,|\, \forall \beta < \kappa: p(\beta) \in Q_\beta, \mathrm{supp}(p) \text{ is finite}\} \\ p \preccurlyeq q & \text{iff} & \forall \beta < \kappa: p(\beta) \leqslant_\beta q(\beta) \\ 1_\kappa & = & (0 | \beta < \kappa) \end{array}$$

We want to show that the product corresponds to a simple iteration. Define a sequence

$$((\check{Q}_\beta, \check{\leqslant}_\beta) | \beta < \kappa) \in M$$

where $\check{Q}_\beta$ is the canonical name for Q_β with respect to a forcing which has the β-sequence $1_\beta = (0|\gamma < \beta)$ as its maximal element. (Note that the definition of $\check{x} = \{(\check{y}, 1_\beta)|y \in x\}$ only depends on 1_β .) Let the sequence $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa) \in M$ be defined from the sequence $((\check{Q}_\beta, \check{\leqslant}_\beta)|\beta < \kappa)$ of names as in the iteration theorem.

Then there is a canonical isomorphism

$$\pi: \prod_{\beta<\kappa} Q_\beta \leftrightarrow P_\kappa$$

defined by: $p \mapsto p'$ where

$$p'(\beta) = \widecheck{p(\beta)}$$

with respect to a partial order with maximal element 1_β. It is tedious but straightforward to check that this defines an isomorphism.

6 Iteration theorems

A main concern of forcing is the preservation of cardinals. There are several criteria for ensuring cardinal preservation or at least the preservation of $\aleph_1$. Iteration theorems take the form: if every $\dot{Q}_\beta$ in $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa)$ is forced to satisfy the preservation criterion then also P_κ satisfies the criterion.

Theorem 30. *Let λ be a regular cardinal. Consider the two-step iteration $(P*\dot{Q}, \preccurlyeq, 1)$ of $(P, \leqslant_P, 0)$ and $\big(\dot{Q}, \dot{\leqslant}, 0\big)$. Assume that $(P, \leqslant_P, 0)$ satisfies the λ-c.c. and $0 \Vdash_P$ "$\big(\dot{Q}, \dot{\leqslant}, 0\big)$ satisfies $\check{\lambda}$-c.c". Then $(P*\dot{Q}, \preccurlyeq, 1)$ satisfies the λ-c.c.*

Proof. We may assume that the assumptions of the theorem are satisfied in some ground model M. It suffices to prove the theorem in M. Work inside M. Let $((p_\alpha, q_\alpha)|\ \alpha < \lambda)$ be a sequence in $(P*\dot{Q}, \preccurlyeq, 1)$. It suffices to find two compatible conditions in this sequence.
(1) There is a condition $p \in P$ such that $p \Vdash \sup\big\{\alpha \mid \check{p}_\alpha \in \dot{G}\big\} = \lambda$ where $\dot{G}$ is the canonical name for a generic filter on P.
Proof. If not, then there is a maximal antichain A in P of conditions q for which there is an ordinal $\gamma_q < \kappa$ such that $q \Vdash \sup\big\{\alpha \mid \check{p}_\alpha \in \dot{G}\big\} = \check{\gamma}_q$. By the κ-c.c., $\mathrm{card}(A) < \lambda$. By the regularity of κ there is $\gamma < \kappa$ such that $\forall q \in A\, \gamma_q < \gamma$. Since A is a maximal antichain,

$$0 = 1_P \Vdash \sup\big\{\alpha \mid \check{p}_\alpha \in \dot{G}\big\} \leqslant \check{\gamma}.$$

But $p_{\gamma+1} \Vdash \check{p}_{\gamma+1} \in \dot{G}$ and $p_{\gamma+1} \Vdash \sup\big\{\alpha \mid \check{p}_\alpha \in \dot{G}\big\} \geqslant \check{\gamma} + 1$. Contradiction. *qed*(1)

Take an M-generic filter G on P such that $p \in G$ and $p \Vdash \sup\ \big\{\alpha \mid \check{p}_\alpha \in \dot{G}\big\} = \lambda$. In $M[G]$ form the sequence $(q_\alpha^G \mid p_\alpha \in G)$; by (1) this sequence has ordertype λ . $\dot{Q}^G$ satisfies the λ-c.c. in $M[G]$ and λ is still a regular cardinal in $M[G]$. So there are $\alpha < \beta < \lambda$ such that q_α^G and q_β^G are compatible in $\dot{Q}^G$. Take $r \in G$ and $q \in \mathrm{dom}\big(\dot{Q}\big)$ such that $r \leqslant p_\alpha, p_\beta$ and $r \Vdash q \leqslant q_\alpha, q_\beta$. Then $(r, q) \in P*\dot{Q}$ and $(r, q) \preccurlyeq (p_\alpha, q_\alpha), (p_\beta, q_\beta)$. $\square$

Theorem 31. *Let* $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ *be the finite support iteration of the sequence* $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$. *Let* λ *be a regular cardinal and suppose that*

$$P_\beta \Vdash \text{“}\dot{Q}_\beta \text{ is } \check{\lambda}\text{-cc”}$$

for all $\beta<\kappa$. *Then every* P_α, $\alpha\leqslant\kappa$ *is* λ*-cc.*

Proof. Again it suffices to prove the theorem in some ground model M. Work inside M. We prove the theorem by induction on $\alpha\leqslant\kappa$. The theorem is trivial for $P_0=\{\emptyset\}$.

Let $\alpha=\beta+1$. One can canonically prove that $P_{\beta+1}\cong P_\beta * \dot{Q}_\beta$. Then P_α is λ-cc by the inductive assumption and the previous theorem.

Finally consider a limit ordinal $\alpha\leqslant\kappa$. Let $A\subseteq P_\alpha$ have cardinality λ. Every condition $p\in A$ has a finite support $\operatorname{supp}(p)$. By the Δ-system lemma, we may suppose that $(\operatorname{supp}(p)\mid p\in A)$ is a Δ-system with some finite kernel d. Take $\beta<\alpha$ such that $d\subseteq\beta$. By the inductive assumption P_β is λ-cc. Take distinct $p,q\in A$ such that $p\restriction\beta, q\restriction\beta$ are compatible in P_β. Take $r\in P_\beta$ such that $r\leqslant_\beta p\restriction\beta, q\restriction\beta$. We then define a compatibility element $s\leqslant_\alpha p,q$ by

$$s(i)=\begin{cases} r(i), \text{ for } i<\beta\\ p(i), \text{ for } \beta\leqslant i<\alpha, i\in\operatorname{supp}(p)\\ q(i), \text{ for } \beta\leqslant i<\alpha, i\notin\operatorname{supp}(p)\end{cases}$$

□

Although the final model $M[G_\kappa]$ is not the union of the models $M[G_\beta]$ it may behave like a union with respect to “small” sets.

Lemma 32. *In a ground model* M *let* $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)$ *be the finite support iteration of the sequence* $((\dot{Q}_\beta,\dot{\leqslant}_\beta)|\beta<\kappa)$ *of limit lenght* κ. *Let* G_κ *be* M*-generic over* P_κ. *Consider a sets* $S\in M$, $X\in M[G_\kappa]$, $X\subseteq S$ *and assume that* $M[G_\kappa]\vDash\operatorname{card}(S)<\operatorname{cof}(\kappa)$. *Then there is* $\alpha<\kappa$ *such that* $X\in M[G_\alpha]$ *where* $G_\alpha=\{p\restriction\alpha\mid p\in G_\kappa\}$.

Proof. Take $\dot{X}\in M$ and $X=\dot{X}^{G_\kappa}$. Without loss of generality we may assume that $1_\kappa\Vdash\dot{X}\subseteq S$. Work in $M[G_\kappa]$. For all $x\in X$ choose a condition $p_x\in G_\kappa$ such that $p_x\Vdash\check{x}\in\dot{X}$. For every $x\in X$ there is some $\alpha_x<\kappa$ such that $\operatorname{supp}(p_x)\subseteq\alpha_x$. Since $\operatorname{card}(S)<\operatorname{cof}(\kappa)$ take an $\alpha<\kappa$ such that $\alpha_x\leqslant\alpha$ for all $x\in X$. We claim that
(1) $X=\left\{x\in S\mid\exists p\in P_\kappa\left(p\restriction\alpha\in G_\alpha\wedge\operatorname{supp}(p)\subseteq\alpha\wedge p\Vdash\check{x}\in\dot{X}\right)\right\}$.
Proof. If $x\in X$ then p_x satisfies the existential condition on the right. Conversely assume that $p\restriction\alpha\in G_\alpha\wedge\operatorname{supp}(p)\subseteq\alpha\wedge p\Vdash\check{x}\in\dot{X}$. Take $q\in G_\kappa$ such that $p\restriction\alpha=q\restriction\alpha$. Then $\operatorname{supp}(p)\subseteq\alpha$ implies that $q\leqslant_\kappa p$. Hence $p\in G_\kappa$ and $x\in X$. *qed*(1)
This proves $X\in M[G_\alpha]$. □

Corollary 33. *In the previous lemma let* P_κ *have the countable chain condition and let* κ *be an uncountable regular cardinal. Then*

$$\mathcal{P}(\delta)\cap M[G_\kappa]=\mathcal{P}(\delta)\cap\bigcup_{\alpha<\kappa}M[G_\alpha]$$

for all $\delta < \kappa$.

7 Forcing Martin's axiom

Martin's axiom postulates the existence of partially generic sets for *all* ccc forcings. Recalling that every Cohen forcing $\mathrm{Fn}(\lambda, 2, \aleph_0)$ is ccc i.e., this amount to a proper class of of forcings to consider. To reduce the class of requirements to a set that can be dealt with in a set-sized iterated forcing, we show that MA_κ "reflects" down to cardinality κ .

Lemma 34. *For infinite cardinals* κ *the following are equivalent:*

a) MA_κ *;*

b) for every ccc forcing Q *whose underlying set is a subset of* κ *and every* $\mathcal{D} \subseteq \mathcal{P}(\kappa)$ *with* $\mathrm{card}(\mathcal{D}) \leqslant \kappa$ *there exists a* $\mathcal{D}$-generic *filter on* Q*.*

Proof. $(a) \rightarrow (b)$ is obvious. For the converse use a Löwenheim-Skolem downward argument. Let $(P, \leqslant, 1)$ be a ccc forcing and let the set $\mathcal{D}$ have cardinality $\leqslant \kappa$. Without loss of generality we may assume that $\mathcal{D} \subseteq \mathcal{P}(P)$ and that every $D \in \mathcal{D}$ is dense in P. Consider the first-order structure

$$(P, \leqslant, 1, (D)_{D \in \mathcal{D}})$$

with a language of cardinality $\leqslant \kappa$. By the Löwenheim-Skolem theorem there is an elementary substructure

$$(Q, \leqslant \cap Q^2, 1, (D \cap Q)_{D \in \mathcal{D}}) \prec (P, \leqslant, 1, (D)_{D \in \mathcal{D}})$$

such that $\mathrm{card}(Q) \leqslant \kappa$. By elementarity $(Q, \leqslant \cap Q^2, 1)$ is a forcing and every $D \cap Q$ is dense in Q. If $A \subseteq Q$ is an antichain in Q then it is an antichain in P. So A is countable and Q is ccc.

We may assume that $Q \subseteq \kappa$. By (b) take a $(D \cap Q)_{D \in \mathcal{D}}$ -generic filter F on Q. We show that

$$G = \{p \in P \mid \exists q \in F q \leqslant p\}$$

is a $\mathcal{D}$-generic filter on P. The filter properties are easy. For the $\mathcal{D}$-genericity consider $D \in \mathcal{D}$. By the $(D \cap Q)_{D \in \mathcal{D}}$ -genericity of F there is $q \in F \cap (D \cap Q)$. Then

$$q \in F \cap (D \cap Q) \subseteq G \cap D \neq \emptyset.$$

$\square$

Notation 35.

1. *If* $(P, \leq_P, 1_P)$ *is a partial order (if* $(B, \leq, \wedge, \vee, 0, 1)$ *is a complete Boolean algebra), we will simply write P(B) instead.*

2. *In an iterated forcing, let* $\pi_{\beta\gamma}: P_\beta \rightarrow P_\gamma$, $\pi_{\beta\gamma}((p_\alpha)_{\alpha<\beta}) = (q_\alpha)_{\alpha<\gamma}$, $q_\alpha = p_\alpha$ *for* $\alpha < \beta$, $q_\alpha = 1$ *for* $\alpha \geq \beta$, *denote the canonical complete embedding. Let* $\pi^*_{\beta\gamma}: V^{P_\beta} \rightarrow V^{P_\gamma}$ *denote the map induced by* $\pi_{\beta\gamma}$.

Theorem 36. *Suppose that M is a ground model. Suppose that $2^{<\kappa} = \kappa > \omega$ in M. There is a ccc forcing $(P, \leq_P, 1_P)$ in M such that for every M-generic filter G on P,* MA *and $2^\omega = \kappa$ hold in $M[G]$.*

Proof. [Proof ideas]

1. There are at most $2^{<\kappa} = \kappa$ many counterexamples to MA.
2. Build $M \subseteq M[G_0] \subseteq M[G_1] \subseteq \ldots M[G_\alpha] \ldots \subseteq M[G]$ for $\alpha < \kappa$ and eliminate 1 counterexample in each step.
3. Ensure that $M[G_\alpha] \vDash 2^{<\kappa} = \kappa$ for all $\alpha < \kappa$.
4. Every forcing of size $<\kappa$ and every set of size $<\kappa$ of maximal antichains of the forcing is in $M[G_\alpha]$ for some $\alpha < \kappa$, since κ is regular. □

Proof. We work in M. Let $h: \kappa \times \kappa \to \kappa$ denote Gödel pairing. Then $h(\alpha, \beta) = \gamma$ implies that $\alpha \leq \gamma$, for all $\alpha, \beta < \kappa$. The β^{th} forcing in $M[G_\alpha]$ will be used in step γ.

We define

1. a finite support iteration $(P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha})_{\alpha \leq \kappa}$ with
   a. P_α ccc and
   b. $|P_\alpha| < \kappa$

   for all $\alpha \leq \kappa$ and
2. P_γ-names $\dot{F}_\gamma$ for all $\gamma < \kappa$ such that $1_{P_\gamma} \Vdash_{P_\gamma}$ " $\dot{F}_\gamma: \kappa \to V$ enumerates all partial orders $(P, \leq_P, 1_P)$ with $P = \lambda$ for some $\lambda < \kappa$".
3. P_γ-names $\dot{Q}_\gamma$ for all $\gamma = h(\alpha, \beta) < \kappa$ such that $1_{P_\gamma} \Vdash_{P_\gamma}$" if $\pi_{\alpha\gamma}(\dot{F}_\alpha)(\beta)$ is c.c.c., then $\dot{Q}_\gamma = \pi_{\alpha\gamma}(\dot{F}_\alpha)(\beta)$, otherwise $|\dot{Q}_\gamma| = 1$"

Suppose that $\gamma < \kappa$ and $\dot{F}_\gamma$, $\dot{Q}_\alpha$ are defined for all $\alpha < \gamma$.

To define $\dot{F}_\gamma$, note that $1_{P_\gamma} \Vdash_{P_\gamma} 2^{<\kappa} = \kappa$, since there are only $(|P_\gamma|^\omega)^\lambda \leq \kappa$ many nice P_γ-names for subsets of cardinals $\lambda < \kappa$ (as in Lemma 80, Models of Set Theory 1).

Choose $\dot{F}_\gamma$ with (2) by the Maximality Principle (Problem 36, Models of Set Theory I).

To define $\dot{Q}_\gamma$, suppose that $\gamma = h(\alpha, \beta)$. Choose a P_γ-name $\dot{Q}_\gamma$ with (3) by the Maximality Principle. Since $1_P \Vdash_{P_\gamma} \dot{Q}_\gamma$ has domain $<\kappa$, we can choose a nice name $\dot{Q}_\gamma$ with $|Q_\gamma| < \kappa$.

Now suppose that G is M-generic for P_κ. Let $G_\alpha := \pi_{\alpha\kappa}^{-1}[G]$ for $\alpha < \kappa$.

Claim. MA_λ for all $\lambda < \kappa$.

Proof. We work in $M[G]$. (It is sufficient to prove MA_λ for c.c.c. partial orders with domain λ for cardinals $\lambda < \kappa$, by a previous lemma.)

Suppose that $(P, \leq_P, 1_P)$ is a c.c.c. partial order with $P = \lambda < \kappa$ and that $\mathcal{D}$ is a set of dense subsets of P with $|\mathcal{D}| \leq \lambda$.

Then $P, \mathcal{D} \in M[G_\alpha]$ for some $\alpha < \kappa$ by a previous lemma. Then $P = \dot{F}_\alpha^{G_\alpha}(\beta)$ for some $\beta < \kappa$ by (2).

Let $\gamma = h(\alpha, \beta)$. Note that P is ccc in $M[G_\gamma]$, since P is ccc in $M[G]$. Then $P = \pi_{\alpha\gamma}(\dot{F}_\alpha)^{G_\gamma}(\beta) = \dot{Q}_\gamma^{G_\gamma}$ by (3).

So there is a $M[G_\gamma]$-generic filter for P in $M[G_\gamma]$. Since $\mathcal{D} \in M[G_\alpha] \subseteq M[G_\gamma]$, the filter is $\mathcal{D}$-generic. □

Claim. MA_λ for all $\lambda < \kappa$.

Proof. We have $2^{<\kappa} = \kappa$ in $M[G]$, since $|P_\kappa| \leq \kappa$ and hence there are $\leq\kappa$ "nice names" for subsets of $\lambda < \kappa$. Moreover MA_λ implies that $2^\omega = 2^\lambda > \lambda$ for all $\lambda < \kappa$, so $2^\omega \geq \kappa$ in $M[G]$. □

□

8 Martin's axiom and generic Σ_1 absoluteness of H_{ω_2}

Definition 37. *Suppose that $\kappa > \omega$ is a cardinal and that Γ is a class of partial orders.*

1. $\mathrm{BFA}_\kappa(\Gamma)$ *postulates that for all $P \in \Gamma$, there is a $\mathcal{D}$-generic filter on P for any set $\mathcal{D}$ of maximal antichains in P of size $\leq\kappa$ with $|\mathcal{D}| \leq \kappa$.*
2. *If P is a partial order, let* $\mathrm{BFA}_\kappa(P) := \mathrm{BFA}_\kappa(\{P\})$.

Remark 38. Suppose that $\kappa > \omega$ is a cradinal. If Γ is a class of forcings such that every element of Γ has the κ^+-c.c., then $\mathrm{BFA}_\kappa(\Gamma) \Longleftrightarrow \mathrm{FA}_\kappa(\Gamma)$.

In particular, $\mathrm{BFA}_{\omega_1}(\mathrm{cc}\,c) \Longleftrightarrow \mathrm{FA}_{\omega_1}(\mathrm{ccc}) \Longleftrightarrow \mathrm{MA}_{\omega_1}$.

We will only consider BFA_κ for complete Boolean algebras.

