LECTURE NOTES 11.-13. NOVEMBER 2013

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1. Consistency of Martin's axiom

Notation 1.1. *(1) If $(P, \leq_P, 1_P)$ is a partial order (if $(B, \leq, \wedge, \vee, 0, 1)$ is a complete Boolean algebra), we will simply write P (B) instead.*

(2) In an iterated forcing, let $\pi_{\beta\gamma}\colon P_\beta \to P_\gamma$, $\pi_{\beta\gamma}((p_\alpha)_{\alpha<\beta}) = (q_\alpha)_{\alpha<\gamma}$, $q_\alpha = p_\alpha$ for $\alpha < \beta$, $q_\alpha = 1$ for $\alpha \geq \beta$, denote the canonical complete embedding. Let $\pi^*_{\beta\gamma}\colon V^{P_\beta} \to V^{P_\gamma}$ denote the map induced by $\pi_{\beta\gamma}$.

Theorem 1.2. *Suppose that M is a ground model. Suppose that $2^{<\kappa} = \kappa > \omega$ in M. There is a c.c.c. forcing $(P, \leq_P, 1_P)$ in M such that for every M-generic filter G on P, MA and $2^\omega = \kappa$ hold in $M[G]$.*

Proof ideas. (1) There are at most $2^{<\kappa} = \kappa$ many counterexamples to MA.

(2) Build $M \subseteq M[G_0] \subseteq M[G_1] \subseteq ...M[G_\alpha]... \subseteq M[G]$ for $\alpha < \kappa$ and eliminate 1 counterexample in each step.

(3) Ensure that $M[G_\alpha] \vDash 2^{<\kappa} = \kappa$ for all $\alpha < \kappa$.

(4) Every forcing of size $< \kappa$ and every set of size $< \kappa$ of maximal antichains of the forcing is in $M[G_\alpha]$ for some $\alpha < \kappa$, since κ is regular.

□

Proof. We work in M. Let $h\colon \kappa \times \kappa \to \kappa$ denote Gödel pairing. Then $h(\alpha, \beta) = \gamma$ implies that $\alpha \leq \gamma$, for all $\alpha, \beta < \kappa$. The β^{th} forcing in $M[G_\alpha]$ will be used in step γ.

We define

(1) a finite support iteration $(P_\alpha, \leq_{P_\alpha}, 1_{P_\alpha})_{\alpha\leq\kappa}$ with

  (i) P_α c.c.c. and

  (ii) $|P_\alpha| < \kappa$

for all $\alpha \leq \kappa$ and

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(2) P_γ-names $\dot{F}_\gamma$ for all $\gamma < \kappa$ such that $1_{P_\gamma} \Vdash_{P_\gamma}$ "$\dot{F}_\gamma \colon \kappa \to V$ enumerates all partial orders $(P, \leq_P, 1_P)$ with $P = \lambda$ for some $\lambda < \kappa$".

(3) P_γ-names $\dot{Q}_\gamma$ for all $\gamma = h(\alpha, \beta) < \kappa$ such that $1_{P_\gamma} \Vdash_{P_\gamma}$ " if $\pi_{\alpha\gamma}(\dot{F}_\alpha)(\beta)$ is c.c.c., then $\dot{Q}_\gamma = \pi_{\alpha\gamma}(\dot{F}_\alpha)(\beta)$, otherwise $|\dot{Q}_\gamma| = 1$"

Suppose that $\gamma < \kappa$ and $\dot{F}_\gamma$, $\dot{Q}_\alpha$ are defined for all $\alpha < \gamma$.

To define $\dot{F}_\gamma$, note that $1_{P_\gamma} \Vdash_{P_\gamma} 2^{<\kappa} = \kappa$, since there are only $(|P_\gamma|^\omega)^\lambda \leq \kappa$ many nice P_γ-names for subsets of cardinals $\lambda < \kappa$ (as in Lemma 80, Models of Set Theory 1).

Choose $\dot{F}_\gamma$ with (2) by the Maximality Principle (Problem 36, Models of Set Theory I).

