LECTURE NOTES 13.-15. JANUARY 2014

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1. Iteration of proper forcing with countable support

Notation 1.1. *If P is a forcing, let $\dot{G}_P$ denote a P-name for the P-generic filter.*

Definition 1.2. *Suppose that P is a forcing and $\dot{x}, \dot{Q}$ are P-names.*

(1) *$\dot{x}$ is a* hereditarily minimal P-name *if there is no P-name $\dot{y}$ with $|tc(\dot{y}| < |tc(\dot{x})|$ and $1_P \Vdash_P \dot{x} = \dot{y}$.*
(2) *$\dot{Q}$ is a* full P-name *if for all $p \in P$, formulas φ, and P-names $\dot{y}$, if $p \Vdash_P \exists z \in \dot{Q}\ \varphi(z)$, then $\exists \dot{z} \in \operatorname{dom}(\dot{Q})\ p \Vdash_P \dot{z} \in \dot{Q} \wedge \varphi(\dot{x})$.*

Lemma 1.3. *Suppose that P is a forcing and that $\dot{Q}$ is a P-name. Let $\tilde{Q}$ denote the class of pairs $(\dot{x}, p)$ such that $\dot{x}$ is a hereditarily minimal P-name and $p \Vdash_P \dot{x} \in \dot{Q}$. Then $\tilde{Q}$ is a set, $\tilde{Q}$ is a full P-name, and $1_P \Vdash_P \dot{Q} = \tilde{Q}$.*

Definition 1.4. *(Iteration) An iteration of length γ is a pair $((P_\beta)_{\beta \leq \gamma}, (\dot{Q}_\beta)_{\beta < \gamma})$ such that for all $\beta \leq \gamma$*

(1) *If $p \in P_\beta$ and $\alpha < \beta$, then $p \upharpoonright \alpha \in P_\alpha$.*
(2) *If $\beta < \gamma$, then $\dot{Q}_\beta$ is a full P_β-name.*
(3) *If $\beta < \gamma$, then $1_{P_\beta} \Vdash_{P_\beta} \dot{Q}_\beta$ is a forcing.*
(4) *If $p \in P_\beta$ and $\alpha < \beta$, then $p(\alpha) \in \operatorname{dom}(\dot{Q})$.*
(5) *If $\beta < \gamma$, then the map $\pi \colon P_{\beta+1} \to P_\beta * \dot{Q}_\beta$, $\pi(h) = (h \upharpoonright \beta, h(\beta))$, is an isomorphism.*
(6) *If $p, q \in P_\beta$, then $p \leq_{P_\beta} q$ if and only if $(p \upharpoonright \alpha) \vDash_{P_\alpha} p(\alpha) \leq_{\dot{Q}_\alpha} q(\alpha)$ for all $\alpha < \beta$.*
(7) *If $\alpha < \beta$, then $\dot{1}_{\dot{Q}_\alpha} := 1_{P_\beta}(\alpha)$ is a $\mathbb{P}$-name for $1_{\dot{Q}_\alpha}$.*
(8) *If $p \in P_\beta$, $\alpha < \beta$, and $q \leq_{P_\alpha} (p \upharpoonright \alpha)$, then $q^\frown(p \upharpoonright [\alpha, \beta)) \in P_\beta$.*
(9) *P_β is separative, i.e. $\forall p, q \in P_\beta\ (p \not\leq q \Rightarrow \exists r \leq p\ r \perp q)$.*

We sometimes write P_γ for the iteration $((P_\beta)_{\beta \leq \gamma}, (\dot{Q}_\beta)_{\beta \leq \gamma})$. As for the two-step iteration we have the following.