Remark 39.

1. Every partial order P is densely embedded into its Boolean completion $B(P)$ (see Problem 25, Models of Set Theory 1).
2. Suppose that M is a ground model. We work in M. Suppose that B is a complete Boolean algebra, φ a formula, and σ a B^*-name. Let

$$[\![\varphi(\sigma)]\!] := [\![\varphi(\sigma)]\!]_{B^*} := \bigvee \{p \in B^* |\ p \Vdash_{B^*}^M \varphi(\sigma)\}.$$

Then $[\![\varphi(\sigma)]\!] \Vdash_{B^*}^M \varphi(\sigma)$ by Problem 18(c).

Lemma 40. *Suppose that B is a complete Boolean algebra and $\kappa > \omega$ is a cardinal. Then* $\mathrm{BFA}_\kappa(B^*)$ *implies that $1_B \Vdash_{B^*} \check{\kappa}$ is a cardinal.*

Proof. Suppose that $\mu < \kappa$ and $p \Vdash_B \dot{f}: \check{\kappa} \to \check{\mu}$ is injective. Let

$$A_\alpha = \{[\![\dot{f}(\check{\alpha}) = \check{\beta}]\!] \in B^* |\ \beta < \mu\}.$$

Then each A_β is a maximal antichain below p. Suppose that G is a filter on B with $G \cap A_\beta \neq \emptyset$ for all $\beta < \kappa$. Let $f: \kappa \to \mu$, $f(\alpha) = \beta$ if $[\![\dot{f}(\check{\alpha}) = \check{\beta}]\!] \in G$. Then f is injective, contradiction. $\square$

We will now use $\mathrm{BFA}_\kappa(B(P)^*)$ to reconstruct the first order theory of a structure with domain κ.

Suppose that M is a ground model. We work in M. Suppose that P is a partial order, $\kappa > \omega$ is a cardinal, $(\dot{R}_\alpha)_{\alpha<\kappa}$ is a sequence of P-names for relations on κ, and $\dot{M}$ is a P-name for the structure $(\kappa, \dot{R}_\alpha)_{\alpha<\kappa}$.

Definition 41. *Suppose that in M, G^* is a filter on P. Let*

1. $\dot{R}_\alpha[G^*] = \{s \in \kappa^{<\omega} | \exists p \in G^* \, p \Vdash_P (\dot{M} \vDash \dot{R}_\alpha(\check{s}))\}$.
2. $\dot{M}[G^*] = (\kappa, \dot{R}_\alpha[G^*])_{\alpha<\kappa}$.

Lemma 42. *We work in M. There is a set $\mathcal{D}^*$ of maximal antichains in $B(P)$ of size $\leq\kappa$ with $|\mathcal{D}^*| \leq \kappa$ such that for every $\mathcal{D}^*$-generic filter G^* on $B(P)$, every formula $\varphi(x_0, ..., x_n)$, and $\alpha_0, ..., \alpha_n < \kappa$*

$$\dot{M}[G^*] \vDash \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n) \Longleftrightarrow \exists p \in G^* \, p \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)).$$

Proof. For $\alpha_0, ..., \alpha_n < \kappa$ and $\ulcorner \varphi \urcorner(x_0, ..., x_n)$ let

$$A_{\ulcorner \varphi \urcorner, \alpha_0, ..., \alpha_n} = \{[\![\dot{M} \vDash \ulcorner \neg\varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)]\!], [\![\dot{M} \vDash \ulcorner \varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n]\!]\}.$$

For $\ulcorner \psi(x, x_0, ..., x_n) \urcorner$ and $\alpha_0, ..., \alpha_n < \kappa$ let

$$A_{\exists, \ulcorner \psi \urcorner, \alpha_0, ..., \alpha_n} = \{[\![\dot{M} \vDash \ulcorner \neg\exists x \, \psi \urcorner(x, \check{\alpha}_0, ..., \check{\alpha}_n)]\!]\} \cup \{[\![\dot{M} \vDash \ulcorner \psi \urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n]\!] | \alpha < \kappa\}.$$

Let $\mathcal{D}^* = \{A_{\ulcorner \varphi \urcorner, \alpha_0, ..., \alpha_n}, A_{\exists, \ulcorner \psi \urcorner, \alpha_0, ..., \alpha_n} | \ulcorner \varphi \urcorner \text{a formula}, \alpha_0, ..., \alpha_n < \kappa\}$.

We prove the claim by induction on (codes for) formulas $\ulcorner \varphi \urcorner$.

For atomic formulas, this holds by the definition of $\dot{M}[G^*]$.

For conjunctions, if $\dot{M}[G^*] \vDash \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner \psi \urcorner(\beta_0, ..., \beta_k)$, then $\exists p, q \in G^*$ $p \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n)$, $q \Vdash_P (\dot{M} \vDash \ulcorner \psi \urcorner(\beta_0, ..., \beta_k)$. Let $r \leq p, q$ in G^*. Then $r \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner \psi \urcorner(\beta_0, ..., \beta_k))$.

If $p \in G^*$ and $p \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n) \wedge \ulcorner \psi \urcorner(\check{\beta}_0, ..., \check{\beta}_k))$, then $M[G^*] \vDash \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner \psi \urcorner(\beta_0, ..., \beta_k)$.

For negations, we have $\dot{M}[G^*] \vDash \neg \ulcorner \varphi \urcorner(\alpha_0, ..., \alpha_n) \Longleftrightarrow \neg\exists p \in G^* \, p \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)) \Longleftrightarrow \exists p \in G^* \, p \Vdash_P (\dot{M} \vDash \neg \ulcorner \varphi \urcorner(\check{\alpha}_0, ..., \check{\alpha}_n))$, since $G^* \cap A_{\ulcorner \varphi \urcorner, \alpha_0, ..., \alpha_n} \neq \emptyset$.

For existential quantifiers, if $\dot{M}[G^*] \vDash \exists x \ulcorner \varphi \urcorner(x, \alpha_0, ..., \alpha_n)$, then there is some $\alpha < \kappa$ with $\dot{M}[G^*] \vDash \ulcorner \varphi \urcorner(\alpha, \alpha_0, ..., \alpha_n)$. So there is some $p \in G^*$ with $p \Vdash_P (\dot{M} \vDash \ulcorner \varphi \urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n))$ and hence $p \Vdash_P (\dot{M} \vDash \exists x \ulcorner \varphi \urcorner(x, \check{\alpha}_0, ..., \check{\alpha}_n))$.

If $p \Vdash_P (\dot{M} \vDash \exists x \ulcorner \psi \urcorner(x, \vec{\sigma}))$ for some $p \in G^*$, then there is some $\alpha < \kappa$ with $p \Vdash_P (\dot{M} \vDash \ulcorner \psi \urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n))$, since $G^* \cap A_{\exists, \ulcorner \psi \urcorner, \alpha_0, ..., \alpha_n} \neq \emptyset$. Then $\dot{M}[G^*] \vDash \ulcorner \varphi \urcorner(\alpha, \alpha_0, ..., \alpha_n)$ by the inductive hypothesis, so $\dot{M}[G^*] \vDash \ulcorner \exists x \, \varphi \urcorner(x, \alpha_0, ..., \alpha_n)$. $\square$

Lemma 43. *Suppose that in M, $\mathrm{BFA}_\kappa(B(P)^*)$ holds and that $1_P \Vdash_P (\check{\kappa}, \dot{R}_0)$ is wellfounded. Then there is a set $\mathcal{D}^*$ of maximal antichains in P of size $\leq\kappa$ with $|\mathcal{D}^*| \leq \kappa$ such that for every $\mathcal{D}^*$-generic filter G^* on P, $(\kappa, \dot{R}_0[G^*])$ is wellfounded.*

Proof. We work in M. For each $\alpha < \kappa$, let $\dot{r}_\alpha$ denote a name for the rank function on $(\alpha, \dot{R}_0 \cap (\alpha \times \alpha))$, i.e.

$$1_p \Vdash_P \dot{r}_\gamma: \check{\gamma} \to \mathrm{Ord}, \forall \beta < \gamma\, \dot{r}_\gamma(\beta) = \sup\{\dot{r}_\gamma(\alpha) + 1 |\, (\alpha, \beta) \in \dot{R}_0\}.$$

Since $\mathrm{BFA}_\kappa(B(P)^*)$ implies that $1_P \Vdash_P \check{\kappa} \in \mathrm{Card}$, we have $1_p \Vdash_P \dot{r}_\alpha: \check{\alpha} \to \check{\kappa}$. Let

$$A_{\alpha,\beta} = \{[\![\dot{r}_\alpha(\check{\beta}) = \check{\gamma}]\!] |\, \gamma < \kappa\}$$

for $\alpha, \beta < \kappa$. Let $\mathcal{D}^* = \{A_{\alpha,\beta} |\, \alpha, \beta < \kappa\}$.

Suppose that G^* is a $\mathcal{D}^*$-generic filter on P. Then

$$\dot{r}_\alpha[G^*] = \{(\beta, \gamma) |\, \beta < \alpha, [\![\dot{\rho}_\alpha(\check{\beta}) = \check{\gamma}]\!] \in G^*\}.$$

Since $G^* \cap A_{\alpha,\beta} \neq \emptyset$ for all $\beta < \kappa$, $\dot{r}_\alpha[G^*]: \alpha \to \kappa$ is a well-defined function. Then $\dot{r}_\alpha[G^*]$ is order preserving from $(\alpha, \dot{R}_0[G^*] \cap (\alpha \times \alpha))$ to $(\kappa, <)$ for each $\alpha < \kappa$, by the last equation.

Since $cof(\kappa) > \omega$, this implies that $(\kappa, \dot{R}_0[G^*])$ is wellfounded. □

Definition 44. *1. A formula φ is*

a. $\Delta_0 = \Sigma_0 = \Pi_0$ if all its quantifiers are bounded.

b. Π_n if it is logically equivalent to a formula of the form $\neg\psi$, where ψ is a Σ_n formula.

c. Σ_{n+1} if it is logically equivalent to a formula of the form

$$\exists x_0, \ldots, x_m\, \psi(x_0, \ldots, x_m, y_0, \ldots, y_l),$$

where ψ is a Π_n formula.

2. Suppose that $(M, R_\alpha, f_\alpha)_{\alpha<\kappa}$ and $(N, S_\alpha, g_\alpha)_{\alpha<\kappa}$ are structures with $M \subseteq N$ and Ψ is a set of (coded) formulas.

a. Let $(M, R_\alpha, f_\alpha)_{\alpha<\kappa} \prec_\Psi (N, S_\alpha, g_\alpha)_{\alpha<\kappa}$ if for every (coded) formula $\ulcorner \varphi(x_0, \ldots, x_m) \urcorner \in \Psi$ and all $y_0, \ldots, y_m \in M$,

$$(M, R_\alpha, f_\alpha)_{\alpha<\kappa} \models \ulcorner \varphi(y_0, \ldots, y_m) \urcorner \Longleftrightarrow (N, S_\alpha, g_\alpha)_{\alpha<\kappa} \models \ulcorner \varphi(y_0, \ldots, y_m) \urcorner.$$

b. Let $M \prec N$ if $M \prec_{\Sigma_n} N$ for all $n < \omega$.

Problem 1. Suppose that κ is an infinite cardinal. Then $H_{\kappa^+} \prec_{\Sigma_1} V$.

Proof. Suppose that $V \models \varphi(x, \vec{y})$, where $\vec{y} \in H_{\kappa^+}$ and φ is a Δ_0 formula. Suppose that $x \in H_{\theta^+}$, $\theta \geq \kappa$.

Let $N \prec H_{\theta^+}$ with $x, tc(\vec{y}) \in N$, $|N| \leq \kappa$. Let $\pi: N \to \bar{N}$ denote the transitive collapse of N. Then $\pi(\vec{y}) = \vec{y}$ and $\bar{N} \subseteq H_{\kappa^+}$. Let $\bar{x} = \pi(x)$.

Then $\bar{N} \vDash \varphi(\bar{x}, \vec{y})$, so $H)_{\kappa^+} \vDash \varphi(\bar{x}, \vec{y})$ by Δ_0-absoluteness between transitive sets. $\Box$

Lemma 45. *Suppose that $\kappa \geq \omega$ is a cardinal. There is a $\Sigma_1^{H_{\kappa^+}}$ definable surjection h: $\mathcal{P}(\kappa) \to H_{\kappa^+}$.*

Proof. Let g: $\kappa \times \kappa \to \kappa$ denote Gödel pairing. Let $f(x)$ denote $\pi(0)$, where π: $\kappa \to V$ is the transitive collapse of $(\kappa, g^{-1}[x])$, if this is wellfounded, and let $f(x) = 0$ otherwise. Then h: $\mathcal{P}(\kappa) \to H_{\kappa^+}$ is a $\Sigma_1^{H_{\kappa^+}}$ definable surjection. $\Box$

Theorem 46. *[Bagaria] Suppose that M is a ground model. Suppose that P is a partial order and that $\kappa > \omega$ is a cardinal in M. The following conditions are equivalent.*

1. $\mathrm{BFA}_\kappa(B(P)^*)$ *holds in M, and*

2. $H_{\kappa^+} \prec_{\Sigma_1} H_{\kappa^+}^{M[G]}$ *for all M-generic filters G on P.*

Proof. Suppose that $\mathrm{BFA}_\kappa(B(P)^*)$ holds in M. Suppose that

$$1_P \Vdash_P^M (H_{\kappa^+} \vDash \ulcorner \exists x\, \varphi \urcorner(x, y_0, ..., y_n)),$$

where φ is a Δ_0 formula and $y_0, ..., y_n \in H_{\kappa^+}^M$.

Suppose that h: $\mathcal{P}(\kappa)^M \to H_{\kappa^+}^M$ is a $\Sigma_1^{H_{\kappa^+}^M}$ definable surjection in M. Suppose that $x_i \in \mathcal{P}(\kappa)^M$ and $h(x_i) = y_i$ for all $i \leq n$. Then

$$1_P \Vdash_P^M (H_{\kappa^+} \vDash \ulcorner \exists x\, \varphi \urcorner(x, \overbrace{h(x_0)}, ..., \overbrace{h(x_n)}))).$$

Let $\dot{N}$ denote a name for the transitive closure of $\{x_0, ..., x_n\}$ and a witness for the statement $\ulcorner \exists x\, \varphi \urcorner(x, \overbrace{h(x_0)}, ..., \overbrace{h(x_n)}))$ in $H_{\kappa^+}^{M[\dot{G}]}$, where $\dot{G}$ is a name for the M-generic filter on P. Suppose that $\dot{\pi}$ is a P-name for an isomorphism $\dot{\pi}$: $(\dot{N}, \in) \to (\check{\kappa}, \dot{E})$ such that $1_P \Vdash_P \dot{\pi}(\check{\alpha}) = 2 \cdot \check{\alpha}$ for all $\alpha < \kappa$. Let $\bar{x} := \{2 \cdot \alpha |\, \alpha \in x\}$ for $x \subseteq \kappa$.

Then $1_P \Vdash_P^M (\check{\kappa}, \dot{E})$ is wellfounded and

$$1_{B(P)^*} \Vdash_{B(P)^*}^M (\dot{N} \vDash \ulcorner \exists x\, \varphi \urcorner(x, \overbrace{h(\bar{x}_0)}, ..., \overbrace{h(\bar{x}_n)})).$$

We choose a set $\mathcal{D}^*$ of maximal antichains in $B(P)^*$ of size $\leq\kappa$ with $|\mathcal{D}^*| \leq \kappa$ by the previous lemmas. There is a $\mathcal{D}^*$-generic filter G^* in M, since $B\ F\ A_\kappa(B(P)^*)$ holds in M. Then

1. $\bar{\kappa}$ is an initial segment of $\mathrm{Ord}^{(\kappa, \dot{E}[G^*])}$,

2. $(\kappa, \dot{E}[G^*]) \vDash \ulcorner \exists x\, \varphi \urcorner(x, h(\bar{x}_0), ..., h(\bar{x}_n))$ by Lemma 42, and

3. the structure $(\kappa, \dot{E}[G^*])$ is wellfounded, by Lemma 43.

Then $\ulcorner \exists x\ \varphi\ \urcorner(x, x_0, ..., x_n)$ holds in the transitive collapse $N \in M$ of $(\kappa, \dot{E}[G^*])$. Since $N \prec_{\Sigma_1} H_{\kappa^+}^M$, the proof is complete.

For the other direction, suppose that in M, $\mathcal{D}$ is a set of maximal antichains in $B(P)^*$ of size $\leq\kappa$ with $|\mathcal{D}| \leq \kappa$. Suppose that Q is an elementary substructure of the Boolean algebra $B(P)$ with $\bigcup\ \mathcal{D} \subseteq Q$ and $|Q| \leq \kappa$. Suppose that π: $\bar{Q} \to Q$ is elementary and $\bar{Q}$, $\pi^{-1}(\mathcal{D}) \in H_{\kappa^+}^M$.

Suppose that G is M-generic for $B(P)^*$. Since Q is a Boolean subalgebra of $B(P)^*$, it is easy to check that $H := G \cap Q$ is a $\mathcal{D}$-generic filter on Q. Then $\bar{H} := \pi^{-1}[H]$ is a $\pi^{-1}[\mathcal{D}]$-generic filter on $\bar{Q}$. Since the existence of such a filter is a Σ_1 statement over H_{κ^+}, there is such a filter $\bar{I} \in M$. Then the upwards closure $I = \{q \in B(P)^* | \exists p \in I\, \pi(p) \leq q\}$ of $\pi[I]$ is a $\mathcal{D}$-generic filter in M. □

9 Ideals and cardinal coefficients

Ideals capture (some aspects of) the notion of *small sets*.

Definition 47. *A set $\mathcal{I} \subseteq \mathcal{P}(R)$ is an* ideal *on R if*

a) if $A, B \in \mathcal{I}$ then $A \cup B \in \mathcal{I}$

b) if $A \in \mathcal{I}$ and $B \subseteq A$ then $B \in \mathcal{I}$

c) if $r \in R$ then $\{r\} \in \mathcal{I}$

d) $R \notin \mathcal{I}$

*An ideal is κ-*complete *if for any family $\mathcal{A} \subseteq \mathcal{I}$, $\operatorname{card}(\mathcal{A}) < \kappa$ holds $\bigcup \mathcal{A} \in \mathcal{I}$. An ideal is σ-*complete *if it is $\aleph_1$-complete.*

We have already considered the following ideals on $\mathbb{R}$:

Definition 48. *$\mathcal{N} = \{X \subseteq \mathbb{R} | X$ has measure zero$\}$ is the ideal of* <u>n</u>ullsets*, the* null ideal*, and $\mathcal{M} = \{X \subseteq \mathbb{R} | X$ is meager$\}$ is the* meager ideal*.*

Both these ideals are σ-complete, see Theorem 12 and Theorem 22. They may have "more" completeness in certain models of set theory. We saw in the mentioned Theorem 12 that under $\mathrm{MA}_{\aleph_1}$ the ideals are $\aleph_2$-complete. On the other hand the continuum hypothesis CH implies that $\mathcal{M}$ is *not* $\aleph_2$-complete. So the value of the completeness of $\mathcal{M}$ is independent of the axioms of ZFC. To study such phenomena one introduces *cardinal characteristics* that capture properties of ideal and that may vary between different models of set theory. Sometimes these coefficients are misleadingsly called cardinal *invariants*.