To define $\dot{Q}_\gamma$, suppose that $\gamma = h(\alpha, \beta)$. Choose a P_γ-name $\dot{Q}_\gamma$ with (3) by the Maximality Principle. Since $1_P \Vdash_{P_\gamma} \dot{Q}_\gamma$ has domain $< \kappa$, we can choose a nice name $\dot{Q}_\gamma$ with $|Q_\gamma| < \kappa$.

Now suppose that G is M-generic for P_κ. Let $G_\alpha := \pi_{\alpha\kappa}^{-1}[G]$ for $\alpha < \kappa$.

Claim 1.3. *$M[G] \vDash MA_\lambda$ for all $\lambda < \kappa$.*

Proof. We work in $M[G]$. (It is sufficient to prove MA_λ for c.c.c. partial orders with domain λ for cardinals $\lambda < \kappa$, by a previous lemma.)

Suppose that $(P, \leq_P, 1_P)$ is a c.c.c. partial order with $P = \lambda < \kappa$ and that $\mathcal{D}$ is a set of dense subsets of P with $|\mathcal{D}| \leq \lambda$.

Then $P, \mathcal{D} \in M[G_\alpha]$ for some $\alpha < \kappa$ by a previous lemma. Then $P = \dot{F}_\alpha^{G_\alpha}(\beta)$ for some $\beta < \kappa$ by (2).

Let $\gamma = h(\alpha, \beta)$. Note that P is c.c.c. in $M[G_\gamma]$, since P is c.c.c. in $M[G]$. Then $P = \pi_{\alpha\gamma}(\dot{F}_\alpha)^{G_\gamma}(\beta) = \dot{Q}_\gamma^{G_\gamma}$ by (3).

So there is a $M[G_\gamma]$-generic filter for P in $M[G_\gamma]$. Since $\mathcal{D} \in M[G_\alpha] \subseteq M[G_\gamma]$, the filter is $\mathcal{D}$-generic. □

Claim 1.4. *$M[G] \vDash 2^\omega = \kappa$.*

Proof. We have $2^{<\kappa} = \kappa$ in $M[G]$, since $|P_\kappa| \leq \kappa$ and hence there are $\leq \kappa$ "nice names" for subsets of $\lambda < \kappa$. Moreover MA_λ implies that $2^\omega = 2^\lambda > \lambda$ for all $\lambda < \kappa$, so $2^\omega \geq \kappa$ in $M[G]$. □

□

2. Martin's axiom and generic Σ_1 absoluteness of H_{ω_2}

Definition 2.1. *Suppose that $\kappa > \omega$ is a cardinal and that Γ is a class of partial orders.*

(i) $BFA_\kappa(\Gamma)$ postulates that for all $P \in \Gamma$, there is a $\mathcal{D}$-generic filter on P for any set $\mathcal{D}$ of maximal antichains in P of size $\leq \kappa$ with $|\mathcal{D}| \leq \kappa$.

(ii) If P is a partial order, let $BFA_\kappa(P) := BFA_\kappa(\{P\})$.

Remark 2.2. *Suppose that $\kappa > \omega$ is a cradinal. If Γ is a class of forcings such that every element of Γ has the κ^+-c.c., then $BFA_\kappa(\Gamma) \iff FA_\kappa(\Gamma)$.*

In particular, BFA_{ω_1}(c.c.c.) $\iff FA_{\omega_1}$(c.c.c.) $\iff MA_{\omega_1}$.

We will only consider BFA_κ for complete Boolean algebras.

Remark 2.3. *(1) Every partial order P is densely embedded into its Boolean completion $B(P)$ (see Problem 25, Models of Set Theory 1).*

(2) Suppose that M is a ground model. We work in M. Suppose that B is a complete Boolean algebra, φ a formula, and σ a B^*-name. Let

$$[\![\varphi(\sigma)]\!] := [\![\varphi(\sigma)]\!]_{B^*} := \bigvee\{p \in B^* \mid p \Vdash^M_{B^*} \varphi(\sigma)\}.$$

Then $[\![\varphi(\sigma)]\!] \Vdash^M_{B^*} \varphi(\sigma)$ by Problem 18(c).