Lemma 1.5. *If M is a ground model, $((P_\beta)_{\beta \leq \gamma}, (\dot{Q}_\beta)_{\beta \leq \gamma})$ is an iteration in M, and G is M-generic for P_γ, then*

(1) *$G_\beta := \{p \upharpoonright \beta \mid p \in G\}$ is M-generic for P_β.*
(2) *$g_\beta := \{\dot{q}^{G_\beta} \mid \exists p \in P_\beta\ p^\frown(\dot{q})^\frown 1_{(\beta,\gamma)} \in G\}$ is $M[G_\beta]$-generic for $\dot{Q}_\beta^G$.*

Definition 1.6. *(μ-support iteration) Suppose that μ is a regular cardinal. Suppose that $((P_\beta)_{\beta \leq \gamma}, (\dot{Q}_\beta)_{\beta \leq \gamma})$ is an iteration.*

(i) *If $p \in P_\beta$, then* $\operatorname{supp}(p) = \{\alpha < \beta \mid p(\alpha) \neq \dot{1}_{\dot{Q}_\alpha}\}$ *is the* support of p.

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(ii) *An iteration* $((P_\beta)_{\beta\leq\gamma},(\dot{Q}_\beta)_{\beta\leq\gamma})$ *is a* $<\mu$-support iteration *if for all limits* $\beta\leq\gamma$, P_β *is the set of sequences* p *satisfying the conditions for the iteration and with* $|\mathrm{supp}(p)|<\mu$.

Remark 1.7. *The finite support iteration defined earlier, together with the extra conditions of full names and separativity, is equivalent to the finite support iteration in the previous definition.*

For an iterated forcing P_γ, the maps $\pi_{\alpha,\beta}\colon V^{P_\alpha}\to V^{P_\beta}$ are defined for $\alpha\leq\beta\leq\gamma$ as for the finite support iteration.

Remark 1.8. *Suppose that* $((P_\beta)_{\beta\leq\gamma},(\dot{Q}_\beta)_{\beta\leq\gamma})$ *is a countable support iteration such that for all* $\beta<\gamma$, $1_{P_\beta}\Vdash_{P_\beta}\dot{Q}_\beta$ *is proper.*

The aim is to show that P_γ *is proper, i.e. for all* $\lambda>2^{|P_\gamma|}$, *all* $(M,\in,<)\prec(H_\lambda,\in,<)$ *with* $P_\gamma\in M$, *and all* $p\in P_\gamma\cap M$, *there is an* (M,P_γ)*-generic condition* q *with* $q\Vdash_{P_\gamma}p\in\dot{G}_{P_\gamma}$ ($\Longleftrightarrow q\leq p$ *since* P_γ *is separative).*

We show this by induction, and the inductive hypothesis is slightly stronger.

(i) q *should extend a given* (M,P_β)*-generic condition* q_0 *for a give* $\beta<\gamma$.
(ii) *instead of* $p\in P_\gamma$ *we work a name* $\dot{p}$ *for a condition in* P_γ.

The preservation of properness in countable support iterations follows from Lemma 1.10.

Lemma 1.9. *Suppose that* P *is proper and* $\dot{Q}$ *is a full* P*-name such that* $1_P\Vdash_P\dot{Q}$ *is proper. Suppose that* $(M,\in,<)\prec(H_\lambda,\in,<)$ *is countable with* $(P*\dot{Q})\in M$. *Suppose that* $q_0\in P$ *is* (M,P)*-generic,* $\dot{p},\dot{p}_0,\dot{p}_1$ *are* P*-names with* $1_P\Vdash_P\dot{p}=(\dot{p}_0,\dot{p}_1)$ *such that*

$$q_0\Vdash_P\dot{p}\in(M\cap(P*\dot{Q}))\wedge\dot{p}_0\in\dot{G}.$$

Then there is a P*-name* $\dot{q}_1$*such that* $(q_0,\dot{q}_1)$ *is* (M,P)*-generic and* $(q_0,\dot{q}_1)\Vdash_{P*\dot{Q}}\dot{p}\in\dot{G}_{P*\dot{Q}}$.