Definition 49. *Let $\mathcal{I}$ be an ideal on R. Define the following cardinal characteristics:*

- $\operatorname{add}(\mathcal{I}) = \min\{\operatorname{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} \notin \mathcal{I}\}$ *is the* additivity (number) *of $\mathcal{I}$;*
- $\operatorname{cov}(\mathcal{I}) = \min\{\operatorname{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \bigcup \mathcal{A} = R\}$ *is the* covering (number) *of $\mathcal{I}$;*
- $\operatorname{non}(\mathcal{I}) = \min\{\operatorname{card}(X) | X \subseteq R, X \notin \mathcal{I}\}$*;*
- $\operatorname{cof}(\mathcal{I}) = \min\{\operatorname{card}(\mathcal{A}) | \mathcal{A} \subseteq \mathcal{I}, \forall B \in \mathcal{I} \exists A \in \mathcal{A} \colon B \subseteq A\}$ *is the* cofinality *of $\mathcal{I}$, a family $\mathcal{A} \subseteq \mathcal{I}$ such that $\forall B \in \mathcal{I} \exists A \in \mathcal{A} \colon B \subseteq A$ is called* cofinal *in $\mathcal{I}$.*

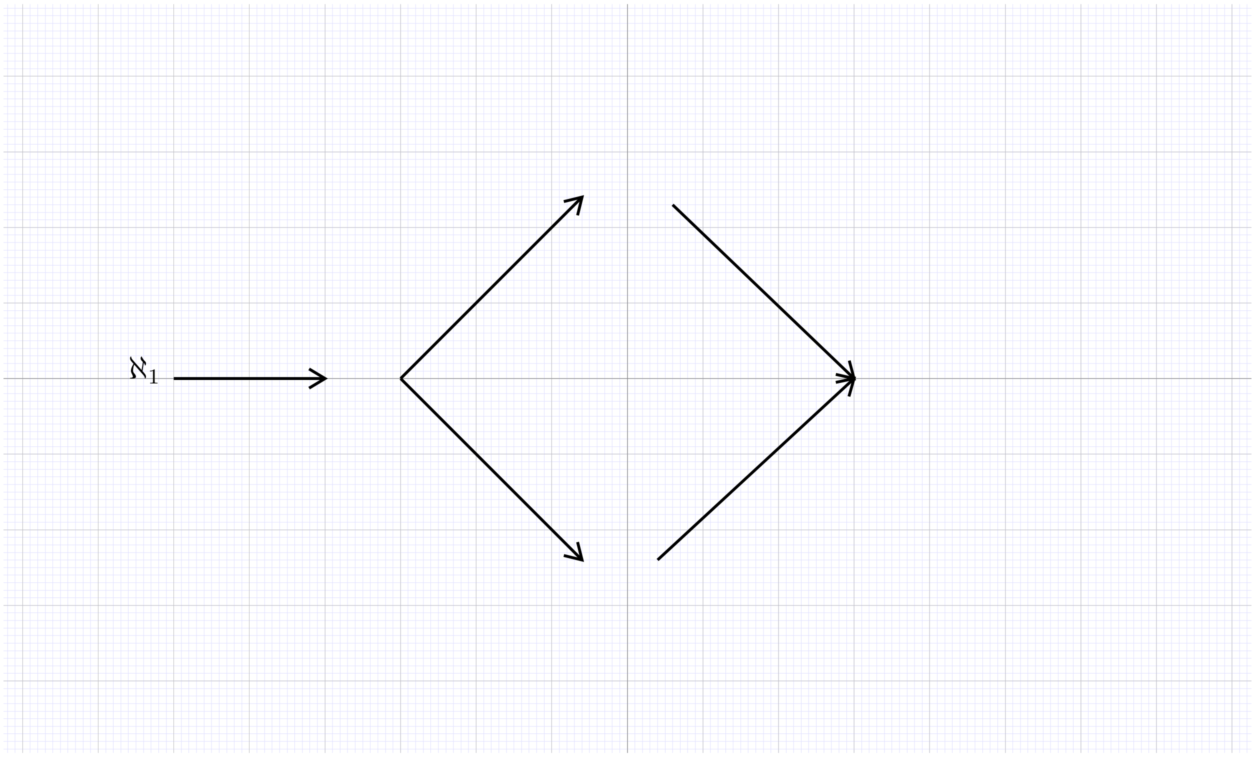
Proposition 50. *Let $\mathcal{I}$ be a σ-complete ideal on $\mathbb{R}$. Then*

$$\aleph_1 \leqslant \operatorname{add}(\mathcal{I}) \leqslant \operatorname{cov}(\mathcal{I}) \leqslant \operatorname{cof}(\mathcal{I}) \leqslant 2^{\aleph_0}$$

and

$$\mathrm{add}(\mathcal{I}) \leqslant \mathrm{non}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$$

This can be pictured by the following diagram:



Proof. The inequalities

$$\aleph_1 \leqslant \mathrm{add}(\mathcal{I}) \leqslant \mathrm{cov}(\mathcal{I}) \text{ and } \mathrm{add}(\mathcal{I}) \leqslant \mathrm{non}(\mathcal{I})$$

are trivial. To show that $\mathrm{cov}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$ consider a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{A})$. Then $\bigcup \mathcal{A} = R$ and so $\mathrm{cov}(\mathcal{I}) \leqslant \mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{I})$.

To show $\mathrm{non}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{I})$ consider again a cofinal family $\mathcal{A} \subseteq \mathcal{I}$ with $\mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{A})$. For each $B \in \mathcal{A}$ choose $x_B \in \mathbb{R} \setminus B \neq \emptyset$. Then $X = \{x_B | B \in \mathcal{A}\}$ has cardinality $\leqslant \mathrm{card}(\mathcal{A}) = \mathrm{cof}(\mathcal{I})$. Assume for a contradiction that $X \in \mathcal{I}$. By cofinality take $B \in \mathcal{A}$ such that $X \subseteq B$. Then $x_B \in X \subseteq B$, contradiction. So $X \notin \mathcal{I}$ and

$$\mathrm{non}(\mathcal{I}) \leqslant \mathrm{card}(X) \leqslant \mathrm{cof}(\mathcal{I}).$$

$\square$

If the continuum hypothesis holds, then all these characteristics are equal to $\aleph_1 = 2^{\aleph_0}$. So it is interesting to study such characteristics in models of ZFC in which $\aleph_1 \neq 2^{\aleph_0}$. The obvious examples to study are models of MA + $\aleph_1 \neq 2^{\aleph_0}$ and the COHEN model for $\aleph_1 \neq 2^{\aleph_0}$.

Theorem 51. *Assume* MA. *Then*

$$\mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) = \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0}$$

and

$$\mathrm{add}(\mathcal{M}) = \mathrm{cov}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0}$$

Proof. Because MA implies $\mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$ (Theorem 12) and $\mathrm{add}(\mathcal{N}) = 2^{\aleph_0}$ (Theorem 22). $\square$

Theorem 52. *Let M be a ground model of* ZFC + CH*, and let $M \vDash \kappa$ is a regular cardinal$>\aleph_1$. In M, let $(P, \leqslant, 1_P) = \mathrm{Fn}(\omega \times \kappa, 2, \aleph_0)$ be the forcing for adding κ* COHEN *reals and let $M[G]$ be a generic extension of M by P. Then in $M[G]$*

$$\aleph_1 = \mathrm{add}(\mathcal{N}) = \mathrm{cov}(\mathcal{N}) < \mathrm{non}(\mathcal{N}) = \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0}.$$

Proof. In $M[G]$, $\mathrm{cov}(\mathcal{N}) = \aleph_1$ since by Problem Sheet 1, 3(a) there is an $\aleph_1$-sequence of measure zero sets whose union is $\mathbb{R}$. $\mathrm{non}(\mathcal{N}) = 2^{\aleph_0}$, since by the argument of Theorem 4 every set of reals of cardinality $<2^{\aleph_0}$ is a measure zero set.

$\square$

Before proving an analogous result for the meager ideal $\mathcal{M}$ we make some preparations concerning "codes" of open sets in $\mathbb{R}$. In a transitive ZFC-model N consider an open set $A \subseteq \mathbb{R}$. A can be represented as

$$A = \bigcup c$$

where $c \in N$ is a set of rational open intervals. To make being a code a definite notion, only the rational endpoints of a rational interval are recorded in a code.

Definition 53. *An* open code *or a* G-code *is a set $c \subseteq [\mathbb{Q}]^2 = \{\{r, s\} \mid r, s \in \mathbb{Q}, r < q\}$. If M is a transitive model of set theory and $c \in M$ then*

$$c^M = \bigcup_{\{r,s\} \in c} (r, s)^M$$

is the interpretation *of the code c in M, where $(r, s)^M = \{t \in \mathbb{R} \cap M \mid r < t < s\}$ is the open interval between r and s as defined in M.*

If $N \supseteq M$ is another transitive model of set theory then $c^M \subseteq c^N$. Indeed if $\mathbb{R} \cap M \neq \mathbb{R} \cap N$ and $c \neq \emptyset$ then $c^M \neq c^N$. Nevertheless one may view c^M and c^N as the "same" open set interpreted in different models. Accordingly, many properties of c^M in M transfer to c^N in N. E.g.,

Lemma 54. *Let $c \in M \subseteq N$ be a G-code. Then c^M is dense open in M if c^N is dense open in N.*

Proof. Let c^M be dense open in M. Consider $r, s \in \mathbb{Q}$, $r < s$. By density take $x \in c^M \cap (r, s)^M$. Then $x \in c^N \cap (r, s)^N$.

Conversely Let c^N be dense open in N. Consider $r, s \in \mathbb{Q}$, $r < s$. By density, $c^N \cap (r, s)^N \neq \emptyset$. Take a rational pair $\{r_0, s_0\} \in c$ such that $(r_0, s_0)^N \cap (r, s)^N \neq \emptyset$. Take $q \in (r_0, s_0)^N \cap (r, s)^N \cap \mathbb{Q}$. Then $q \in c^M \cap (r, s)^M$. $\square$

Note that a set $X \subseteq \mathbb{R}$ is nowhere dense iff the complement of X contains a dense open set. A set $A \subseteq \mathbb{R}$ is meager iff the complement of A contains a countable intersection of dense open sets. Let us "code" countable intersections of open sets as follows.

Definition 55. *A* G_δ-code *is a countable set* d *of* G*-codes. The* interpretation *of* d *is the set in a model* M *is*

$$d^M = \bigcap_{c \in d} c^M.$$

To explain the notations G and G_δ note that in HAUSDORFF's times, open sets were called "Gebiet" with a "G" and countable intersections ("Durchschnitt") were denoted by subscripts δ. We show that COHEN reals "avoid" meager sets from the ground model.

Lemma 56. *Let* M *be a ground model and let* $M[z] = M[H]$ *be a generic extension of* M *by the standard* COHEN *forcing* $P = \mathrm{Fn}(\omega, 2, \aleph_0)$*: let* H *be* M*-generic for* P *and let* $z = \bigcup H \in {}^\omega 2$ *be the associated* COHEN *real. Consider a set* $X \in M$ *which is meager in the ground model and let* $d \in M$ *be a* G_δ*-code for a countable intersection of dense open sets such that* $X \cap d^M = \emptyset$*. Then* $z \in d^{M[z]}$*.*

Proof. Let us identify $\mathbb{R}$ with ${}^\omega 2$, linearly ordered lexicographically, and let us identify $\mathbb{Q}$ with the elements of $\mathbb{R}$ which are eventually 0. Consider $c \in d$. Define, in M,

$$D = \{p \in P \,|\, \exists (r, s) \in c \forall y \in \mathbb{R}\, (y \supseteq p \rightarrow y \in (r, s)\}.$$

(1) D is dense in P.
Proof. Let $q \in P$. Since c^M is dense, there exists a real $y_0 \supseteq q$ such that $y_0 \in c^M$. Take $(r, s) \in c$ such that $y_0 \in (r, s)$. Take $p \in P$, $p \supseteq q$ such that $\forall y \in \mathbb{R}\, (y \supseteq p \rightarrow y \in (r, s))$. Then $p \in D$ and D is dense. *qed*(1)

By genericity take $p \in D \cap H$. Then $z \supseteq p$ and by the definition of D there is $(r, s) \in c$ so that

$$z \in (r, s) \subseteq c^{M[z]}.$$

Since this holds for every $c \in d$:

$$z \in \bigcap_{c \in d} c^{M[z]} = d^{M[z]}.$$

$\square$

We can now continue to prove $\aleph_1 = \mathrm{add}(\mathcal{M}) = \mathrm{non}(\mathcal{M}) < \mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0}$ in the cohen extension $M[G]$.

Lemma 57. $M[G] \vDash \mathrm{non}(\mathcal{M}) = \aleph_1$.

Proof. In $M[G]$ define the sequence $(z_i | i < \kappa)$ of COHEN reals $z_i: \omega \rightarrow 2$ by

$$z_i(n) = (\bigcup G)(n, i).$$

We claim that $A = \{z_i | i < \omega_1\} \notin \mathcal{M}^{M[G]}$. Assume not and let $d \in M[G]$ be a G_δ-code for a countable intersection of dense open sets so that

$$A \cap d^{M[G]} = \emptyset.$$

By previous lemmas take a countable $X \subseteq \kappa, X \in M$ such that $d \in M[G \upharpoonright X]$. Take $i \in \omega_1 \setminus X$. Then $d \in M[G \upharpoonright (\kappa \setminus \{i\})]$. We have

$$M[G] = M[G \upharpoonright (\kappa \setminus \{i\})][G \upharpoonright \{i\}] = M[G \upharpoonright (\kappa \setminus \{i\})][z_i]$$

where z_i is a COHEN real with respect to the model $M[G \upharpoonright (\kappa \setminus \{i\}]$. By the previous Lemma

$$z_i \in d^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d^{M[G]}$$

contradicting that $A \cap d^{M[G]} = \emptyset$. □

Lemma 58. $M[G] \vDash \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.

Proof. Assume for a contradiction that $(A_\xi | \xi < \lambda)$, $\lambda < \kappa$ is a sequence of meager sets such that $\mathbb{R} = \bigcup_{\xi < \lambda} A_\xi$. For each $\xi < \lambda$ choose a G_δ-code d_ξ such that $A_\xi \cap d_\xi^{M[G]} = \emptyset$. By Lemma 53 take $X \subseteq \kappa$, $\mathrm{card}(X) = \mathrm{card}(\lambda) + \aleph_0$ such that

$$\forall \xi < \lambda \colon d_\xi \in M[G \upharpoonright X].$$

Take $i \in \kappa \setminus X$. Then

$$\forall \xi < \lambda \colon d_\xi \in M[G \upharpoonright (\kappa \setminus \{i\})].$$

As above

$$z_i \in d_\xi^{M[G \upharpoonright (\kappa \setminus \{i\})][z_i]} = d_\xi^{M[G]}$$

for all $\xi < \lambda$. Hence

$$z_i \notin \bigcup_{\xi < \lambda} A_\xi = \mathbb{R},$$

contradiction. □

10 The CICHON diagram

We want to relate cardinal characteristics of the ideals $\mathcal{N}$ and $\mathcal{M}$ in a joint diagram called the CICHON diagram. We first have to define two more characteristics.

Definition 59.

a) *Define the partial ordering* $\leqslant^*$ *of* eventual domination *on* ω^ω *by*

$$f \leqslant^* g \text{ iff } \exists m < \omega \forall n \in [m, \omega) \colon f(n) \leqslant g(n).$$

b) *The* bounding number *is*

$$\mathfrak{b} = \min \{\mathrm{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F \colon f \nleqslant^* g\},$$

i.e., the smallest cardinality of an unbounded family in $\leqslant^*$.

c) *The* dominating number *is*

$$\mathfrak{d} = \min \{\mathrm{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F \colon f \leqslant^* g\},$$

i.e., the smallest cardinality of a cofinal (or dominating) family in $\leqslant^*$.

Lemma 60. $\mathfrak{b} \leqslant \mathfrak{d}$.

Proof. Every cofinal family is unbounded. ∎

The following diagram records provable relations between the cardinal characteristics introduced so far. An arrow $\longrightarrow$ stands for the $\leqslant$-relation between cardinals. Some inequalities have already been proved:

$$\begin{tikzcd}
\operatorname{cov}(\mathcal{N}) \arrow[r, "\text{Th. 62}"] & \operatorname{non}(\mathcal{M}) \arrow[r, "\text{L. 50}"] & \operatorname{cof}(\mathcal{M}) \arrow[r] & \operatorname{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r, "\text{L. 60}"] & \mathfrak{d} \arrow[u] & \\
\operatorname{add}(\mathcal{N}) \arrow[uu, "\text{L. 50}"'] \arrow[r] & \operatorname{add}(\mathcal{M}) \arrow[u] \arrow[r, "\text{L. 50}"'] & \operatorname{cov}(\mathcal{M}) \arrow[u] \arrow[r, "\text{Th. 62}"'] & \operatorname{non}(\mathcal{N}) \arrow[uu, "\text{L. 50}"']
\end{tikzcd}$$

It is remarkable that there are inequalities connecting the ideals $\mathcal{N}$ and $\mathcal{M}$.

Lemma 61. *There are sets $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$, i.e., $\mathbb{R}$ is the (disjoint) union of two sets which are both "small".*

Proof. We work with the standard reals $\mathbb{R}$. Let $(q_n | n < \omega)$ enumerate the rational numbers. For $m < \omega$ let

$$U_m = \bigcup_{n > m} (q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n}).$$

U_m is dense open in $\mathbb{R}$ and

$$\sum_{n > m} \operatorname{length}((q_n - \frac{1}{2^n}, q_n + \frac{1}{2^n})) = \sum_{n > m} \frac{2}{2^n} = \frac{2}{2^m}.$$

Let $A = \bigcap_{m \in \omega} U_m$. By the calculation of the sum of interval lengths, A is a measure zero set, i.e., $A \in \mathcal{N}$.

$\mathbb{R} \setminus U_m$ is nowhere dense. Then $B = \bigcup_{m \in \omega} (\mathbb{R} \setminus U_m)$ is meager, i.e., $B \in \mathcal{M}$. Moreover

$$z \notin A \leftrightarrow \exists m < \omega \colon z \notin U_m \leftrightarrow \exists m < \omega \colon z \in (\mathbb{R} \setminus U_m) \leftrightarrow z \in B.$$

∎

Theorem 62. (ROTHBERGER, 1938) $\operatorname{cov}(\mathcal{M}) \leqslant \operatorname{non}(\mathcal{N})$ *and* $\operatorname{cov}(\mathcal{N}) \leqslant \operatorname{non}(\mathcal{M})$.