Lemma 2.4. *Suppose that B is a complete Boolean algebra and $\kappa > \omega$ is a cardinal. Then $BFA_\kappa(B^*)$ implies that $1_B \Vdash_{B^*} \check{\kappa}$ is a cardinal.*

Proof. Suppose that $\mu < \kappa$ and $p \Vdash_B \dot{f}\colon \check{\kappa} \to \check{\mu}$ is injective. Let

$$A_\alpha = \{[\![\dot{f}(\check{\alpha}) = \check{\beta}]\!] \in B^* \mid \beta < \mu\}.$$

Then each A_β is a maximal antichain. Suppose that G is a filter on B with $G \cap A_\beta \neq \emptyset$ for all $\beta < \kappa$. Let $f\colon \kappa \to \mu$, $f(\alpha) = \beta$ if $[\![\dot{f}(\check{\alpha}) = \check{\beta}]\!] \in G$. Then f is injective, contradiction. □

We will now use $BFA_\kappa(B(P)^*)$ to reconstruct the first order theory of a structure with domain κ.

Suppose that M is a ground model. We work in M. Suppose that P is a partial order, $\kappa > \omega$ is a cardinal, $(\dot{R}_\alpha)_{\alpha<\kappa}$ is a sequence of P-names for relations on κ, and $\dot{M}$ is a P-name for the structure $(\kappa, \dot{R}_\alpha)_{\alpha<\kappa}$.

Definition 2.5. *Suppose that in M, G^* is a filter on P. Let*

(1) $\dot{R}_\alpha[G^*] = \{s \in \kappa^{<\omega} \mid \exists p \in G^* \ p \Vdash_P (\dot{M} \vDash \dot{R}_\alpha(\check{s}))\}$.
(2) $\dot{M}[G^*] = (\kappa, \dot{R}_\alpha[G^*])_{\alpha<\kappa}$.

Lemma 2.6. *We work in M. There is a set $\mathcal{D}^*$ of maximal antichains in $B(P)$ of size $\leq \kappa$ with $|\mathcal{D}^*| \leq \kappa$ such that for every $\mathcal{D}^*$-generic filter G^* on $B(P)$, every formula $\varphi(x_0, ..., x_n)$, and $\alpha_0, ..., \alpha_n < \kappa$*

$$\dot{M}[G^*] \vDash \ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n) \Longleftrightarrow \exists p \in G^* \ p \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)).$$

Proof. For $\alpha_0, ..., \alpha_n < \kappa$ and $\ulcorner\varphi\urcorner(x_0, ..., x_n)$ let

$$A_{\ulcorner\varphi\urcorner, \alpha_0, ..., \alpha_n} = \{[\![\dot{M} \vDash \ulcorner\neg\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)]\!], [\![\dot{M} \vDash \ulcorner\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n]\!]\}.$$

For $\ulcorner\psi(x, x_0, ..., x_n)\urcorner$ and $\alpha_0, ..., \alpha_n < \kappa$ let

$$A_{\exists, \ulcorner\psi\urcorner, \alpha_0, ..., \alpha_n} = \{[\![\dot{M} \vDash \ulcorner\neg\exists x\psi\urcorner(x, \check{\alpha}_0, ..., \check{\alpha}_n)]\!]\} \cup \{[\![\dot{M} \vDash \ulcorner\psi\urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n]\!] \mid \alpha < \kappa\}.$$

Let $\mathcal{D}^* = \{A_{\ulcorner\varphi\urcorner, \alpha_0, ..., \alpha_n}, A_{\exists, \ulcorner\psi\urcorner, \alpha_0, ..., \alpha_n} \mid \ulcorner\varphi\urcorner \text{ a formula}, \alpha_0, ..., \alpha_n < \kappa\}$.

We prove the claim by induction on (codes for) formulas $\ulcorner\varphi\urcorner$.

For atomic formulas, this holds by the definition of $\dot{M}[G^*]$.