Proof. Suppose that G is V-generic on P with $q_0\in G$. Let $p=\dot{p}^G$ and $Q=\dot{Q}^G$. Then $p\in M\cap R$ and $p=(p_0,\dot{p}_1)$ with $p_0\in G$. Since $\dot{p}_1\in M$ and P is proper, we have $p_1:=\dot{p}_1^G\in M[G]\cap Q$, by considering a bijection between $\mathrm{dom}(P)$ and an ordinal in M.

We have $M[G]\prec H_\lambda[G]=H_\lambda^{V[G]}$ by Lemma 95. Since Q is proper, there is an $(M[G],Q)$-generic condition $q_1\leq p_1$ in $V[G]$. Suppose that $\dot{q}_1$ is a P-name for q_1. This uses that $\dot{Q}$ is full.

Since q_0 is (M,P)-generic by the assumption and since $q_0\Vdash_P$ "$\dot{q}_1$ is $(M[\dot{G}_P],\dot{Q})$-generic", $(q_0,\dot{q}_1)$ is $(M,P*\dot{Q})$-generic by Lemma 104. Since $q_0\Vdash_P\dot{p}_0\in\dot{G}_P$ and $q_0\Vdash_P\dot{q}_1\leq\dot{p}_1$, we have $(q_0,\dot{q}_1)\Vdash_{P*\dot{Q}}\dot{p}\in\dot{G}_{P*\dot{Q}}$. □

Lemma 1.10. *Suppose that* $((P_\beta)_{\beta\leq\gamma},(\dot{Q}_\beta)_{\beta\leq\gamma})$ *is a countable support iteration such that for all* $\beta<\gamma$, $1_{P_\beta}\Vdash_{P_\beta}\dot{Q}_\beta$ *is proper. Suppose that* $\lambda>2^{|P|}$, $(M,\in,<)\prec(H_\lambda,\in,<)$ *is countable with* $P_\gamma\in M$.

Then for every $\beta_0\in\gamma\cap M$, *every* (M,P_{β_0})*-generic condition* $q\in P_\gamma$, *and every* P_{β_0}*-name* $\dot{p}_0$ *with*

$$q_0\Vdash_{P_{\beta_0}}\dot{p}\in(P_\gamma\cap M)\wedge\dot{p}_0\restriction\beta\in\dot{G}_{P_{\beta_0}},$$

there is an (M,P_γ)*-generic condition* $q\in P_\gamma$ *such that* $q\restriction\beta_0=q_0$ *and* $q\Vdash_{P_{\beta_0}}\dot{p}_0\in\dot{G}_\gamma$.

Proof. Note that $((P_\beta)_{\beta\leq\gamma}) \in M$ and hence $P_\beta \in M$ for all $\beta \in M$. We prove the claim by induction on $\gamma \in Ord$. If γ is a successor ordinal, the claim follows from the previous lemma.

Suppose that γ is a limit ordinal. Suppose that $(\beta_n)_{n\in\omega}$ is a strictly increasing sequence of ordinals in $\gamma \cap M$ cofinal in $\gamma \cap M$ with $\beta_0 = \beta$. Suppose that $(D_n)_{n\in\omega}$ is an enumeration of all dense subsets $D \subseteq P_\gamma$ with $D \in M$.

Construction 1.11. *We will construct a sequence $(q_n, \dot{p}_n)_{n\in\omega}$ such that for all $n \in \omega$*

(1_n) (a) $q_n \in P_{\beta_n}$,
(b) $q_{n+1} \restriction \beta_n = q_n$,
(2_n) (a) $\dot{p}_n$ is a P_{β_n}-name,
(b) $\dot{p}_0 = \dot{p}$,
(3_n) (a) $q_n \Vdash_{P_{\beta_n}} \dot{p}_n \in (\dot{P}_\gamma \cap M)$,
(b) $q_n \Vdash_{P_{\beta_n}} \dot{p}_n \leq \pi_{\beta_{n-1},\beta_n}(\dot{p}_{n-1})$ if $n \geq 1$,
(c) $q_n \Vdash_{P_{\beta_n}} \dot{p}_n \in D_{n-1}$ if $n \geq 1$,
(4_n) $q_n \Vdash_{P_{\beta_n}} \dot{p}_n \restriction \beta_n \in \dot{G}_{P_{\beta_n}}$.
(5_n) q_n is (M, P_{β_n})-generic.