Proof. Let $A \in \mathcal{N}$ and $B \in \mathcal{M}$ such that $A \cup B = \mathbb{R}$ as in the preceding Lemma.
(1) Let $X \notin \mathcal{M}$. Then $X + A = \{x + a | x \in X, a \in A\} = \mathbb{R}$.
Proof. Let $z \in \mathbb{R}$. Then $z - X \nsubseteq B$. Take $x \in X$ such that $z - x \in A$. Then $z \in x + A \in X + A$. *qed*(1)

Now take $X \notin \mathcal{M}$ with $\mathrm{card}(X) = \mathrm{non}(\mathcal{M})$. Then

$$\mathbb{R} = X + A = \bigcup_{x \in X} (x + A).$$

The right hand side is a covering of $\mathbb{R}$ by $\leqslant \mathrm{card}(X)$ many sets in $\mathcal{N}$. So $\mathrm{cov}(\mathcal{N}) \leqslant \mathrm{card}(X) = \mathrm{non}(\mathcal{M})$.

The proof of the other inequality proceeds in the same way, with $\mathcal{M}$ and $\mathcal{N}$ interchanged. □

Before we prove further inequalities in the CICHON diagram let us check the values in the diagram in the models of set theory considered so far.

If we assume MA or CH then we know already that all entries except possible $\mathfrak{b}$ or $\mathfrak{d}$ are equal to $2^{\aleph_0}$.

Lemma 63. *Assume* MA *. Then* $\mathfrak{b} = 2^{\aleph_0}$ *(and so* $\mathfrak{d} = 2^{\aleph_0}$*).*

Proof. Let $F \subseteq {}^\omega\omega$ and $\mathrm{card}(F) < 2^{\aleph_0}$. It suffices to show that F is bounded in the structure $({}^\omega\omega, \leqslant^*)$. Define HECHLER *forcing* by

$$P = \{(a, A) | a \in {}^{<\omega}\omega, A \subseteq {}^\omega\omega, \mathrm{card}(A) < \aleph_0\}$$

with

$$(a', A') \leqslant (a, A) \text{ iff } a' \supseteq a, A' \supseteq A, \text{ and } \forall n \in \mathrm{dom}(a') \setminus \mathrm{dom}(a)\, \forall f \in A\colon a'(n) > f(n)$$

and $1_P = (\emptyset, \emptyset)$.

(1) HECHLER forcing has the ccc.

Proof. If $(a, A), (a, B) \in P$ with the same “stem” a, then they are compatible:

$$(a, A \cup B) \leqslant (a, A), (a, B).$$

So if $\mathcal{C}$ is an antichain in P, then the map $(a,\ A) \mapsto a$ is injective on $\mathcal{C}$. Since there are only countably many possible stems a, $\mathrm{card}(\mathcal{C}) \leqslant \aleph_0$. *qed*(1)

For every $f \in {}^\omega\omega$ set

$$D_f = \{(a, A) \in P | f \in A\}.$$

(2) D_f is dense in P.

Proof. Since $(a, A \cup \{f\}) \leqslant (a, A)$ and $(a, A \cup \{f\}) \in D_f$. *qed*(2)

For every $n < \omega$ set

$$D_n = \{(a, A) \in P | n \in \mathrm{dom}(a)\}.$$

(3) D_n is dense in P.

Proof. Let $(b, B) \in P$. Define $a\colon n + 1 \to \omega$ by

$$a(i) = \begin{cases} b(i), \text{ if } i \in \mathrm{dom}(b) \\ \max\{f(i) | f \in B\} + 1 \end{cases}$$

Then $(a, B) \leqslant (b, B)$ and $(a, A) \in D_n$. *qed*(3)

By MA take a $\{D_f | f \in F\} \cup \{D_n\}$-generic filter G on P. Let

$$h = \bigcup \{a | (a, A) \in G\}.$$

Then $h\colon \omega \rightarrow \omega$, since G meets every D_n.
(4) $\forall f \in F\colon f \leqslant^* h$, i.e., F is bounded.
Proof. Let $f \in F$. Take $(a, A) \in G \cap D_f$. Let $m = \mathrm{dom}(a)$. Consider $n \in [m, \omega)$. Let $(a', A') \in G$ such that $n \in \mathrm{dom}(a')$. Since all elements of G are compatible we may assume that $(a', A') \leqslant (a, A)$. Then

$$h(n) = a'(n) > f(n).$$

Hence $h \geqslant^* f$. □

So under MA or CH all entries in the CICHON diagram are equal to $2^{\aleph_0}$.

In the COHEN model for $2^{\aleph_0} = \kappa > \aleph_1$ we have from our previous analysis:

$$\begin{tikzcd}
\mathrm{cov}(\mathcal{N}) = \aleph_1 \arrow[r] & \mathrm{non}(\mathcal{M}) = \aleph_1 \arrow[r] & \mathrm{cof}(\mathcal{M}) = 2^{\aleph_0} \arrow[r] & \mathrm{cof}(\mathcal{N}) = 2^{\aleph_0} \\
 & \mathfrak{b} \arrow[u] \arrow[r] & \mathfrak{d} \arrow[u] & \\
\mathrm{add}(\mathcal{N}) = \aleph_1 \arrow[uu] \arrow[r] & \mathrm{add}(\mathcal{M}) = \aleph_1 \arrow[u] \arrow[r] & \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0} \arrow[u] \arrow[r] & \mathrm{non}(\mathcal{N}) = 2^{\aleph_0} \arrow[uu]
\end{tikzcd}$$

We now determine that the values of $\mathfrak{b}$ and $\mathfrak{d}$ are consistent with the diagram:

Theorem 64. *Let M be a ground model of* ZFC + CH*, and let $M \vDash \kappa$ is a regular cardinal$>\aleph_1$. Let $M[G]$ be a generic extension of M by the partial order for adjoining κ* COHEN *reals using finite conditions. Then, in $M[G]$, $\mathfrak{b} = \aleph_1$ and $\mathfrak{d} = 2^{\aleph_0}$.*

Proof. We show that the first $\aleph_1$ COHEN reals are unbounded. On the other hand no family $<2^{\aleph_0}$ can be cofinal in ${}^\omega\omega$ since there will always be a COHEN real which is not dominated. □

11 $\mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{d}$

To give an impression of non-trivial proofs of inequalities in Cichon's diagram we show that $\mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{d}$. Recall that

$$\mathfrak{d} = \min\,\{\mathrm{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F\colon f \leqslant^* g\},$$

is the smallest cardinality of a cofinal (or dominating) family in $\leqslant^*$.

It is convenient to introduce the following quantifiers:

$$\exists^\infty n \varphi(n) \quad \text{for} \quad \forall m \in \omega \exists n \in \omega (n > m \wedge \varphi(n)) \text{ “there are infinitely many”}$$

$$\forall^\infty n \varphi(n) \quad \text{for} \quad \exists m \in \omega \forall n \in \omega (n > m \rightarrow \varphi(n)) \text{ “for all but finitely many”.}$$

The following theorem links the meager ideal to a combinatorial property in ${}^\omega\omega$:

Theorem 65. $\mathrm{cov}(\mathcal{M}) = \min \{\mathrm{card}(F) | \, F \subseteq {}^\omega\omega \, and \, \forall g \in {}^\omega\omega \exists f \in F \forall^\infty n \, f(n) \neq g(n)\}$.

The family F on the RHS can be considered to be “cofinal” for the relation $\forall^\infty n \, f(n) \neq g(n)$ of being *eventually different*. Since $f <^* g$ implies $\forall^\infty n \; f(n) \neq g(n)$ the Theorem implies the desired inequality.

$$\begin{aligned} \mathrm{cov}(\mathcal{M}) &= \min \{\mathrm{card}(F) | \, F \subseteq {}^\omega\omega \text{and } \forall g \in {}^\omega\omega \exists f \in F \forall^\infty n \, f(n) \neq g(n)\} \\ &\leqslant \min \{\mathrm{card}(F) | F \subseteq {}^\omega\omega, \forall g \in {}^\omega\omega \exists f \in F : f \leqslant^* g\} \\ &= \mathfrak{d} \end{aligned}$$

Theorem 65 will follow from the following

Lemma 66. *For every infinite cardinal κ the following are equivalent:*

1. $\mathbb{R}$ *is not the union of less than κ-many meager sets;*
2. $\forall F \in [{}^\omega\omega]^{<\kappa} \exists g \in {}^\omega\omega \forall f \in F \exists^\infty n \, f(n) = g(n)$*;*
3. $\forall F \in [{}^\omega\omega]^{<\kappa} \forall G \in [[\omega]^\omega]^{<\kappa} \exists g \in {}^\omega\omega \forall f \in F \forall X \in G \exists^\infty n \in X \, f(n) = g(n)$*.*

Note that *a*) expresses that $\kappa \leqslant \mathrm{cov}(\mathcal{M})$; *b*) expresses that $\kappa \leqslant$ the RHS in Theorem 65. This implies the equality in Theorem 65.

Proof. (1) → (2): Assume $F \subseteq {}^\omega\omega$ and $|F| < \kappa$. For $f \in F$ let $G_f = \{g \in {}^\omega\omega | \; \exists^\infty n \; f(n) = g(n)\}$. Every $G_f = \bigcap_{m \in \omega} \{g \in {}^\omega\omega | \; \exists n > m \; f(n) = g(n)\}$ is a G_δ-set because every $\{g \in {}^\omega\omega | \; \exists n > m \; f(n) = g(n)\}$ is open. Moreover, every G_f is dense in $\mathbb{R}$. Hence $\mathbb{R} - G_f$ is meager for all $f \in F$. So

$$\bigcup_{f \in F} (\mathbb{R} - G_f) \neq \mathbb{R}.$$

Take

$$g \in \mathbb{R} \setminus \bigcup_{f \in F} (\mathbb{R} - G_f) = \bigcap \{G_f | \, f \in F\}.$$

Then $\forall f \in F \exists^\infty n \, f(n) = g(n)$, as required.

(2) → (3): Let $F = \{f_\alpha | \; \alpha < \lambda\}$, $\lambda < \kappa$, be a family of functions $f_\alpha \colon \omega \rightarrow \omega$ and $G = \{X_\alpha | \; \alpha < \lambda\}$ be a family of infinite subsets of ω. Let $\langle x_\alpha^n | \; n \in \omega \rangle$ be the monotone enumeration of X_α. Let $Q = \{s \colon \mathrm{dom}(s) \rightarrow \omega | \; \mathrm{dom}(s) \subseteq \omega \text{ is finite}\}$ the set of all finite partial functions from ω to ω; this is a version of Cohen forcing for a single Cohen real. For $\alpha, \beta < \lambda$ define a function $h_{\alpha,\beta} \colon \omega \rightarrow Q$ by

$$h_{\alpha,\beta}(n) = f_\beta \upharpoonright \{x_\alpha^0, x_\alpha^1, \ldots, x_\alpha^n\}$$

for all $n\in\omega$. Since Q is countable, we have by (2)

$$\forall F\in[{}^{\omega}Q]^{<\kappa}\exists h\in{}^{\omega}Q\forall f\in F\exists^{\infty}n\, f(n)=h(n).$$

In particular, there exists a function $h\in{}^{\omega}Q$, such that

$$\forall\alpha,\beta<\lambda\exists^{\infty}n\, h_{\alpha,\beta}(n)=h(n).$$

Recursively choose a sequence $\langle x_n|\, n\in\omega\rangle$, such that

$$x_n\in\mathrm{dom}(h(n))-\{x_0,x_1,\dots,x_{n-1}\}$$

for all $n\in\omega$. Let $g{:}\,\omega\rightarrow\omega$ be a function, such that for all $n<\omega$

$$g(x_n)=h(n)(x_n).$$

To check (3) for g consider $\alpha,\beta<\lambda$. There are infinitely many n with $h_{\alpha,\beta}(n)=h(n)$. For such n the corresponding x_n satisfies

$$x_n\in\mathrm{dom}(h(n))=\mathrm{dom}(h_{\alpha,\beta}(n))=\{x_\alpha^0,x_\alpha^1,\dots,x_\alpha^n\}\subseteq X_\alpha$$

and

$$f_\beta(x_n)=h_{\alpha,\beta}(n)(x_n)=h(n)(x_n)=g(x_n).$$

Thus $\exists^{\infty}n\in X_\alpha\, f_\beta(n)=g(n)$

$(3)\rightarrow(1)$: $\square$

Proof:
$(1)\rightarrow(2)$: Assume $F\subseteq\omega^\omega$ and $|F|<\kappa$. For $f\in F$ let $G_f=\{g\in\omega^\omega|\ \exists^{\infty}n\ f(n)=g(n)\}$. Every $G_f=\bigcap_{m\in\omega}\ \{g\in\omega^\omega|\ \exists n>m\ f(n)=g(n)\}$ is a G_δ-set because every $\{g\in\omega^\omega|\ \exists n>m\ f(n)=g(n)\}$ is open. Moreover, every G_f is dense in $\langle\mathsf{R}\rangle$. Hence $\langle\mathsf{R}\rangle-G_f$ is for all $f\in F$ meager. So $\bigcap\ \{G_f|\ f\in F\}\neq\emptyset$. But if $g\in\bigcap\ \{G_f|\ f\in F\}$, then $\forall f\in F\exists^{\infty}n\, f(n)=g(n)$.
$(2)\rightarrow(3)$: Let $F=\{f_\alpha|\ \alpha<\lambda\}$, $\lambda<\kappa$, be a family of functions $f_\alpha{:}\,\omega\rightarrow\omega$ and $G=\{X_\alpha|\ \alpha<\lambda\}$ be a family of infinite subsets of ω. Let $\langle x_\alpha^n|\ n\in\omega\rangle$ be the monotone enumeration of X_α. For $\alpha,\beta<\lambda$ define a function $h_{\alpha,\beta}$ by

$$h_{\alpha,\beta}=f_\beta\restriction\{x_\alpha^0,x_\alpha^1,\dots,x_\alpha^n\}$$

for all $n\in\omega$. Let $\Phi=\{s{:}\, d\,o\,m(s)\rightarrow\omega|\ d\,o\,m(s)\subseteq\omega$ finite$\}$. Since Φ is countable, we have by (2)

$$\forall F\in[\Phi^\omega]^{<\kappa}\exists h\in\Phi^\omega\forall f\in F\exists^{\infty}n\, f(n)=h(n).$$

In particular, there exists a function $h\in\Phi^\omega$, such that

$$\forall\alpha,\beta<\lambda\exists^{\infty}n\, h_{\alpha,\beta}(n)=h(n).$$

Pick inductively a sequence $\langle x_n|\, n\in\omega\rangle$, such that

$$x_n\in d\,o\,m(h(n))-\{x_0,x_1,\dots,x_{n-1}\}$$

for all $n\in\omega$. Let $g{:}\,\omega\rightarrow\omega$ be a function, such that

$$g(x_n)=h(n)(x_n)$$

holds for all n with $h_{\alpha,\beta}(n) = h(n)$. Then g is a witness for (3) because

$$f_\beta(x_n) = h_{\alpha,\beta}(n)(x_n) = h(n)(x_n) = g(x_n)$$

for all n with $h_{\alpha,\beta}(n) = h(n)$.
(3) $\to$ (1): Let $\langle F_\alpha | \, \alpha < \lambda \rangle$ with $\lambda < \kappa$ be a family of meager sets. We want to show that $\bigcup \{F_\alpha | \ \alpha < \lambda\} \neq \langle \mathsf{R} \rangle$. Since every F_α is meager, $F_\alpha = \bigcup \{F_\alpha^n | \ n \in \omega\}$ where every F_α^n is nowhere dense. By definition also the topological closure $c\,l(F_\alpha^n)$ is nowhere dense. So we can assume w.l.o.g. that $\langle F_\alpha | \, \alpha < \lambda \rangle$ is a family of closed nowhere dense sets.
For $\alpha < \lambda$ let

$$s_n^\alpha = m\,i\,n\{s \in 2^{<\omega} | \, \forall t \in 2^{<n}[t^\frown s] \cap F_\alpha = \emptyset\}$$

where the minimum is taken with respect to a fixed enumeration of $2^{<\omega}$ and $[s] = \{f \in 2^\omega | \ s \subseteq f\}$. Why does this minimum exist? Consider first an arbitrary $t \in 2^{<\omega}$. Then there exists $t \subseteq s \in 2^{<\omega}$, such that

$$[s] \cap F_\alpha = \emptyset. \quad (*)$$

Because: $[t]$ is open. Since F_α is nowhere dense, $[t] \nsubseteq F_\alpha$. Pick $f \in [t] \setminus F_\alpha$. But $2^\omega - F_\alpha$ is open. So there exists a neighbourhood $[s]$ of f such that $[s] \subseteq 2^\omega - F_\alpha$.
Now we can construct recursively an $s \in 2^{<\omega}$ such that

$$\forall t \in 2^{<n}[t^\frown s] \cap F_\alpha = \emptyset.$$

To do so, let $2^{<n} = \{t_k | \ k \leq m\}$. For t_0 pick an s_0 as in $(*)$. If s_k is already defined, consider $t_{k+1}^\frown s_k$ and pick for it an s_{k+1} as in $(*)$. Then s_m is as wanted.
Back to the s_n^α from above. By (3) there exists a sequence $\langle s_n | \, n \in \omega \rangle$ such that

$$\forall \alpha < \lambda \exists^\infty n \, s_n^\alpha = s_n.$$

For $\alpha < \lambda$ let $X_\alpha = \{n \in \omega | \, s_n^\alpha = s_n\}$.
Lemma
There exists an increasing sequence $\langle k_n | \, n \in \omega \rangle$ such that
(1) $\sum_{j \leq k_n} |s_j| < k_{n+1}$ for all $n \in \omega$
(2) $\forall \alpha < \lambda \exists^\infty n \, x_\alpha \cap [k_{2n}, k_{2n+1}[\neq \emptyset$.
Proof: For every finite $A \subseteq \lambda$ define $f_A \colon \omega \to \omega$ by

$$f_A(n) = m\,i\,n\{m \in \omega | \, \forall \alpha \in A[n, m[\cap X_\alpha \neq \emptyset\}$$

and for every $k \in \omega$ let

$$f'_{A,k}(0) = k \quad \text{and} \quad f'_{A,k}(n+1) = f_A(f'_{A,k}(n))$$

for all $n \in \omega$.
By (3) $\lambda < \mathfrak{d}$. So there exists a strictly increasing function $f \colon \omega \to \omega$ such that
1. $\forall A \in [\lambda]^{<\omega} \forall k \exists^\infty n \, f'_{A,k}(n) < f(n)$
2. $\sum_{j \leq f(n)} |s_j| < f(n+1)$.
We can find such an f because $|\{f'_{A,k} | \ A \in [A]^{<\omega}, k \in \omega\}| = |\lambda|$. So there is by definition of $\mathfrak{d}$ an f which is not dominated by any $f'_{A,k}$. That is $f \nleq^* f'_{A,k}$ for all $A \in [\lambda]^{<\omega}$, $k \in \omega$, i.e. $\exists^\infty n \, f'_{A,k}(n) < f(n)$. Once we have found such an f we can recursively ensure 2.
We have

$$\forall A \in [\lambda]^{<\omega} \exists^\infty n \exists k \, f(n) \leq k \leq f_A(k) \leq f(n+1) \quad (*).$$