For conjunctions, if $\dot{M}[G^*] \vDash \ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner\psi\urcorner(\beta_0, ..., \beta_k)$, then $\exists p, q \in G^* \ p \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n)$, $q \Vdash_P (\dot{M} \vDash \ulcorner\psi\urcorner(\beta_0, ..., \beta_k)$. Let $r \leq p, q$ in G^*. Then $r \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner\psi\urcorner(\beta_0, ..., \beta_k))$.

If $p \in G^*$ and $p \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n) \wedge \ulcorner\psi\urcorner(\check{\beta}_0, ..., \check{\beta}_k))$, then $M[G^*] \vDash \ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n) \wedge \ulcorner\psi\urcorner(\beta_0, ..., \beta_k)$.

For negations, we have $\dot{M}[G^*] \vDash \neg\ulcorner\varphi\urcorner(\alpha_0, ..., \alpha_n) \Longleftrightarrow \neg\exists p \in G^* \ p \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n)) \Longleftrightarrow \exists p \in G^* \ p \Vdash_P (\dot{M} \vDash \neg\ulcorner\varphi\urcorner(\check{\alpha}_0, ..., \check{\alpha}_n))$, since $G^* \cap A_{\ulcorner\varphi\urcorner, \alpha_0, ..., \alpha_n} \neq \emptyset$.

For existential quantifiers, if $\dot{M}[G^*] \vDash \exists x \ulcorner\varphi\urcorner(x, \alpha_0, ..., \alpha_n)$, then there is some $\alpha < \kappa$ with $\dot{M}[G^*] \vDash \ulcorner\varphi\urcorner(\alpha, \alpha_0, ..., \alpha_n)$. So there is some $p \in G^*$ with $p \Vdash_P (\dot{M} \vDash \ulcorner\varphi\urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n))$ and hence $p \Vdash_P (\dot{M} \vDash \exists x \ \ulcorner\varphi\urcorner(x, \check{\alpha}_0, ..., \check{\alpha}_n))$.

If $p \Vdash_P (\dot{M} \vDash \exists x \ \ulcorner\psi\urcorner(x, \vec{\sigma}))$ for some $p \in G^*$, then there is some $\alpha < \kappa$ with $p \Vdash_P (\dot{M} \vDash \ulcorner\psi\urcorner(\check{\alpha}, \check{\alpha}_0, ..., \check{\alpha}_n))$, since $G^* \cap A_{\exists, \ulcorner\psi\urcorner, \alpha_0, ..., \alpha_n} \neq \emptyset$. Then $\dot{M}[G^*] \vDash \ulcorner\varphi\urcorner(\alpha, \alpha_0, ..., \alpha_n)$ by the inductive hypothesis, so $\dot{M}[G^*] \vDash \ulcorner\exists x \ \varphi\urcorner(x, \alpha_0, ..., \alpha_n)$. □

Lemma 2.7. *Suppose that in M, $BFA_\kappa(B(P)^*)$ holds and that $1_P \Vdash_P (\check{\kappa}, \dot{R}_0)$ is wellfounded. Then there is a set $\mathcal{D}^*$ of maximal antichains in P of size $\leq \kappa$ with $|\mathcal{D}^*| \leq \kappa$ such that for every $\mathcal{D}^*$-generic filter G^* on P, $(\kappa, \dot{R}_0[G^*])$ is wellfounded.*

Proof. We work in M. For each $\alpha < \kappa$, let $\dot{r}_\alpha$ denote a name for the rank function on $(\alpha, \dot{R}_0 \cap (\alpha \times \alpha))$, i.e.

$$1_p \Vdash_P \dot{r}_\gamma \colon \check{\gamma} \to Ord,\ \forall \beta < \gamma\ \dot{r}_\gamma(\beta) = \sup\{\dot{r}_\gamma(\alpha) + 1 \mid (\alpha, \beta) \in \dot{R}_0\}.$$

Since $BFA_\kappa(B(P)^*)$ implies that $1_P \Vdash_P \check{\kappa} \in Card$, we have $1_p \Vdash_P \dot{r}_\alpha \colon \check{\alpha} \to \check{\kappa}$. Let

$$A_{\alpha,\beta} = \{[\![\dot{r}_\alpha(\check{\beta}) = \check{\gamma}]\!] \mid \gamma < \kappa\}$$

for $\alpha, \beta < \kappa$. Let $\mathcal{D}^* = \{A_{\alpha,\beta} \mid \alpha, \beta < \kappa\}$.