Suppose that we have already constructed $(q_n, \dot{p}_n)$ satisfying Conditions $(1_n) - (5_n)$.

Claim 1.12. *There is a condition $q_{n+1} \in P_{\beta_{n+1}}$ extending q_n and a P_{β_n}-name $\dot{p}_{n+1}$ satisfying the Conditions $(1_{n+1}) - (5_{n+1})$.*

Proof. Suppose that G is V-generic on P_{β_n} with $q_n \in G$. Let $p_n = \dot{p}_n^G$. Then $p_n \in P_\gamma \cap M$ and $p_n \restriction \beta_n \in G$ by (3_n)(a) and (4_n).

Let $D'_n = \{r \restriction \beta_n \in P_{\beta_n} \mid r \in D_n\}$. It is easy to check that D'_n is dense in P_{β_n}, since D_n is dense in P_γ. Since $(p_n \restriction \beta_n), D_n \in M$ and $M \prec H_\lambda$, there is some $p \leq (p_n \restriction \beta_n)$ with $p \in D'_n \cap M$. Then there is some $p_{n+1} \in P_\gamma$ with

(i) $p_{n+1} \restriction \beta_n = p_n$ and
(ii) $p_{n+1} \in D_n$.

Suppose that p'_{n+1} is a P_{β_n}-name for p_{n+1} such that q_n forces (i),(ii) for p'_{n+1}. Let $\dot{p}_{n+1} = \pi_{\beta_n,\beta_{n+1}}(p'_{n+1})$. The conditions $(1_{n+1}) - (3_{n+1})$ hold for any $q_{n+1} \in P_{\beta_{n+1}}$ extending q_n.

We have

$$q_n \Vdash_{P_{\beta_n}} (\dot{p}_{n+1} \restriction \beta_{n+1}) \in (P_{\beta_{n+1}} \cap M) \wedge (\dot{p}_{n+1} \restriction \beta_n) \in \dot{G}_{\beta_n}.$$

The statement of the lemma holds for (β_n, β_{n+1}) instead of (β, γ) by the inductive hypothesis. So there is an $(M, P_{\beta_{n+1}})$-generic condition $q_{n+1} \in P_{\beta_{n+1}}$ with $q_{n+1} \restriction \beta_{n+1} = q_n$ and $q_{n+1} \Vdash_{P_{\beta_{n+1}}} (\dot{p}_{n+1} \restriction \beta_{n+1}) \in \dot{G}_{\beta_{n+1}}$. This shows (4_{n+1}) and (5_{n+1}). □

Now let $q \in P_\gamma$ denote the limit of $(q_n)_{n\in\omega}$. Then $q \in P_\gamma$ and $q \restriction \beta_0 = q_0$.

Claim 1.13. *$q \Vdash_{P_\gamma} \dot{p}_n \in \dot{G}_\gamma$ for all $n \in \omega$.*

Proof. Suppose that G is V-generic on P_γ and let $p_n = \dot{p}_n^G$. Let $G_\alpha = \{p \restriction \alpha \mid p \in G\}$ for $\alpha \leq \gamma$. Let $\delta = \sup(M \cap \gamma)$. Since $p_n \restriction \beta_m \in G_{\beta_m} \cap M$ for all $m \geq n$ by Condition (4), we have $p_n \restriction \delta \in G_\delta$. Since $p_n \in M$ by Condition (3_n)(a), $\mathrm{supp}(p_n) \subseteq M$. Hence $p_n \in G$. □

Claim 1.14. *q is (M, P_γ)-generic.*

Proof. It is sufficient to show that $D_n \cap M$ is predense below q for all n.