Otherwise there were A and m such that

$$\forall n \geq m\, f(n+1) < f_A(f(n)).$$

So for $k = f(m)$ and all $m \in \omega$

$$f(n) \leq f(n+m) < f_A(f(m)) < f_A(f_A(f(m))) < \ldots < f'_{A,k}(n)$$

would hold. But this contradicts the choice of f.
Define $X'_A = \{n \in \omega |\, \exists k\, f(n) \leq k \leq f_A(k) < f(n+1)\}$. Then by $(*)$ X'_A is infinite for all $A \in [\lambda]^{<\omega}$. Consider $X^0 = \{2n |\, n \in \omega\}$ and $X^1 = \{2n+1 |\, n \in \omega\}$. Then ($X'_A \cap X^0$ is infinite for all $A \in [\lambda]^{<\omega}$) or ($X'_A \cap X^1$ is infinite for all $A \in [\lambda]^{<\omega}$). [If otherwise $|X'_A \cap X^0| < \omega$ and $|X'_B \cap X^1| < \omega$, then $|X'_A \cap X'_B| = |X'_{A\cup B}| = |X'_{A\cup B} \cap X^0| + |X'_{A\cup B} \cap X^1| \leq |X'_A \cap X^0| + |X'_B \cap X^1| < \omega$ which contradicts $(*)$.]
If $|X'_A \cap X^0| = \omega$, then set $k_n = f(n)$. Otherwise set $k_n = f(n+1)$.
□ (Lemma)
Lemma
There exists $X \subseteq \omega$ such that $|X \cap [k_{2n}, k_{2n+1}[| \leq 1$ for all $n \in \omega$ and $X \cap X_\alpha$ is infinite for every $\alpha < \lambda$.
Proof: For $\alpha < \lambda$ and $n \in \omega$ define

$f_\alpha(n) = min(X_\alpha \cap [k_{2n}, k_{2n+1}[)$ if $X_\alpha \cap [k_{2n}, k_{2n+1}[\neq \emptyset$

$f_\alpha(n) = 0$ otherwise
and $Y_\alpha = \{n \in \omega |\, f_\alpha(n) \neq \emptyset\}$. By (3) there exists a $g \in \omega^\omega$ such that

$$\forall \alpha < \lambda \exists^\infty n \in Y_\alpha g(n) = f_\alpha(n).$$

Hence the claim holds for $X = \{g(n) |\, n \in \omega\}$. □
Now we can prove "(3) → (1)". Let $\langle x_n |\, n \in \omega \rangle$ be the monotone enumeration of X from the previous lemma. Set

$$x = s_{x_0}^\frown s_{x_1}^\frown s_{x_2}^\frown \ldots.$$

We show that $x \notin \bigcup \{F_\alpha |\, \alpha < \lambda\}$. Let $\alpha < \lambda$. It follows from the above construction that there exists $x_n \in X \cap [k_{2n}, k_{2n+1}[$ such that $s_{x_n} = s^\alpha_{x_n}$. But $\sum_{j<n} |s_{x_j}| < k_{2n} < x_n$ and (by the definition of $\langle s^\alpha_n |\, n \in \omega \rangle$) $[s_{x_0}^\frown \ldots {}^\frown s_{x_{n-1}}^\frown s_{x_n}] = [s_{x_0}^\frown \ldots {}^\frown s_{x_{n-1}}^\frown s^\alpha_{x_n}]$ is disjoint from F_α. Hence $x \notin F_\alpha$. □

12 Additivity and cofinality of measure and category

We want to prove the inequalities

$$\mathrm{add}(\mathcal{N}) \leqslant \mathrm{add}(\mathcal{M}) \text{ and } \mathrm{cof}(\mathcal{M}) \leqslant \mathrm{cof}(\mathcal{N})$$

in the CICHON diagram. In this section we shall assume that $\mathcal{M}$ and $\mathcal{N}$ are ideals on ${}^\omega 2$ equipped with the standard topology $\mathcal{T}$, standard metric d and standard measure μ. First we introduce the general tool of GALOIS-TUKEY reductions.

Definition 67. *Let* $(P, \leqslant_P)$ *and* $(Q, \leqslant_Q)$ *be weak partial orders (reflexive and transitive) and let* $f: P \to Q$ *and* $f^*: Q \to P$ *be maps satisfying*

$$\forall p \in P \forall q \in Q(f(p) \leqslant_Q q \to p \leqslant_P f^*(q)).$$

Then we say that f, f^* *is a* (GALOIS-)TUKEY reduction *of* P *to* Q*, and we write* $P \preccurlyeq_T Q$*. This situation can be pictured by*

$$\begin{array}{ccc} P & \xleftarrow{f^*} & Q \\ \leqslant_P & & \leqslant_Q \\ P & \xrightarrow{f} & Q \end{array}$$

Obviously, $\preccurlyeq_T$ *is a (class-sized) weak partial order on weak partial orders. If ideals* $\mathcal{I}$ *are taken as partial orders* $(\mathcal{I}, \subseteq)$ *then we write* $\mathcal{I} \preccurlyeq_T \mathcal{J}$ *for* $(\mathcal{I}, \subseteq) \preccurlyeq_T (\mathcal{J}, \subseteq)$*.*

Remark 68. As an intuition one may say that f translates questions $p?$ in P into questions $f(p)?$ in Q. Then if $q!$ is an answer to $f(p)?$ ($f(p) \leqslant_Q q$) we have that $f^*(q)!$ is an answer to the original question $p?$ ($p \leqslant_P f^*(q)$). Such schemata arrise in many parts of mathematics and beyond where problems in one domain are reduced to problems in another domain. The classic example is the reduction of field-theoretic questions to group-theoretic question in GALOIS theory.

TUKEY reductions have the following consequence for cardinal characteristics.

Lemma 69. *Let* $\mathcal{I}, \mathcal{J}$ *be ideals with* $\mathcal{I} \preccurlyeq_T \mathcal{J}$*. Then* $\mathrm{add}(\mathcal{I}) \geqslant \mathrm{add}(\mathcal{J})$ *and* $\mathrm{cof}(\mathcal{I}) \leqslant \mathrm{cof}(\mathcal{J})$*.*

Proof. The proofs are straightforward. □

So for our purposes it suffices to show that $\mathcal{M} \preccurlyeq_T \mathcal{N}$. The proof will proceed via an auxiliary weak partial order $(\mathcal{C}, \subseteq^*)$ such that

$$\mathcal{M} \preccurlyeq_T \mathcal{C} \preccurlyeq_T \mathcal{N}.$$

We shall define maps

$$\begin{array}{ccccc} \mathcal{M} & \xleftarrow{\varphi_1^*} & \mathcal{C} & \xleftarrow{\varphi_2^*} & \mathcal{N} \\ \subseteq & & \subseteq^* & & \subseteq \\ \mathcal{M} & \xrightarrow{\varphi_1} & \mathcal{C} & \xrightarrow{\varphi_2} & \mathcal{N} \end{array}$$

to connect between category-theoretic smallness and measure-theoretic smallness.

Definition 70. *Let*

$$\mathcal{C} = \left\{ S \in ([\omega]^{<\omega})^\omega \mid \sum_{n<\omega} \frac{|S(n)|}{n^2} < \infty \right\}.$$

We use n^2 *as a function with a (modest) growth rate when n goes to* ∞*. Since we are only interested in eventual behaviour of sequences, we can for convenience agree that* $\frac{a}{0} = 0$*. Define a weak partial order* $\subseteq^*$ *on* $\mathcal{C}$ *by*

$$S_1 \subseteq^* S_2 \text{ iff } \forall m < \omega \exists n_0 < \omega\, (n_0 > m \wedge \forall n < \omega (n \geqslant n_0 \to S_1(n) \subseteq S_2(n))).$$

Lemma 71.

Lemma 72. $\mathcal{C} \preccurlyeq_T \mathcal{N}$.

Proof. Take a family $(G_{i,j} \,|\, i, j \in \omega)$ of μ-independent open sets $G_{i,j} \subseteq {}^{\omega}2$ such that

$$\mu(G_{i,j}) = \frac{1}{i^2}\,.$$

Define $\varphi_2 \colon \mathcal{C} \to \mathcal{N}$ by

$$\varphi_2(S) = \bigcap_{n<\omega} \bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k}\,.$$

Since the infinite sum$\sum_{m<\omega} \frac{|S(m)|}{m^2}$ converges,

$$\mu(\varphi_2(S)) \leqslant \mu\Big(\bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k}\Big) \leqslant \sum_{m<n<\omega} \Big(|S(m)| \cdot \frac{1}{m^2}\Big) \overset{n\to\infty}{\longrightarrow} 0\,.$$

So $\mu(\varphi_2(S)) = 0$.

To define $\varphi_2^* \colon \mathcal{N} \to \mathcal{C}$ consider $G \in \mathcal{N}$. Then $\mu({}^{\omega}2 \setminus G) = 1$. Since measurable sets can be approximated from within by closed sets, take a closed, and hence compact $K^G \subseteq {}^{\omega}2$ such that $K^G \cap G = \emptyset$ and $\mu(K^G) > 0$. The set

$$K' = K^G \setminus \bigcup \{N_p \,|\, p \in \mathrm{Fn}(\omega, 2, \aleph_0), \mu(K^G \cap N_p) = 0\}$$

has the same measure $\mu(K') = \mu(K^G)$, is closed and compact and satisfies

$$\forall U \in \mathcal{T}\, (K' \cap U \neq \emptyset \to \mu(K' \cap U) > 0).$$

So we may assume that

$$\forall U \in \mathcal{T}\, (K^G \cap U \neq \emptyset \to \mu(K^G \cap U) > 0).$$

Let $(U_n | n < \omega)$ be an enumeration of all basic open sets N_p in ${}^{\omega}2$ such that $K^G \cap N_p \neq \emptyset$. For $n, i \in \omega$ define

$$A_{n,i}^G = \{j < \omega \,|\, K^G \cap U_n \cap G_{i,j} = \emptyset\}.$$

Then for $i < \omega$

$$K^G \cap U_n \subseteq \bigcap_{j \in A_{n,i}^G} ({}^{\omega}2 \setminus G_{i,j}),$$

and

$$K^G \cap U_n \subseteq \bigcap_{i<\omega} \bigcap_{j \in A_{n,i}^G} ({}^{\omega}2 \setminus G_{i,j}).$$

Hence by the μ-independence of the $G_{i,j}$ and their complements,

$$0 < \mu(K^G \cap U_n) \leqslant \prod_{i<\omega} \prod_{j \in A_{n,i}^G} \mu({}^{\omega}2 \setminus G_{i,j}) = \prod_{i<\omega} \prod_{j \in A_{n,i}^G} \Big(1 - \frac{1}{i^2}\Big).$$

Taking multiplicative inverses in $\mathbb{R}$ observe that

$$\Big(1 - \frac{1}{i^2}\Big)^{-1} = 1 + \frac{1}{i^2} + \frac{1}{i^4} + \ldots > 1 + \frac{1}{i^2}\,.$$

Then

$$\infty > \left(\prod_{i<\omega} \prod_{j \in A^G_{n,i}} (1 - \frac{1}{i^2})\right)^{-1} \geqslant \left(\prod_{i<\omega} \prod_{j \in A^G_{n,i}} (1 + \frac{1}{i^2})\right),$$

i.e., the infinite product on the right converges. By standard techniques from analysis, using e.g. logarithms, this implies that the corresponding infinite sum converges:

$$\sum_{i<\omega} \sum_{j \in A^G_{n,i}} \frac{1}{i^2} = \sum_{i<\omega} \frac{|A^G_{n,i}|}{i^2} < \infty.$$

Thus $S_n := (A^G_{n,i} | i < \omega) \in \mathcal{C}$ for all $n < \omega$. By a previous lemma, we can choose $S_G = \varphi_2^*(G) \in \mathcal{C}$ such that

$$\forall n < \omega\colon S_n \subseteq^* S_G = \varphi_2^*(G).$$

Finally we have to show that φ_2, φ_2^* are a TUKEY reduction of $\mathcal{C}$ to $\mathcal{N}$. So let $S \in \mathcal{C}$ and $G \in \mathcal{N}$ such that $\varphi_2(S) \subseteq G$, i.e.,

$$\varphi_2(S) = \bigcap_{n<\omega} \bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k} \subseteq G.$$

Using the above construction of $\varphi_2^*(G)$ we have

$$K^G \cap \bigcap_{n<\omega} \bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k} = \emptyset.$$

Every set $\bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k}$ is open, also in the relative topology on the compact set K^G. We can apply the BAIRE category theorem in the measure space (K^G, d); K^G is closed in $\mathcal{T}$ and thus (K^G, d) is a complete metric space. If all the $\bigcup_{m>n} \bigcup_{k \in S(m)} G_{m,k}$ were dense open then their intersection in K^G would be non-empty. Thus we can take some $n_0 < \omega$ such that $\bigcup_{m>n_0} \bigcup_{k \in S(m)} G_{m,k}$ is not dense in K^G. Take some basic open set U_l such that $U_l \cap K^G \neq \emptyset$ and

$$(U_l \cap K^G) \cap \bigcup_{m>n_0} \bigcup_{k \in S(m)} G_{m,k} = \emptyset.$$

Consider $m > n_0$. Since

$$A^G_{l,m} = \{k < \omega \,|\, K^G \cap U_l \cap G_{m,k} = \emptyset\}$$

we have

$$S(m) \subseteq A^G_{l,m} = S_l(m).$$

Furthermore for sufficiently high $m < \omega$

$$S(m) \subseteq S_l(m) \subseteq S_G(m).$$

Hence

$$S \subseteq^* S_G = \varphi_2^*(G)$$

as required. $\square$

In the previous proof, the μ-independent family $(G_{i,j}\,|i,\,j\in\omega)$ was the principle tool for converting situations in $\mathcal{C}$ to $\mathcal{N}$ and vice versa. The next lemma will provide such a device for converting between $\mathcal{M}$ and $\mathcal{C}$.

Lemma 73. *For every nonempty open set $U\subseteq{}^{\omega}2$ there is a countable family $\mathcal{V}$ of subsets of U such that*

a) every dense open subset of ${}^{\omega}2$ contains a member of $\mathcal{V}$;

b) the intersection of any n^2 elements of $\mathcal{V}_n$ is nonempty.

Proof. Let $(U_n|n<\omega)$ be an enumeration of the closed nonempty open subsets of ${}^{\omega}2$. Note that these are compact, and by compactness a union of *finitely* many neighbourhoods N_p .

For $k\in\omega$ let

$$A_k=\left\{n\geqslant k\,|\forall I\subseteq k(\bigcap_{i\in I}U_i\neq\emptyset\rightarrow U_n\cap\bigcap_{i\in I}U_i\neq\emptyset)\right\}.$$

Let

$$\mathcal{V}=\{\bigcup_{i<n^2}U_{m_i}\mid\text{there is a sequence } m_0,\ldots,m_{n^2-1}\text{ such that } m_0\in\omega,\forall i<n^2-1\colon m_{i+1}\in A_{m_i}\}.$$

Obviously $\mathcal{V}$ consists of closed nonempty open subsets of ${}^{\omega}2$ and is at most countable.
a) Let V be dense open in ${}^{\omega}2$.
(1) $A_k\cap\{n\in\omega|U_n\subseteq V\}\neq\emptyset$.
Proof. Let $\tilde{I}=\{I\subseteq k|\bigcap_{i\in I}U_i\neq\emptyset\}$. By the density of V choose, for each $I\in\tilde{I}$, a basic neighbourhoods $N_{p_I}\subseteq V\cap\bigcap_{i\in I}U_i$. Then $\bigcup_{I\in\tilde{I}}N_{p_I}$ is nonempty, closed and open. By varying the p_I one can arrange that $\bigcup_{I\in\tilde{I}}N_{p_I}=U_n$ for some $n\geqslant k$. Then $n\in A_k$ and $U_n\subseteq V$. *qed*(1)

By (1), one can recursively choose a sequence $m_0,\ldots,m_{n^2-1}$ such that $m_0\in\omega,\forall i<n^2-1\colon m_{i+1}\in A_{m_i}$ and $U_{m_0},U_{m_1},\ldots,U_{m_{n^2-1}}\subseteq V$. Then $\bigcup_{i<n^2}U_{m_i}\in\mathcal{V}$ and $\bigcup_{i<n^2}U_{m_i}\subseteq V$.
b) Let $V_0,\ldots,V_{n^2-1}\in\mathcal{V}$, where for $j<n^2$:

$$V_j=\bigcup_{i<n^2}U_{m_i^j}\text{ for some sequence } m_0^j,\ldots,m_{n^2-1}^j\text{ such that } m_0^j\in\omega,\forall i+1<n^2\colon m_{i+1}^j\in A_{m_i^j}.$$

We can assume that the V_j are permuted in a way that m_0^0 is minimal, then m_1^1 is minimal, then m_2^2 is minimal, etc. Then for each $i+1<n^2$

$$m_i^i\leqslant m_i^{i+1}\wedge m_{i+1}^{i+1}\in A_{m_i^{i+1}}\subseteq A_{m_i^i}.$$

By the definition of the sets A_k we have

$$U_{m_0^0}\neq\emptyset,U_{m_0^0}\cap U_{m_1^1}\neq\emptyset,U_{m_0^0}\cap U_{m_1^1}\cap U_{m_2^2}\neq\emptyset,\text{etc.}$$

This implies

$$\bigcap_{j<n^2}V_j\supseteq\bigcap_{j<n^2}U_{m_j^j}\neq\emptyset.$$

$\square$

Lemma 74. $\mathcal{M}\preccurlyeq_T\mathcal{C}$.

Proof. Let $(U_n|n<\omega)$ be an enumeration of the closed nonempty open subsets of ${}^\omega 2$. For $n<\omega$ choose a family $\mathcal{V}_n=(V_m^n|m<\omega)$ of subsets of U_n as in the previous lemma:
(1) every dense open subset of ${}^\omega 2$ contains a member of $\mathcal{V}_n$, and
(2) the intersection of any n^2 elements of $\mathcal{V}_n$ is nonempty.

It suffices to define a TUKEY map φ_1: $\mathcal{M}\to\mathcal{C}$ for a cofinal subset of $\mathcal{M}$. Note that every meager set is contained in an *increasing* union of *closed* nowhere dense sets.