Suppose that G^* is a $\mathcal{D}^*$-generic filter on P. Then

$$\dot{r}_\alpha[G^*] = \{(\beta, \gamma) \mid \beta < \alpha,\ [\![\dot{\rho}_\alpha(\check{\beta}) = \check{\gamma}]\!] \in G^*\}.$$

Since $G^* \cap A_{\alpha,\beta} \neq \emptyset$ for all $\beta < \kappa$, $\dot{r}_\alpha[G^*] \colon \alpha \to \kappa$ is a well-defined function. Then $\dot{r}_\alpha[G^*]$ is order preserving from $(\alpha, \dot{R}_0[G^*] \cap (\alpha \times \alpha))$ to $(\kappa, <)$ for each $\alpha < \kappa$, by the last equation.

Since $cof(\kappa) > \omega$, this implies that $(\kappa, \dot{R}_0[G^*])$ is wellfounded. □

Definition 2.8. *(1) A formula φ is*

(i) $\Delta_0 = \Sigma_0 = \Pi_0$ if all its quantifiers are bounded.

(ii) Π_n if it is logically equivalent to a formula of the form $\neg\psi$, where ψ is a Σ_n formula.

(iii) Σ_{n+1} if it is logically equivalent to a formula of the form

$$\exists x_0, ..., x_m \psi(x_0, ..., x_m, y_0, ..., y_l),$$

where ψ is a Π_n formula.

(2) Suppose that $(M, R_\alpha, f_\alpha)_{\alpha<\kappa}$ and $(N, S_\alpha, g_\alpha)_{\alpha<\kappa}$ are structures with $M \subseteq N$ and Ψ is a set of (coded) formulas.

(i) Let $(M, R_\alpha, f_\alpha)_{\alpha<\kappa} \prec_\Psi (N, S_\alpha, g_\alpha)_{\alpha<\kappa}$ if for every (coded) formula $\ulcorner\varphi(x_0, ..., x_m)\urcorner \in \Psi$ and all $y_0, ..., y_m \in M$,

$$(M, R_\alpha, f_\alpha)_{\alpha<\kappa} \models \ulcorner\varphi(y_0, ..., y_m)\urcorner \Longleftrightarrow (N, S_\alpha, g_\alpha)_{\alpha<\kappa} \models \ulcorner\varphi(y_0, ..., y_m)\urcorner.$$

(ii) Let $M \prec N$ if $M \prec_{\Sigma_n} N$ for all $n < \omega$.

Lemma 2.9. *Suppose that $\kappa \geq \omega$ is a cardinal. There is a $\Sigma_1^{H_{\kappa^+}}$ definable surjection $h\colon \mathcal{P}(\kappa) \to H_{\kappa^+}$.*

Proof. Let $g\colon \kappa\times\kappa \to \kappa$ denote Gödel pairing. Let $f(x)$ denote $\pi(0)$, where $\pi\colon \kappa \to V$ is the transitive collapse of $(\kappa, g^{-1}[x])$, if this is wellfounded, and let $f(x) = 0$ otherwise. Then $h\colon \mathcal{P}(\kappa) \to H_{\kappa^+}$ is a $\Sigma_1^{H_{\kappa^+}}$ definable surjection. □

Theorem 2.10 (Bagaria)**.** *Suppose that M is a ground model. Suppose that P is a partial order and that $\kappa > \omega$ is a cardinal in M. The following conditions are equivalent.*

(1) $BFA_\kappa(B(P)^*)$ holds in M, and
(2) $H_{\kappa^+} \prec_{\Sigma_1} H_{\kappa^+}^{M[G]}$ for all M-generic filters G on P.

Proof. Suppose that $BFA_\kappa(B(P)^*)$ holds in M. Suppose that

$$1_P \Vdash_P^M (H_{\kappa^+} \models \ulcorner \exists x\ \varphi \urcorner (x, y_0, ..., y_n)),$$

where φ is a Δ_0 formula and $y_0, ..., y_n \in H_{\kappa^+}^M$.