Otherwise there is a V-generic filter G on P_γ with $q \in G$ and $G \cap D_n \cap M = \emptyset$. But $q \Vdash_{P_\gamma} \pi_{\beta_{n+1},\gamma}(\dot{p}_{n+1}) \in (\dot{G}_{P_{\beta_{n+1}}} \cap D_n \cap M)$ by Conditions (4_{n+1}), (3_{n+1})(c), and (3_{n+1})(a). □

This completes the proof of the lemma. □

Corollary 1.15. *If P_γ is a countable support iteration of proper forcings, then forcing with P_γ preserves ω_1.*

2. Equivalent definitions of proper forcing

Definition 2.1. *Suppose that S is an uncountable set and $\kappa > \omega$ is a cardinal. Suppose that $A \subseteq [S]^{<\kappa} = \{X \subseteq S \mid |X| < \kappa\}$ or $A \subseteq [S]^{\kappa} = \{X \subseteq S \mid |X| = \kappa\}$.*

1. *A is* unbounded *if for all $x \in [S]^{\kappa}$ (or $[S]^{\kappa}$), there is some $y \in A$ with $x \subseteq y$.*
2. *A is* closed *if for all chains $(x_\alpha)_{\alpha<\gamma}$ with $x_\alpha \subseteq x_\beta$ for $\alpha < \beta$, if $\bigcup_{\alpha<\kappa} x_\alpha \in [S]^{<\kappa}$ (or $[S]^{\kappa}$, then $\bigcup_{\alpha<\kappa} x_\alpha \in A$.*
3. *A is* directed *if for all $x, y \in A$, there is some $z \in A$ with $x \cup y \subseteq z$.*
4. *A is* stationary *if $A \cap C \neq \emptyset$ for every club $C \subseteq [S]^{<\kappa}$ (or $[S]^{\kappa}$).*
5. *If $f \colon [S]^{<\omega} \to [S]^{<\kappa}$ is a function, $x \in [S]^{<\kappa}$ is a* closure point *of f if $f(e) \subseteq x$ for all $e \in [x]^{<\omega}$. Let $C_f \subseteq [S]^{<\kappa}$ denote the club of closure points of f.*
6. *If $F \colon [S]^{<\omega} \to S$ is a function, $x \in [S]^{\omega}$ is a* closure point *of F if $F(e) \in x$ for all $e \in [x]^{<\omega}$. Let $C_F \subseteq [S]^{\omega}$ denote the club of closure points of F.*

Lemma 2.2. *Suppose that $\kappa > \omega$ is a cardinal and that $C \subseteq [S]^{<\kappa}$ is closed. If $A \subseteq C$ is directed and $|A| < \kappa$, then $\bigcup A \in C$.*

Proof. Suppose that $A = \{x_\alpha \mid \alpha < \lambda\}$ is a counterexample of minimal size $\lambda < \kappa$.

We construct a weakly increasing sequence $(A_\alpha)_{\alpha<\lambda}$ of directed subsets of A of size $< \lambda$ with $x_\alpha \in A_\alpha$. Then $\bigcup A_\alpha \in C$ for $\alpha < \kappa$, since λ is minimal. So $\bigcup A = \bigcup_{\alpha<\kappa}(\bigcup A_\alpha) \in C$. □

Lemma 2.3. *Suppose that S is an uncountable set and $\kappa > \omega$ is a regular cardinal. For every club $C \subseteq [S]^{<\kappa}$, there is a function $f \colon [S]^{<\omega} \to [S]^{<\kappa}$ such that $C_f \subseteq C$.*

Proof. We construct $f \colon [S]^{<\omega} \to [S]^{<\kappa}$ with $e \subseteq f(e) \in C$ for all $e \in [S]^{<\omega}$ and $f(e_0) \subseteq f(e_1)$ if $e_0 \subseteq e_1$ by induction on $|e|$.

We claim that $C_f \subseteq C$. If $x \in C_f$, then $x = \bigcup\{f(e) \mid e \in [x]^{<\omega}\} \in C$ by the previous lemma. □