So let $F\in\mathcal{M}$, $F=\bigcup_{n<\omega} F_n$ where $F_0\subseteq F_1\subseteq\dots$ are closed and nowhere dense. Then define $\varphi_1(F)=S_F{:}\,\omega\to\omega$ by

$$S_F(n)=\{\min\{k<\omega|F_n\cap V_k^n=\emptyset\}\}.$$

Note that the complement of F_n is dense open, and then by (1), the minimum exists and $S_F(n)$ is a singleton subset of ω. $S_F(n)\in[\omega]^1\subseteq[\omega]^{<\omega}$, and so $S_F\in([\omega]^{<\omega})^\omega$. And $S_F\in\mathcal{C}$ since

$$\sum_{n<\omega}\frac{|S_F(n)|}{n^2}=\sum_{n<\omega}\frac{1}{n^2}<\infty.$$

Hence $\varphi_1{:}\,\mathcal{M}\to\mathcal{C}$ is welldefined (on a cofinal subset of $\mathcal{M}$).

Now define $\varphi_1^*{:}\,\mathcal{C}\to\mathcal{M}$ by

$$\varphi_1^*(S)=F^S={}^\omega 2\setminus\bigcap_{n\in\omega}\bigcup_{m>n}\bigcap_{i\in S(m)}V_i^m.$$

Note that $D_n:=\bigcup_{m>n}\bigcap_{i\in S(m)}V_i^m$ is open.
(3) D_n is dense.
Proof. Since

$$\sum_{m<\omega}\frac{|S(m)|}{m^2}<\infty$$

we must have $|S(m)|\leqslant m^2$ for sufficiently large $m<\omega$. Let $O\subseteq{}^\omega 2$ be nonempty and open. Take some $m<\omega$ such that $m>n$, $|S(m)|\leqslant m^2$, and $U_m\subseteq O$. By the intersection property of $\mathcal{V}_m$

$$\emptyset\neq\bigcap_{i\in S(m)}V_i^m\subseteq U_m\subseteq O.$$

So $D_n=\bigcup_{m>n}\bigcap_{i\in S(m)}V_i^m$ intersects O and is thus dense. *qed*(3)

Then F^S is the complement of a countable intersection of dense open sets, hence F^S is meager. So $\varphi_1^*{:}\,\mathcal{C}\to\mathcal{M}$ is welldefined.

Finally we have to show that φ_1,φ_1^* are a TUKEY reduction of $\mathcal{M}$ to $\mathcal{C}$. So let $F\in\mathcal{M}$ as above and let $\varphi_1(F)=S_F\subseteq^* S$. Take $n_0<\omega$ such that

$$\forall m\in[n_0,\omega){:}\ S_F(m)\subseteq S(m).$$

For $i\in S_F(m)$, $F_m\cap V_k^m=\emptyset$. Consider $n\geqslant n_0$. Then

$$F_n\cap\bigcup_{m>n}\bigcap_{i\in S(m)}V_i^m\subseteq F_n\cap\bigcup_{m>n}\bigcap_{i\in S_F(m)}V_i^m=\emptyset,$$

$$\bigcup_{n<\omega}F_n\cap\bigcap_{n\in\omega}\bigcup_{m>n}\bigcap_{i\in S(m)}V_i^m=\emptyset,$$

and so

$$F = \bigcup_{n<\omega} F_n \subseteq {}^{\omega}2 \setminus \bigcap_{n\in\omega} \bigcup_{m>n} \bigcap_{i\in S(m)} V_i^m = \varphi_1^*(S).$$

$\square$

Theorem 75. *$\mathcal{M} \preccurlyeq_T \mathcal{N}$ and hence* $\mathrm{add}(\mathcal{N}) \leqslant \mathrm{add}(\mathcal{M})$ *and* $\mathrm{cof}(\mathcal{M}) \leqslant \mathrm{cof}(\mathcal{N})$.

13 Forcing with sets of positive measure

Definition 76. *Let $(\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \mathbb{R})$ be the following forcing:*

a) $\boldsymbol{B} = \{A \subseteq \mathbb{R} | A$ *is closed and has positive measure*$\}$*;*

b) $A \leqslant_{\boldsymbol{B}} B$ *iff* $A \setminus B \in \mathcal{N}$.

To consider closed sets in various models of set theory, define codes for closed sets (as we defined codes for open sets before).

Definition 77. *An F-*code *is a countable set d of rational open intervals. The* interpretation *of d is the closed set*

$$d^V = \mathbb{R} \setminus \bigcup d.$$

Let M be a ground model and form $(\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \mathbb{R})^M$. Let G be M-generic on $(\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \mathbb{R})^M$.

Lemma 78. *The intersection*

$$X = \bigcap \{d^{M[G]} | d \text{ is an } F\text{-code and } d^M \in G\}$$

is a singleton $\{r_G\}$ with $r_G \in \mathbb{R}$. We call r_G the random real *adjoined by G. Moreover the generic filter can be reconstructed from r_G as*

$$G = \{d^M | d \text{ is an } F\text{-code and } r_G \in d^{M[G]}\}.$$

Proof. (1) The family $\{d^{M[G]} | d$ is an F-code and $d^M \in G\}$ has the finite intersection property, intersections of finitely many elements of the family are nonempty.
Proof. Let $d_0, \dots, d_{n-1}$ be F-codes and $d_0^M, \dots, d_{n-1}^M \in G$. Then there is some F-code d such that $d^M \leqslant_{\boldsymbol{B}} d_0^M, \dots, d_{n-1}^M$. Then $d^M \setminus d_0^M, \dots, d^M \setminus d_{n-1}^M \in \mathcal{N}$. Since d^M is a set of positive measure,

$$d^M \setminus ((d^M \setminus d_0^M) \cup \dots \cup (d^M \setminus d_{n-1}^M)) \neq \emptyset.$$

Take $r \in d^M \setminus ((d^M \setminus d_0^M) \cup \dots \cup (d^M \setminus d_{n-1}^M))$. Then $r \in d_0^M \cap \dots \cap d_{n-1}^M$. Furthermore, $r \in d_0^{M[G]} \cap \dots \cap d_{n-1}^{M[G]}$. *qed*(1)

Every closed set of positive measure has a *compact* subset of positive measure. By density some $d^M \in G$ is compact. Then also $d^{M[G]}$ is compact. Now a family of closed sets with the finite intersection property which contains a compact set has a nonempty intersection.

To prove that the intersection X is a singleton assume that $r, r' \in X$ and $r < r'$. Take $q \in \mathbb{Q}$ such that $r < q < r'$. The set

$$D = \{A \in \boldsymbol{B} | A \subseteq (-\infty, r) \text{ or } A \subseteq (r, \infty)\} \in M$$

is dense in $\boldsymbol{B}$. Let $A \in D \cap G$. Without restriction assume that $A \subseteq (-\infty, r)$. Let $A = d^M$. Then $d^{M[G]} \subseteq (-\infty, r)^{M[G]}$ and so

$$r' \in X \subseteq d^{M[G]} \subseteq (-\infty, r).$$

This contradicts $q < r'$.

Hence there is a unique random real determined by G. To show that

$$G = \{d^M \,|\, d \text{ is an } F\text{-code and } r_G \in d^{M[G]}\}$$

consider $d^M \in G$. Then $r_G \in d^{M[G]}$ and so d^M is an element of the right hand side.

Conversely assume that $r_G \in d^{M[G]}$. Define a set

$$D' = \{e^M \in \boldsymbol{B} \,|\, e \text{ is an } F\text{-code and } (e^M \subseteq d^M \vee e^M \cap d^M = \emptyset)\}.$$

(2) D' is dense in $\boldsymbol{B}$.
Proof. Let $e_0^M \in \boldsymbol{B}$.
Case 1. $e_0^M \cap d^M \in \boldsymbol{B}$. Then take a code e such that $e^M = e_0^M \cap d^M \in D'$.
Case 2. $e_0^M \cap d^M \in \mathcal{N}$. Take an open set O such that $e_0^M \cap d^M \subseteq O$ and $\mu(O) < \mu(e_0^M)$. Then $e_0^M \setminus O \subseteq e_0^M$ is a closed set of positive measure, and if $e^M = e_0^M \setminus O$ then $e^M \in D'$.
qed(2)

Take $e^M \in D' \cap G$. Assume that $e^M \cap d^M = \emptyset$. By the absoluteness of such properties, $e^{M[G]} \cap d^{M[G]} = \emptyset$. But $r_G \in e^{M[G]} \cap d^{M[G]}$, contradiction. Hence $e^M \subseteq d^M$ and so $d^M \in G$. $\square$

Lemma 79. *$\boldsymbol{B}$ has the countable chain condition.*

Proof. Standard. $\square$

Lemma 80. *$\boldsymbol{B}$ is* ${}^\omega\omega$-bounding*, i.e., if G is M-generic on $\boldsymbol{B}$ then*

$$\forall f \in M[G], f\colon \omega \to \omega \,\exists h \in M, h\colon \omega \to \omega \,\forall n < \omega\colon f(n) \leqslant g(n).$$

Proof. OK $\square$

14 Iterating random forcing with finite supports

Let $\boldsymbol{B} = (\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \emptyset)$ be the canonical term for random forcing. Use the Iteration Theorem 23 to define an iterated forcing $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ from names $((\dot{Q}_\beta, \dot{\leqslant}_\beta) | \beta < \kappa)$, where $\kappa = \aleph_2$. Define both sequences by simultaneous induction.

For the initial case and the successor case assume that $\alpha < \kappa$ and $((P_{\alpha'}, \leqslant_{\alpha'}, 1_{\alpha'}) | \alpha' \leqslant \alpha)$ and $((\dot{Q}_{\alpha'}, \dot{\leqslant}_{\alpha'}) | \alpha' < \alpha)$ are defined satisfying
(1) $\mathrm{card}(P_\alpha) \leqslant \aleph_1$ and P_α satisfies the countable chain condition;
(2) $1_{P_\alpha} \Vdash \mathrm{CH}$.

Then $1_{P_\alpha} \Vdash \mathrm{card}\{A \subseteq \mathbb{R} | A \text{ is closed}\} = \aleph_1$. Using the maximality principle choose a P_α-name $\dot{h}_\alpha$ such that $1_{P_\alpha} \Vdash \dot{h}_\alpha \colon \check{\aleph}_1 \to \{A \subseteq \mathbb{R} | A \text{ is closed}\}$ is surjective. $\dot{h}_\alpha(\check{\xi})$ is a name for the ξ-th closed set, from the perspective of P_α. We can then choose P_α-names $\dot{Q}_\alpha$, $\dot{\leqslant}_\alpha$ such that
(3) $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) = (\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \emptyset)$;
(4) $\mathrm{dom}(\dot{Q}_\alpha) \subseteq \{\dot{h}_\alpha(\check{\xi}) | \xi < \aleph_1\}$.
Then
(5) $\mathrm{card}(P_{\alpha+1}) \leqslant \mathrm{card}(P_\alpha) \cdot \mathrm{card}(\mathrm{dom}(\dot{Q}_\alpha)) \leqslant \aleph_1$ and $P_{\alpha+1}$ satisfies the countable chain condition;
(6) $1_{P_{\alpha+1}} \Vdash \mathrm{CH}$: since $\mathrm{card}(P_{\alpha+1}) \leqslant \aleph_1$ and $P_{\alpha+1}$ is ccc the number of canonical $P_{\alpha+1}$-names for reals is

$$\leqslant \mathrm{card}({}^\omega({}^\omega(P_{\alpha+1}))) \leqslant \aleph_1^{\aleph_0 \cdot \aleph_0} = \aleph_1$$

using CH and the HAUSDORFF recursion formula.

For the limit case assume that $\alpha \leqslant \kappa$ is a limit ordinal and that $((P_{\alpha'}, \leqslant_{\alpha'}, 1_{\alpha'}) | \alpha' \leqslant \alpha)$ and $((\dot{Q}_{\alpha'}, \dot{\leqslant}_{\alpha'}) | \alpha' < \alpha)$ are defined so that for $\alpha' < \alpha$
(7) $\mathrm{card}(P_{\alpha'}) \leqslant \aleph_1$ and $P_{\alpha'}$ satisfies the countable chain condition;
(8) $1_{P_{\alpha'}} \Vdash \mathrm{CH}$.

Case 1: $\alpha < \kappa$. Since finite support iterations at limit stages are basically unions of the previous stages,

$$\mathrm{card}(P_\alpha) \leqslant \sum_{\alpha' < \alpha} \mathrm{card}(P_{\alpha'}) \leqslant \sum_{\alpha' < \alpha} \aleph_1 \leqslant \aleph_1 \cdot \aleph_1 = \aleph_1 .$$

P_α has the ccc by the corresponding iteration theorem. Concerning CH, the number of canonical P_α-names for reals is

$$\leqslant \mathrm{card}({}^\omega({}^\omega(P_\alpha))) \leqslant \aleph_1^{\aleph_0 \cdot \aleph_0} = \aleph_1 .$$

This means that the iteration up to $P_\kappa = P_{\aleph_2}$ can be continued.
Case 2: $\alpha = \kappa$. Then

$$\mathrm{card}(P_\kappa) \leqslant \sum_{\alpha < \kappa} \mathrm{card}(P_\alpha) \leqslant \sum_{\alpha < \aleph_2} \aleph_1 \leqslant \aleph_1 \cdot \aleph_2 = \aleph_2 .$$

The number of canonical P_κ-names for reals is

$$\leqslant \mathrm{card}({}^\omega({}^\omega(P_\kappa))) \leqslant \aleph_2^{\aleph_0 \cdot \aleph_0} = \aleph_2 .$$

Fix a ground model M of ZFC + CH and define the above iteration $((P_\alpha, \leqslant_\alpha, 1_\alpha) | \alpha \leqslant \kappa)$ with $\kappa = \aleph_2^M$ within M. Let G_κ be M-generic on P_κ. We study the model $M[G_\kappa]$. For $\alpha < \kappa$, the set $G_\alpha = \{p \upharpoonright \alpha \,|\, p \in G_\kappa\}$ is M-generic on P_α. Since $1_\alpha \Vdash_{P_\alpha} (\dot{Q}_\alpha, \dot{\leqslant}_\alpha, \emptyset) = (\boldsymbol{B}, \leqslant_{\boldsymbol{B}}, \emptyset)$,

$$\dot{Q}_\alpha^{M[G_\alpha]} = \boldsymbol{B}^{M[G_\alpha]}.$$

So $H_\alpha = \{p(\alpha)^{M[G_\alpha]} | p \in G_\kappa\}$ is random generic over the model $M[G_\alpha]$. The associated random real $r_\alpha = r_{H_\alpha}$ satisfies

$$r_\alpha \notin M[G_\alpha].$$

Lemma 81.

a) *Cardinals are absolute between M and $M[G_\kappa]$.*

b) $M[G_\kappa] \vDash 2^{\aleph_0} = \aleph_2$.

Proof. *a*) is implied by P_κ having the ccc. Since there are $\leqslant \aleph_2^M$ canonical P_κ-names for reals, $M[G_\kappa] \vDash 2^{\aleph_0} \leqslant \aleph_2$. On the other hand $(r_\alpha|\alpha < \kappa) \in M[G_\kappa]$ is a sequence of pairwise disjoint reals implies that $M[G_\kappa] \vDash 2^{\aleph_0} \geqslant \aleph_2$. □

We now study the distribution of $\aleph_1$'s and $\aleph_2$'s in the CICHON diagram within the model $M[G_\kappa]$.

Lemma 82. *In* $M[G_\kappa]$, $\mathrm{cov}(\mathcal{N}) = \aleph_2$.

Proof. Consider a sequence $(N_\xi|\xi < \lambda)$, $\lambda < \kappa$ of measure zero sets in $M[G_\kappa]$. For each $\xi < \lambda$ pick a G_δ-code g_ξ such that $N_\xi \subseteq g_\xi^{M[G_\kappa]}$ and $g_\xi^{M[G_\kappa]}$ is a measure zero set. A G_δ-code is basically a countable set of rational numbers. Pick a P_κ-name $\dot{g}_\xi \in M$, $g_\xi = \dot{g}_\xi^{G_\kappa}$ of the form

$$\{(\check{r}, p)|r \in \mathbb{Q}, p \in A_{\xi,r}\}$$

where each $A_{\xi,r}$ is a countable antichain in P_κ. Take some $\alpha < \lambda$ such that

$$\forall \xi < \lambda \forall r \in \mathbb{Q} \forall p \in A_{\xi,r}: \mathrm{supp}(p) \subseteq \alpha.$$

Let $\xi < \lambda$. Then

$$\begin{aligned} g_\xi &= \dot{g}_\xi^{G_\kappa} \\ &= \{r \in \mathbb{Q}|\exists p \in A_{\xi,r}(p \in G_\kappa \wedge (\check{r}, p) \in \dot{g}_\xi)\} \\ &= \{r \in \mathbb{Q}|\exists p \in A_{\xi,r}(p \upharpoonright \alpha \in G_\alpha \wedge (\check{r}, p) \in \dot{g}_\xi)\} \in M[G_\alpha] \end{aligned}$$

So all the codes g_ξ occur in $M[G_\alpha]$. The real r_α is $M[G_\alpha]$-generic on $\boldsymbol{B}^{M[G_\alpha]}$. Since r_α avoids every measure zero set in $M[G_\alpha]$

$$r_\alpha \notin g_\xi^{M[G_\alpha]}.$$

By the absoluteness of this property,

$$r_\alpha \notin g_\xi^{M[G_\kappa]}.$$

This means that

$$r_\alpha \notin \bigcup_{\xi<\lambda} g_\xi^{M[G_\kappa]}{}_{\xi<\lambda} \supseteq \bigcup_{\xi<\lambda} N_\xi.$$

This means that less than κ measure zero sets are not sufficient to cover all the reals. Hence in $M[G_\kappa]$, $\mathrm{cov}(\mathcal{N}) = \kappa = 2^{\aleph_0}$. □

To show that $\mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$ we first show that one can extract COHEN reals from nearly any finite support iteration.

Lemma 83. *Let M be a ground model and, within M, let $((P_\alpha, \leqslant_\alpha, 1_\alpha)|\alpha \leqslant \kappa)$ be a finite support iteration of the sequence $((\dot{Q}_\beta, \dot{\leqslant}_\beta)|\beta < \kappa)$ where λ is a limit ordinal. Assume that for $\alpha < \kappa$ there are $\dot{a}_\beta, \dot{b}_\beta \in \mathrm{dom}(\dot{Q}_\beta)$ such that*

$$1_\alpha \Vdash \dot{a}_\beta \in \dot{Q}_\beta \wedge \dot{b}_\beta \in \dot{Q}_\beta \wedge \dot{a}_\beta \perp \dot{b}_\beta.$$

Let G_κ be M-generic on P_κ and let $\alpha<\kappa$. We know from before that $G_\alpha=\{p\restriction\alpha\,|\,p\in G_\kappa\}$ is M-generic on P_α and $H_\alpha=\{p(\alpha)^{G_\alpha}|p\in G_\kappa\}$ is $M[G_\alpha]$-generic on $\dot{Q}_\alpha^{G_\alpha}$. Now for $\alpha<\kappa$ there is $C\in G_\kappa$ which is $M[G_\alpha]$-generic on the COHEN *forcing* $\mathrm{Fn}(\omega,2,\aleph_0)$.