Suppose that $h\colon \mathcal{P}(\kappa)^M \to H_{\kappa^+}^M$ is a $\Sigma_1^{H_{\kappa^+}^M}$ definable surjection in M. Suppose that $x_i \in \mathcal{P}(\kappa)^M$ and $h(x_i) = y_i$ for all $i \leq n$. Then

$$1_P \Vdash_P^M (H_{\kappa^+} \models \ulcorner \exists x\ \varphi \urcorner (x, h(\check{x}_0), ..., h(\check{x}_n))).$$

Let $\dot{N}$ denote a name for the transitive closure of $\{x_0, ..., x_n\}$ and a witness for the statement $\ulcorner \exists x\ \varphi \urcorner (x, h(\check{x}_0), ..., h(\check{x}_n)))$ in $H_{\kappa^+}^{M[\dot{G}]}$, where $\dot{G}$ is a name for the M-generic filter on P. Suppose that $\dot{\pi}$ is a P-name for an isomorphism $\dot{\pi}\colon (\dot{N}, \in) \to (\check{\kappa}, \dot{E})$ such that $1_P \Vdash_P \dot{\pi}(\check{\alpha}) = 2 \cdot \check{\alpha}$ for all $\alpha < \kappa$. Let $\bar{x} := \{2 \cdot \alpha \mid \alpha \in x\}$ for $x \subseteq \kappa$.

Then $1_P \Vdash_P^M (\check{\kappa}, \dot{E})$ is wellfounded and

$$1_{B(P)^*} \Vdash_{B(P)^*}^M (\dot{N} \models \ulcorner \exists x\ \varphi \urcorner (x, h(\check{\bar{x}}_0), ..., h(\check{\bar{x}}_n)).$$

We choose a set $\mathcal{D}^*$ of maximal antichains in $B(P)^*$ of size $\leq \kappa$ with $|\mathcal{D}^*| \leq \kappa$ by the previous lemmas. There is a $\mathcal{D}^*$-generic filter G^* in M, since $BFA_\kappa(B(P)^*)$ holds in M. Then

(1) $\bar{\kappa}$ is an initial segment of $Ord^{(\kappa, \dot{E}[G^*])}$,
(2) $(\kappa, \dot{E}[G^*]) \models \ulcorner \exists x\ \varphi \urcorner (x, h(\bar{x}_0), ..., h(\bar{x}_n))$ by Lemma 2.6, and
(3) the structure $(\kappa, \dot{E}[G^*])$ is wellfounded, by Lemma 2.7.

Then $\ulcorner \exists x\ \varphi \urcorner(x, x_0, ..., x_n)$ holds in the transitive collapse $N \in M$ of $(\kappa, \dot{E}[G^*])$. Since $N \prec_{\Sigma_1} H^M_{\kappa^+}$, the proof is complete.

For the other direction, suppose that in M, $\mathcal{D}$ is a set of maximal antichains in $B(P)^*$ of size $\leq \kappa$ with $|\mathcal{D}| \leq \kappa$. Suppose that Q is an elementary substructure of the Boolean algebra $B(P)$ with $\bigcup \mathcal{D} \subseteq Q$ and $|Q| \leq \kappa$. Suppose that $\pi \colon \bar{Q} \to Q$ is elementary and $\bar{Q}, \pi^{-1}(\mathcal{D}) \in H^M_{\kappa^+}$.

Suppose that G is M-generic for $B(P)^*$. Since Q is a Boolean subalgebra of $B(P)^*$, it is easy to check that $H := G \cap Q$ is a $\mathcal{D}$-generic filter on Q. Then $\bar{H} := \pi^{-1}[H]$ is a $\pi^{-1}[\mathcal{D}]$-generic filter on $\bar{Q}$. Since the existence of such a filter is a Σ_1 statement over H_{κ^+}, there is such a filter $\bar{I} \in M$. Then the upwards closure $I = \{q \in B(P)^* \mid \exists p \in I\ \pi(p) \leq q\}$ of $\pi[I]$ is a $\mathcal{D}$-generic filter in M. $\square$