Proof. Define

$$C=\{c\in\mathrm{Fn}(\omega,2,\aleph_0)\,|\,\forall n\in\mathrm{dom}(c)\,(c(n)=1\leftrightarrow\dot{a}_{\alpha+n}^{G_{\alpha+n}}\in H_{\alpha+n})\}\in M[G_{\alpha+\omega}]\subseteq M[G_\alpha].$$

(1) C is a filter on $\mathrm{Fn}(\omega,2,\aleph_0)$.
(2) C is $M[G_\alpha]$-generic on $\mathrm{Fn}(\omega,2,\aleph_0)$.
Proof. Let $D\in M[G_\alpha]$ be dense in $\mathrm{Fn}(\omega,2,\aleph_0)$. Take $\dot{D}\in M$ such that $\dot{D}^{G_\alpha}=D$, and $q\in G_\alpha$ such that $q\Vdash\dot{D}$ is dense in $\mathrm{Fn}(\omega,2,\aleph_0)$. Take $p\in G_\alpha$ such that $p\restriction\alpha=q$. Define

$$\begin{aligned}D'=\{\,p'\in P_\kappa\;\;|\;\;&\exists c\in\mathrm{Fn}(\omega,2,\aleph_0)\,((p'\restriction\alpha\Vdash\check{c}\in\dot{D})\wedge\\ &\forall n\in\mathrm{dom}(c)\,(c(n)=1\rightarrow p'\restriction(\alpha+n)\Vdash p'(\alpha+n)\leqslant_{\alpha+n}\dot{a}_{\alpha+n})\wedge\\ &\forall n\in\mathrm{dom}(c)\,(c(n)=0\rightarrow p'\restriction(\alpha+n)\Vdash p'(\alpha+n)\perp_{\alpha+n}\dot{a}_{\alpha+n})\,\}.\end{aligned}$$

We show that D' is dense in P_κ below p. Consider $q\leqslant_\kappa p$. Let

$$N=\max\{n<\omega\,|\,\alpha+n\in\mathrm{supp}(q)\}.$$

Choose $q_N\leqslant_\kappa q$ such that

$$q_N\restriction(\alpha+N)\Vdash q_N(\alpha+N)\dot{\leqslant}_{\alpha+N}\dot{a}_{\alpha+N}\text{ or }q_N\restriction(\alpha+N)\Vdash q_N(\alpha+N)\perp_{\alpha+N}\dot{a}_{\alpha+N}$$

and $\mathrm{supp}(q_N)\cap(\alpha+N,\alpha+\omega)=\emptyset$. Recursively continue to choose $q_{N-i-1}\leqslant_\kappa q_{N-i}$ such that

$$q_{N-i-1}\restriction(\alpha+(N-i-1))\Vdash q_{N-i-1}(\alpha+(N-i-1))\dot{\leqslant}_{\alpha+(N-i-1)}\dot{a}_{\alpha+(N-i-1)}$$

or

$$q_{N-i-1}\restriction(\alpha+(N-i-1))\Vdash q_{N-i-1}(\alpha+(N-i-1))\perp_{\alpha+(N-i-1)}\dot{a}_{\alpha+(N-i-1)}.$$

Also arrange that $\mathrm{supp}(q_{N-i-1})\cap(\alpha+N,\alpha+\omega)=\emptyset$. Define $d\colon N+1\rightarrow2$ by

$$d(n)=1\text{ iff }q_0\restriction(\alpha+n)\Vdash q_0(\alpha+n)\dot{\leqslant}_{\alpha+n}\dot{a}_{\alpha+n}.$$

Since $q_0\restriction\alpha\Vdash\dot{D}$ is dense in $\mathrm{Fn}(\omega,2,\aleph_0)$, take $c\in\mathrm{Fn}(\omega,2,\aleph_0)$, $c\supseteq d$ and $q^*\leqslant q_0\restriction\alpha$ such that

$$q^*\Vdash\check{c}\in\dot{D}.$$

Define $p'\in P_\kappa$ by

$$p'(\beta)=\begin{cases}q^*(\beta),\text{ if }\beta<\alpha\\ q_0(\beta),\text{ if }\beta\in[\alpha,\alpha+N]\\ \dot{a}_{\alpha+n},\text{ if }\beta=\alpha+n,\,n\in\mathrm{dom}(c)\setminus\mathrm{dom}(d),\text{ and }c(n)=1\\ \dot{b}_{\alpha+n},\text{ if }\beta=\alpha+n,\,n\in\mathrm{dom}(c)\setminus\mathrm{dom}(d),\text{ and }c(n)=0\\ q_0(\beta),\text{ if }\beta>\alpha+N.\end{cases}$$

Then $p'\in D'$, and so D' is dense below p.

Now let $p' \in D' \cap G_\kappa$. Take $c \in \mathrm{Fn}(\omega, 2, \aleph_0)$ as in the definition of D'. Then $p' \upharpoonright \alpha \Vdash \check{c} \in \dot{D}$ and so $c \in \dot{D}^{G_\alpha} = D$. We want to show that $c \in C$: let $n \in \mathrm{dom}(c)$. We have to show:

$$c(n) = 1 \leftrightarrow \dot{a}_{\alpha+n}^{G_{\alpha+n}} \in H_{\alpha+n}$$

Case 1: $n \in \mathrm{dom}(d)$.
Case 1.1:

$$\begin{aligned} c(n) = 1 &\Rightarrow d(n) = 1 \\ &\Rightarrow q_0 \upharpoonright (\alpha+n) \Vdash q_0(\alpha+n) \dot{\leqslant}_{\alpha+n} \dot{a}_{\alpha+n} \\ &\Rightarrow q_0(\alpha+n)^{G_{\alpha+n}} \dot{\leqslant}_{\alpha+n}^{G_{\alpha+n}} (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \text{ and } q_0(\alpha+n)^{G_{\alpha+n}} \in H_{\alpha+n} \\ &\Rightarrow (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \in H_{\alpha+n} \end{aligned}$$

Case 1.2:

$$\begin{aligned} c(n) = 0 &\Rightarrow d(n) \neq 1 \\ &\Rightarrow \text{not: } q_0 \upharpoonright (\alpha+n) \Vdash q_0(\alpha+n) \dot{\leqslant}_{\alpha+n} \dot{a}_{\alpha+n} \\ &\Rightarrow q_N \upharpoonright (\alpha+n) \Vdash q_n(\alpha+n) \perp_{\alpha+n} \dot{a}_{\alpha+n} \\ &\Rightarrow q_0(\alpha+n)^{G_{\alpha+n}} \perp_{\alpha+n} (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \text{ and } q_0(\alpha+n)^{G_{\alpha+n}} \in H_{\alpha+n} \\ &\Rightarrow (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \notin H_{\alpha+n} \end{aligned}$$

Case 2: $n \notin \mathrm{dom}(d)$.
Case 2.1:

$$\begin{aligned} c(n) = 1 &\Rightarrow p'(\alpha+n) = \dot{a}_{\alpha+n} \\ &\Rightarrow (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \in H_{\alpha+n} \end{aligned}$$

Case 2.2:

$$\begin{aligned} c(n) = 1 &\Rightarrow p'(\alpha+n) = \dot{b}_{\alpha+n} \\ &\Rightarrow (\dot{b}_{\alpha+n})^{G_{\alpha+n}} \in H_{\alpha+n} \text{ and } \dot{a}_{\alpha+n}^{G_{\alpha+n}} \perp \dot{b}_{\alpha+n}^{G_{\alpha+n}} \\ &\Rightarrow (\dot{a}_{\alpha+n})^{G_{\alpha+n}} \notin H_{\alpha+n} \end{aligned}$$

□

Lemma 84. $M[G] \vDash \mathrm{cov}(\mathcal{M}) = 2^{\aleph_0}$.

Proof. Consider a sequence $(A_\xi | \xi < \lambda)$, $\lambda < \kappa$ of meager sets in $M[G_\kappa]$. For each $\xi < \lambda$ pick a G_δ-code d_ξ for a countable intersection of dense open sets such that $A_\xi \cap w_\xi^{M[G_\kappa]} = \emptyset$. As in the proof of Lemma 82 there is $\alpha < \lambda$ such that all the codes w_ξ are elements of $M[G_\alpha]$. Take a real $r \in M[G_\kappa]$ which is $M[G_\alpha]$-generic on $\mathrm{Fn}(\omega, 2, \aleph_\omega)$. By Lemma 56,

$$r \in w_\xi^{M[G_\kappa]}$$

for all $\xi < \lambda$. Then

$$r \notin \bigcup_{\xi < \lambda} A_\xi ,$$

hence $(A_\xi|\xi<\lambda)$ does not cover $\mathbb{R}$. $\square$

Now we show that $\mathfrak{b}=\aleph_1$ by showing that ${}^\omega\omega\cap M$ is unbounded in ${}^\omega\omega\cap M[G_\kappa]$. We shall show inductively that ${}^\omega\omega\cap M$ is unbounded in ${}^\omega\omega\cap M[G_\alpha]$ for every $\alpha<\kappa$. We shall obtain this from the following approximation property between models of set theory.

Definition 85. *Let $M\subseteq N$ be transitive models of set theory. Then M* captures *N if*

$$\forall f\in{}^\omega\omega\cap N\,\exists g\in{}^\omega\omega\cap M\,\forall h\in{}^\omega\omega\cap M\,(h\leqslant^* f\to h\leqslant^* g).$$

Lemma 86. *If M* captures *N then every $\leqslant^*$-unbounded family H in M is $\leqslant^*$-unbounded in N.*

Proof. Assume that F is $\leqslant^*$-bounded in N by $f:\omega\to\omega$, $f\in N$. Let $g\in{}^\omega\omega\cap M$ as in the previous definition. Then F is $\leqslant^*$-bounded in M. $\square$

Lemma 87. *Let $M\subseteq M[H]$ be a generic extension with a forcing P which is ${}^\omega\omega$-bounding. Then M captures $M[H]$.*

Proof. Let $f\in{}^\omega\omega\cap N$. By the bounding property take $g\in{}^\omega\omega\cap M$ such that $f\leqslant^* g$. Then $h\leqslant^* f$ implies $h\leqslant^* g$ by the transitivity of $\leqslant^*$. $\square$

So the ground model captures the extension if one takes a finite iterate of random forcing. We show that one can also get across limit stages.

Lemma 88. *Let M be a ground model and let $((P_\alpha,\leqslant_\alpha,1_\alpha)|\alpha\leqslant\kappa)\in M$ be the finite support iteration of the sequence $((\dot{Q}_\alpha,\dot{\leqslant}_\alpha)|\alpha<\kappa)\in M$ where κ is a limit ordinal. Assume that*

$$\forall\alpha<\kappa\colon 1_\alpha\Vdash_{P_\alpha}\dot{Q}_\alpha \text{ has the countable chain condition.}$$

Let G_κ be M-generic on P_κ and let $M[G_\alpha]$ with $G_\alpha=\{p\restriction\alpha\,|\,p\in G_\kappa\}$ be the sequence of intermediate models. Assume that M captures $M[G_\alpha]$ for $\alpha<\kappa$.

Then M captures $M[G_\kappa]$.

Proof. P_κ satisfies the countable chain condition. If $\operatorname{cof}(\kappa)>\omega$ then every function $f\in{}^\omega\omega\cap M[G_\kappa]$ is an element of $M[G_\alpha]$ for some $\alpha<\kappa$. Since M captures $M[G_\alpha]$, there is a function $g\in{}^\omega\omega\cap M$ capturing f.

So we can assume that $\operatorname{cof}(\kappa)=\omega$. Let $(\kappa_n|n<\omega)\in{}^\omega\kappa$ be cofinal in κ. Consider $f\in{}^\omega\omega\cap M[G_\kappa]$. Take a name $\dot{f}\in M$ such that $\dot{f}^{G_\kappa}=f$. Assume without loss of generality that $1_\kappa\Vdash\dot{f}:\omega\to\omega$. Let $\prec$ be a wellorder of P_κ.

Let $n<\omega$. In $M[G_{\kappa_n}]$ define a function $f_n:\omega\to\omega$ by setting $f_n(i)=j$ iff

$$\exists p\in P_\kappa(p\restriction\kappa_n\in G_{\kappa_n}\wedge p\Vdash\dot{f}(i)=j\wedge\forall p'\in P_\kappa\forall k<\omega((p'\prec p\wedge p'\restriction\kappa_n\in G_{\kappa_n})\to\neg p'\Vdash\dot{f}(i)=k)).$$

Since M captures $M[G_{\kappa_n}]$ take $g_n \in {}^{\omega}\omega \cap M$ such that

$$\forall h \in {}^{\omega}\omega \cap M \, (h \leqslant^* f_n \rightarrow h \leqslant^* g_n).$$

We can choose g_n minimal in some wellorder of ${}^{\omega}\omega$. The preceding construction implicitely defines a function $g_*: \omega \rightarrow {}^{\omega}\omega \cap M$, $n \mapsto g_n$ with a canonical name $\dot{g}_*$ such that $1_\kappa \Vdash \dot{g}_*: \omega \mapsto ({}^{\omega}\omega \cap M)\check{}$. Since P_κ satisfies the countable chain condition, one can find a *countable* subsets $W \in M$, $W \subseteq {}^{\omega}\omega \cap M$ such that $1_{P_\kappa} \Vdash \dot{g}_*: \omega \mapsto \check{W}$.

Take $g \in {}^{\omega}\omega \cap M$ such that $\forall g' \in W: g' \leqslant^* g$. We show that g captures f.

Consider $h \in {}^{\omega}\omega \cap M$ such that $h \leqslant^* f$. Take $q \in G_\kappa$ and $k < \omega$ such that

$$q \Vdash \forall l < \omega \, (l > k \rightarrow \check{h}(l) \leqslant \dot{f}(l)).$$

Take $n < \omega$ such that $\operatorname{supp}(q) \subseteq \kappa_n$.
(1) $h \leqslant^* f_n$.
Proof. Argue in $M[G_{\kappa_n}]$. Let $i \in (k, \omega)$ and let $f_n(i) = j$. Take $p \in P_\kappa$ such that $p \upharpoonright \kappa_n \in G_{\kappa_n}$ and $p \Vdash \dot{f}(i) = j$. Then q and p are compatible in P_κ and we take $r \leqslant_\kappa q, p$. Then

$$r \Vdash \check{h}(i) \leqslant \dot{f}(i) = j = (f_n(i))\check{}\,.$$

Hence $h(i) \leqslant f_n(i)$. *qed*(1)
(2) $h \leqslant^* g_n$, by (1) and since g_n "captures" f_n.

Since $g_n \in W$ we have

$$h \leqslant^* g_n \leqslant^* g.$$

Hence g captures f. □

Lemma 89. $M[G_\kappa] \vDash \mathfrak{b} = \aleph_1$.

Proof. In M, ${}^{\omega}\omega \cap M$ is trivially $\leqslant^*$-unbounded in $\leqslant^*$. By the previous lemmas, ${}^{\omega}\omega \cap M$ is $\leqslant^*$-unbounded in $M[G_\kappa]$. By CH^M, $M[G_\kappa] \vDash \operatorname{card}({}^{\omega}\omega \cap M) \leqslant \aleph_1$. □

Concerning the other entries of the Cichon diagram

$$\begin{tikzcd}
\operatorname{cov}(\mathcal{N}) \arrow[r, "\text{Th. 62}"] & \operatorname{non}(\mathcal{M}) \arrow[r, "\text{L. 50}"] & \operatorname{cof}(\mathcal{M}) \arrow[r, "\text{Th. 52}"] & \operatorname{cof}(\mathcal{N}) \\
 & \mathfrak{b} \arrow[u] \arrow[r, "\text{L. 60}"] & \mathfrak{d} \arrow[u] & \\
\operatorname{add}(\mathcal{N}) \arrow[uu, "\text{L. 50}"'] \arrow[r, "\text{Th. 52}"] & \operatorname{add}(\mathcal{M}) \arrow[u] \arrow[r, "\text{L. 50}"'] & \operatorname{cov}(\mathcal{M}) \arrow[u] \arrow[r, "\text{Th. 62}"'] & \operatorname{non}(\mathcal{N}) \arrow[uu, "\text{L. 50}"']
\end{tikzcd}$$

we getproved:

$$
\begin{tikzcd}
\mathrm{cov}(\mathcal{N})\aleph_2 \arrow[r, "\text{Th. 62}"] & \mathrm{non}(\mathcal{M})\aleph_2 \arrow[r, "\text{L. 50}"] & \mathrm{cof}(\mathcal{M})\aleph_2 \arrow[r, "\text{Th.52}"] & \mathrm{cof}(\mathcal{N})\aleph_2 \\
 & \mathfrak{b}\aleph_1 \arrow[u] \arrow[r, "\text{L. 60}"] & \mathfrak{d}\aleph_2 \arrow[u] & \\
\mathrm{add}(\mathcal{N}) \arrow[uu, "\text{L. 50}"'] \arrow[r, "\text{Th.52}"'] & \mathrm{add}(\mathcal{M}) \arrow[u] \arrow[r, "\text{L. 50}"'] & \mathrm{cov}(\mathcal{M})\aleph_2 \arrow[u, "\text{Bernhard gezeigt}"'] \arrow[r, "\text{Th. 62}"'] & \mathrm{non}(\mathcal{N})\aleph_2 \arrow[uu, "\text{L. 50}"']
\end{tikzcd}
$$

The only value in the diagram that is not yet determined by the results of the lecture is $\mathrm{add}(\mathcal{M})$. We work with a GALOIS-TUKEY connection.

Lemma 90. *There is a* GALOIS-TUKEY *reduction*

$$({}^{\omega}\omega, \leqslant^*) \preccurlyeq_T (\mathcal{M}, \subseteq).$$

by a connection

$$\begin{array}{ccc} {}^{\omega}\omega & \xleftarrow{\varphi^*} & \mathcal{M} \\ \leqslant^* & & \subseteq \\ {}^{\omega}\omega & \xrightarrow{\varphi} & \mathcal{M} \end{array}$$

This implies

Lemma 91. $\mathrm{add}(\mathcal{M}) \leqslant \mathfrak{b}$.

Proof. Let $\mathcal{F} \subseteq {}^{\omega}\omega$ with $\mathrm{card}(\mathcal{F}) < \mathrm{add}(\mathcal{M})$. It suffices to see that $\mathcal{F}$ is bounded in $\leqslant^*$. By the additivity of $\mathcal{M}$, take $F \in \mathcal{M}$ such that

$$\forall f \in \mathcal{F} \colon \varphi(f) \subseteq F.$$

Then

$$\forall f \in \mathcal{F} \colon f \leqslant^* \varphi^*(F),$$

hence $\mathcal{F}$ is $\leqslant^*$-bounded. □

Proof. (of Lemma 90) For $f \in {}^{\omega}\omega$ define $f' \colon \omega \to \omega$ by

$$f'(n) = \max\{f(j) | j \leqslant n\} + 1.$$

Define $\varphi \colon {}^{\omega}\omega \to \mathcal{M}$ by

$$\varphi(f) = \{x \in {}^{\omega}\omega \,|\, x \leqslant^* f'\}.$$

This definition is justified by
(1) $\varphi(f)$ is nowhere dense (in ${}^{\omega}\omega$), hence $\varphi(f) \in \mathcal{M}$.
Proof. Let $N_s = \{g \in {}^{\omega}\omega | s \subseteq g\}$ be a basic open set in ${}^{\omega}\omega$, where $s \in \mathrm{Fn}(\omega, \omega, \aleph_0)$. Let $s' = s \cup \{(n, f'(n) + 1)\}$ where $n \notin \mathrm{dom}(s)$. Then $N_{s'} \subseteq N_s$ and $N_{s'} \cap \varphi(f) = \emptyset$. *qed*(1)

It suffices to define the "inverse" function φ^* on an $\subseteq$-cofinal subset of $\mathcal{M}$. Note that every meager set is contained in a meager set $F = \bigcup_{n<\omega} F_n$ where every F_n is *closed* and nowhere dense. We may also assume that

$$F_0 \subseteq F_1 \subseteq \ldots$$

Now define $k_n \in \omega$ and $s_n \in {}^{<\omega}\omega$ by recursion for $n < \omega$. Set $k_0 = 0$. Assume that k_n is already defined. Then choose s_n such that

$$\forall t \in {}^{\leqslant k_n}(k_n)\, \forall i \leqslant n\colon N_{t^\frown s_n} \cap F_i = \emptyset.$$

Set

$$k_{n+1} = k_n + |s_n| + \max\{s_n(i)\colon i \in \mathrm{dom}(s_n)\} + 1.$$

Then define $\varphi^*(F) = f_F\colon \omega \to \omega$ by

$$f_F(n) = \max\{s_n(i) | i \in \mathrm{dom}(s_n)\}.$$

(2) φ, φ^* form a GALOIS-TUKEY reduction, i.e.: Assume $\varphi(f) \subseteq F$. Then $f \leqslant^* f_F$.
Proof. Suppose that $f \nleqslant^* f_F$. It suffices to prove $\varphi(f) \nsubseteq F$ by constructing $x \in \varphi(f) \setminus F$. $f \nleqslant^* f_F$ implies that there is an infinite set $Z \subseteq \omega$ where $f(n) > f_F(n)$. Let $Z = \{z_i | i < \omega\}$ where $z_0 \leqslant z_1 \leqslant \ldots$. Then define

$$x = \underbrace{\underbrace{0^\frown 0^\frown 0^\frown \ldots ^\frown 0}_{k_{z_0}} {}^\frown s_{z_0} {}^\frown 0^\frown \ldots ^\frown 0}_{k_{z_1}} {}^\frown s_{z_1} {}^\frown 0^\frown \ldots$$

By the choice of $s_{x_0}, s_{x_1}, \ldots$ we have that

$$x \notin F_0, F_1, \ldots.$$

and so
(2.1) $x \notin F$.
(2.2) $x \in \varphi(f)$.
Proof. We have to show that $x(n) \leqslant f'(n)$ for all but finitely many $n < \omega$. This is clear if $x(n) = 0$. Otherwise, $x(n) = s_{z_i}(m)$ for some i with $k_{z_i} \leqslant n$. In the latter case

$$x(n) \leqslant f_F(i) \leqslant f(i) \leqslant f'(n),$$

since ... □

15 Proper Forcing

Definition 92. $H_\lambda = \{x \mid \mathrm{card}(\mathrm{TC}(x)) < \lambda\}$. *We assume that every H_λ has a chosen wellorder* $<$.

Definition 93. $(M, \in, <) \prec (H_\lambda, \in, <)$ *iff for every* $\varphi \in \mathrm{Fml}(\in, <)$ *and every* $\vec{a} \in \mathrm{Asn}(M)$

$$(M, \in, <) \vDash \varphi[\vec{a}] \text{ iff } (H_\lambda, \in, <) \vDash \varphi[\vec{a}].$$

We simply write $M \prec H_\lambda$ instead of $(M, \in, <) \prec (H_\lambda, \in, <)$.

Definition 94. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. Let G be V-generic on P. Then define*

$$M[G] = \{x^G \mid x \in M\}.$$

This definition will relate to the notions of a generic condition and properness.

Lemma 95. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. Let G be V-generic on P. Then $H_\lambda[G] = H_\lambda^{V[G]}$ and*

$$M[G] \prec H_\lambda[G].$$

Proof. Let $x \in H_\lambda[G]$. Let $\dot{x} \in H_\lambda$ and $\dot{x}^G = x$. By the definition of the interpretation function

$$\mathrm{TC}(x) \subseteq \{\dot{y}^G \mid \dot{y} \in \mathrm{TC}(\dot{x})\}$$

Hence

$$V[G] \vDash \mathrm{card}(\mathrm{TC}(x)) \leqslant \mathrm{card}(\mathrm{TC}(\dot{x})) < \lambda$$

and $x \in H_\lambda^{V[G]}$.

Conversely, let $x \in H_\lambda^{V[G]}$. ...

□

Definition 96. *Let $M \prec H_\lambda$ and let $(P, \leqslant) \in M$ be a forcing. $q \in P$ is (M, P)-*generic *iff for every $D \in M$ which is dense in P, $D \cap M$ is* predense *below q, i.e.,*

$$\forall q_1 \leqslant q \exists q_2 \leqslant q_1 \exists d \in D \cap M q_2 \leqslant d.$$

Lemma 97. *$q \in P$ is (M, P)-*generic *iff for every $D \in M$ which is dense in P there is a P-name $\dot{p}$ such that*

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

Proof. Let $q \in P$ be (M, P)-generic and let $D \in M$ be dense in P. Let G be V-generic on P with $q \in G$. By the definition of being (M, P)-generic the set

$$\{q_2 \mid \exists d \in D \cap M q_2 \leqslant d\}$$

is dense below q. By the genericity of G take $q_2 \in G$ such that $\exists d \in D \cap M q_2 \leqslant d$. Take $p \in D \cap M$ such that $q_2 \leqslant p$. Then $p \in D \cap M \cap G$. Thus

$$q \Vdash \exists p p \in D \cap M \cap G.$$

By the maximality principle there is a P-name $\dot{p}$ such that

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

For the converse, assume the RHS of the equivalence. To show that q is (M, P)-generic consider $D \in M$ which is dense in P. Let $\dot{p}$ be a P-name such that

$$q \Vdash \dot{p} \in D \cap M \cap \dot{G}.$$

To show that $D\cap M$ is predense below q let $q_1\leqslant q$. $q_1\Vdash\dot{p}\in D\cap M\cap\dot{G}$. Take a condition $q_2\leqslant q_1$ and a $d\in D\cap M$ such that

$$q_2\Vdash\dot{p}=\check{d}\wedge\check{d}\in\dot{G}\,.$$

Then q_2 and d must be compatible in P. Take $q_3\leqslant q_2, d$. q_3 and d witness the predensity of $D\cap M$. □

Lemma 98. *A condition $q\in P$ is (M,P)-generic iff*

$$q\Vdash M\big[\dot{G}\big]\cap\mathrm{Ord}=M\cap\mathrm{Ord}\,.$$

Proof. Let $q\in P$ be $(M,\,P)$-generic. Let G be V-generic on P with $q\in G$. Let $\alpha\in M[G]\cap\mathrm{Ord}$. Take a P-name $\dot{\alpha}\in M$ such that $\alpha=\dot{\alpha}^G$. We may assume that $1_P\Vdash\dot{\alpha}\in\mathrm{Ord}$. The set

$$D=\{d\in P\,|\,\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\}\in M$$

is dense in P. By assumption, $D\cap M$ is predense below $q\in G$. So there is $d\in D\cap M\cap G$.

$$H_\lambda\vDash\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\,.$$

Since $M\prec H_\lambda$

$$M\vDash\exists\beta\in\mathrm{Ord}\,d\Vdash\dot{\alpha}=\check{\beta}\,.$$

Take $\beta\in M\cap\mathrm{Ord}$ such that $d\Vdash\dot{\alpha}=\check{\beta}$. Then $\alpha=\dot{\alpha}^G=\beta\in M$.

Conversely let $q\in P$ *not* be (M,P)-generic. Take a dense set $D\in M$ such that $D\cap M$ is *not* predense below q. Let $A\subseteq D$ be a maximal antichain with $A\in M$. Define a P-name for an ordinal by

$$\dot{\alpha}=\{(\check{\beta},a)\,|\,a\in A\text{ is the }\beta\text{-th element of }H_\lambda\text{ in the chosen wellorder of }H_\lambda\}\in M.$$

Since $D\cap M$ is not predence below q take $q_1\leqslant q$ which is incompatible with every element of $D\cap M$. Let G be V-generic with $q_1\leqslant q$. Let $\alpha=\dot{\alpha}^G$. This is due to the fact that there is $a\in A\cap G$ such that a is the α-th element of H_λ. Assume for a contradiction that $\alpha\in M\cap\mathrm{Ord}$. Then $a\in A\cap M\cap G\subseteq D\cap M\cap G$. But then q_1 is compatible with $a\in D\cap M$, contradiction. Thus

$$M[G]\cap\mathrm{Ord}\neq M\cap\mathrm{Ord}.$$

□

Definition 99. *A forcing $(P,\leqslant)$ is* proper *iff for every $\lambda>2^{\mathrm{card}(P)}$ and every countable $M\prec H_\lambda$ with $P\in M$ and every $p\in P\cap M$ there is $q\leqslant p$ which is (M,P)-generic.*

Lemma 100. *$(P,\leqslant)$ is proper iff for every $\lambda>2^{\mathrm{card}(P)}$ and every countable $M\prec H_\lambda$ with $P\in M$ and every $p\in P\cap M$ there is $q\leqslant p$ such that for every V-generic G with $q\in G$*

$$M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord}\,.$$

Theorem 101. *Let $V[G]$ be a generic extension by a proper forcing $(P,\leqslant)$. Then*

a) for every $a\in([\mathrm{Ord}]^\omega)^{V[G]}$ there is $b\in([\mathrm{Ord}]^\omega)^V$ such that $a\subseteq b$;

b) $\aleph_1^{V[G]} = \aleph_1^V$.

Proof. (*a*) Let $a \in ([\mathrm{Ord}]^\omega)^{V[G]}$ and take $\dot{f} \in V$, $\dot{f}^G\colon \omega \to a$. Take $p \in G$ such that $p \Vdash \dot{f}\colon \omega \to \mathrm{Ord}$. Take $\lambda \in \mathrm{Card}$ sufficiently high with $p, P, \dot{f} \in H_\lambda$. Since $(P, \leqslant)$ is proper the set

$$D = \left\{q \in P \mid \text{there is a countable } M \prec H_\lambda \text{ with } p, P, \dot{f} \in M \text{ and } q \leqslant p \text{ is } (M, P)\text{-generic}\right\}$$

is dense in P below p. By the genericity of G take $q \in D \cap G$ and a countable $M \prec H_\lambda$ such that $p, P, \dot{f} \in M$ and $q \leqslant p$ is (M, P)-generic. By Lemma 98

$$M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}.$$

Set $b = M \cap \mathrm{Ord} \in ([\mathrm{Ord}]^\omega)^V$.

(1) $a \subseteq b$.

Proof. Let $x \in a$. Let $x = \dot{f}^G(n)$. By the maximality principle there is a canonical name $\dot{x} \in H_\lambda$ such that $p \Vdash \dot{x} = \dot{f}(\check{n})$. Since $M \prec H_\lambda$ we may assume $\dot{x} \in M$. Then

$$x = \dot{x}^G \in M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord} = b$$

(*b*) follows immediately from (*a*). □

Many important forcings are proper:

Lemma 102. *If* $(P, \leqslant)$ *is ccc then it is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$, $M \prec H_\lambda$ countable with $P \in M$, and $p \in P \cap M$. We show that p itself is an (M, P)-generic condition. Let G be V-generic for P with $p \in G$. It suffices to show that $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$. Let $\alpha \in M[G] \cap \mathrm{Ord}$. Take $\dot{\alpha} \in M$ such that $\alpha = \dot{\alpha}^G$ and $\Vdash \dot{\alpha} \in \mathrm{Ord}$. Let

$$A = \{\beta \in \mathrm{Ord} \mid \exists r \leqslant p . r \Vdash \dot{\alpha} = \check{\beta}\}$$

be a set of possible interpretations of $\dot{\alpha}$. We can define a function $A \to P$, $\beta \mapsto r_\beta$ such that $r_\beta \Vdash \dot{\alpha} = \check{\beta}$. $\{r_\beta \mid \beta \in A\}$ is an antichain in P. By the ccc, $\{r_\beta \mid \beta \in A\}$ is at most countable and so A is at most countable. $A \in M$ and $M \vDash A$ is countable. So $A \subseteq M$. Thus

$$\alpha \in A \subseteq M.$$

□

Lemma 103. *If* $(P, \leqslant)$ *is countably complete then it is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$, $M \prec H_\lambda$ countable with $P \in M$, and $p \in P \cap M$. Let $(x_n \mid n < \omega)$ be an enumeration of M. Define sequences $(p_n \mid n < \omega) \subseteq P \cap M$ and $(\alpha_n \mid n < \omega) \subseteq \mathrm{Ord}$ such that

$$p \geqslant p_0 \geqslant p_1 \geqslant \ldots$$

Choose a condition $p_0 \leqslant p$, $p_0 \in M$ and $\alpha_0 \in \mathrm{Ord}$ such that $p_0 \Vdash x_0 = \check{\alpha}_0$ if that is possible; otherwise let $p_0 = p$ and $a_0 = 0$. If $p_n \in P \cap M$ is defined, choose a condition $p_{n+1} \leqslant p_n$, $p_{n+1} \in M$ and $\alpha_{n+1} \in \mathrm{Ord}$ such that $p_{n+1} \Vdash x_{n+1} = \check{\alpha}_{n+1}$ if that is possible; otherwise let $p_{n+1} = p_n$ and $a_{n+1} = 0$. Note that $(\alpha_n \mid n < \omega) \subseteq M$ since every α_n is definable from p_n, $x_n \in M$.

By the countable completeness of P take $q\in P$ such that $\forall n<\omega q\leqslant p_n$. We show that q is an (M,P)-generic condition. Let G be V-generic for P with $q\in G$. It suffices to show that $M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord}$. Let $\alpha\in M[G]\cap\mathrm{Ord}$. Take $x_n\in M$ such that $\alpha=x_n^G$ and some $r\in G$ such that $r\Vdash x_n=\check{\alpha}$. By the definition of $(p_n\mid n<\omega)$ this means that $p_n\Vdash x_n=\check{\alpha}_n$. Since $r,p_n\in G$ are compatible

$$\alpha=\alpha_n\in M.$$

□

16 2-step iterations of proper forcing

Lemma 104. *Let P be a forcing and $\Vdash_P\dot{Q}$ is a forcing. Let $M\prec H_\lambda$ and $P{*}\dot{Q}\in M$. Then $(q_0,\dot{q}_1)$ is $\big(M,P{*}\dot{Q}\big)$-generic iff q_0 is (M,P)-generic and*

$$q_0\Vdash\dot{q}_1\text{ is }\big(M\big[\dot{G}\big],\dot{Q}\big)\text{-generic.}$$

Proof. Let $(q_0,\dot{q}_1)$ be $\big(M,P{*}\dot{Q}\big)$-generic. We first show that q_0 is (M,P)-generic. Let G be V-generic on P such that $q_0\in G$. It suffices to show that $M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord}$. Let H be $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G\in H$. One can check that

$$G{*}H=\big\{(p_0,\dot{p}_1)\in P{*}\dot{Q}\mid p_0\in G\text{ and }\dot{p}_1^G\in H\big\}$$

is V-generic on $P{*}\dot{Q}$ with $(q_0,\dot{q}_1)\in G{*}H$. Since $(q_0,\dot{q}_1)$ is $\big(M,P{*}\dot{Q}\big)$-generic

$$M\cap\mathrm{Ord}\subseteq M[G]\cap\mathrm{Ord}\subseteq M[G{*}H]\cap\mathrm{Ord}=M\cap\mathrm{Ord}.$$

To show that

$$q_0\Vdash\dot{q}_1\text{ is }\big(M\big[\dot{G}\big],\dot{Q}\big)\text{-generic}$$

it suffices to see that

$$V[G]\vDash\dot{q}_1^G\text{ is }\big(M[G],\dot{Q}^G\big)\text{-generic.}$$

Again take any H being $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G\in H$. One has to check that

$$M[G][H]\cap\mathrm{Ord}=M[G]\cap\mathrm{Ord}.$$

$$M[G]\cap\mathrm{Ord}\subseteq M[G][H]\cap\mathrm{Ord}=M[G{*}H]\cap\mathrm{Ord}=M\cap\mathrm{Ord}\subseteq M[G]\cap\mathrm{Ord}.$$

For the converse assume that q_0 is (M,P)-generic and

$$q_0\Vdash\dot{q}_1\text{ is }\big(M\big[\dot{G}\big],\dot{Q}\big)\text{-generic.}$$

Let $G{*}H$ be V-generic on $P{*}\dot{Q}$ such that $(q_0,\dot{q}_1)\in G{*}H$. Then G is V-generic on P such that $q_0\in G$ and H is $V[G]$-generic on $\dot{Q}^G$ such that $\dot{q}_1^G\in H$. By the assumptions,

$$M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord}\text{ and }M[G][H]\cap\mathrm{Ord}=M[G]\cap\mathrm{Ord}.$$

Together

$$M[G{*}H]\cap\mathrm{Ord}=M[G][H]\cap\mathrm{Ord}=M[G]\cap\mathrm{Ord}=M\cap\mathrm{Ord},$$

i.e., $(q_0, \dot{q}_1)$ is $\left(M, P{*}\dot{Q}\right)$-generic. $\square$

Lemma 105. *If P is proper and $\Vdash_P \dot{Q}$ is proper then $P{*}\dot{Q}$ is proper.*

Proof. Let $\lambda > 2^{\mathrm{card}(P)}$ and let $M \prec H_\lambda$ be countable with $P{*}\dot{Q} \in M$. Let $(p_0, \dot{p}_1) \in P{*}\dot{Q} \cap M$. By the properness of P take $p \leqslant p_0$ which is (M, P)-generic.

$$p \Vdash \exists q \leqslant \dot{p}_1\, q \text{ is } \left(M\left[\dot{G}_P\right], \dot{Q}\right)\text{-generic.}$$

Take $q_0 \leqslant p$ and $\dot{q}_1 \in \mathrm{dom}\left(\dot{Q}\right)$ such that

$$q_0 \Vdash \dot{q}_1 \leqslant \dot{p}_1 \text{ amd } \dot{q}_1 \text{ is } \left(M\left[\dot{G}_P\right], \dot{Q}\right)\text{-generic.}$$

By the previous Lemma, $(q_0, \dot{q}_1) \leqslant (p_0, \dot{p}_1)$ is $\left(M, P{*}\dot{Q}\right)$-generic. $\square$