L-like models with large cardinals and a quasi lower bound on the consistency strenth of PFA

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presenting joint work with Sy Friedman

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Theorem (Goldstern-Shelah, 1995)

1. *If there is a reflecting cardinal, then BPFA holds in a proper forcing extension of the universe.*
2. *If BPFA holds, then ω_2 is reflecting in L.*

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This theorem shows that the consistency strength of BPFA is exactly that of a reflecting cardinal.

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Theorem (Baumgartner, 1984)

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We want to introduce the idea of a quasi lower bound and use this idea together with L-like outer models to verify a quasi lower bound result on the consistency strength of PFA.

L-like principles and the Outer Model Programme

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- $\diamondsuit$ ($\diamondsuit_\kappa$ for every regular, uncountable κ)
- $\Box_\kappa$ for various κ, global $\Box$
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The Outer Model Programme

Basic Idea: Starting from a model of ZFC with large cardinals, obtain L-like properties in a forcing extension and preserve large cardinals.

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Theorem (Friedman, 2007)

- Con(ω-*superstrong*) $\rightarrow$ Con($GCH + \omega$-*superstrong*)
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Limitations for $\square$

- If κ is subcompact, $\square_\kappa$ fails. (Jensen)
- If κ is supercompact, $\square_\lambda$ fails for every $\lambda \geq \kappa$. (Solovay)

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Theorem (folklore, Cummings, Friedman)

Given a model of ZFC, there is a forcing extension which preserves many large cardinals (in particular all subcompacts) and satisfies the following L-like properties:

- *GCH*
- $\square_\lambda$ *for every singular* λ
- $\square$ *on the singular cardinals*

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But for our intended application on the consistency strength of PFA, we need an even more L-like model...

Condensation

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If $M \prec (L_\alpha, \in)$, *then for some* $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}, \in)$.

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Models of the form L[A]

To define our desired Condensation property, we will assume that we are in a model of the form V=L[A] where A is a function from Ord $\times$ Ord $\rightarrow$ Ord such that for any ordinal α, range$(A \restriction (\alpha \times \alpha)) \subseteq \alpha$.

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If M is a substructure of $(L_\alpha[A], \in, A)$, we say that M *condenses* if for some $\bar{\alpha} \leq \alpha$, $M \cong (L_{\bar{\alpha}}[A], \in, A)$.

Generalized Condensation Principles

Local Club Condensation

Assume $V = L[A]$. If α has uncountable cardinality κ and $\mathcal{A} = (L_\alpha[A], \in, A, \dots)$ is a structure for a countable language, then there exists a continuous chain $\langle \mathcal{B}_\gamma \colon \omega \leq \gamma < \kappa \rangle$ of condensing substructures of $\mathcal{A}$ whose domains B_γ have union $L_\alpha[A]$, each B_γ has cardinality $\operatorname{card} \gamma$ and contains γ as a subset.

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Acceptability

Assume V=L[A]. For any ordinals $\gamma \geq \delta$, if there is a new subset of δ in $L_{\gamma+1}[A]$, then

$$H^{L_{\gamma+1}[A]}(\delta) = L_{\gamma+1}[A].$$

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We say that A witnesses Local Club Condensation or Acceptability respectively in the above.

Our L-like model

Theorem (Friedman, H)

Starting with a model containing an ω-superstrong cardinal κ, we can perform a cofinality-preserving class forcing to obtain a generic extension of the form $L[A]$ such that A witnesses both Local Club Condensation and Acceptability and the ω-superstrength of κ is preserved.

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Starting with a proper class of subcompacts, we may preserve those.

Consistency of Fragments of PFA

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A notion of forcing P is said to be λ-linked if there is a function $f : P \longrightarrow \lambda$ such that two conditions p_0 and p_1 in P are compatible whenever $f(p_0) = f(p_1)$.

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PFA for $2^{\aleph_0}$-linked forcing is equiconsistent with a Σ^2_1-indescribable cardinal. A Σ^2_1-indescribable gap $[\kappa, \kappa^+)$ gives rise to a model of PFA for $(2^{\aleph_0})^+$-linked forcing.

Σ^2_1-indescribable gaps $[\kappa, \kappa^+)$ are just slightly larger than subcompacts - they are subcompact limits of subcompacts (and a little more).

A Reversal

Theorem (Neeman)

Assume V is a proper (forcing) extension of a fine structural model M and satisfies PFA for $(2^{\aleph_0})^+$-linked forcing. Then there is a Σ^2_1 indescribable gap $[\kappa, \kappa^+)$ in M.

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Our quasi lower bound result

By basically taking the contraposition of the last theorem, we obtain the following:

Theorem (Friedman, H)

It is consistent that there is a model with a proper class of subcompacts but no proper (forcing) extension satisfies PFA for $(2^{\aleph_0})^+$*-linked forcing.*

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A proper class of subcompacts is a quasi lower bound for PFA for $(2^{\aleph_0})^+$-linked forcing with respect to proper (forcing) extensions.

Impact on PFA 1/2

Definition

We say that a large cardinal property is *local* if it is preserved by sufficiently distributive forcing.

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Proof: Starting over a given model with a proper class of local large cardinals, force $\square_\kappa$ for a proper class of cardinals κ while preserving a proper class of the given local large cardinals. As PFA implies that $\square_\kappa$ fails for every $\kappa \geq \omega_2$, former cannot hold in any further set-generic extension.

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Lemma (H, 2013)

Given a supercompact cardinal, it is possible to obtain a model with a proper class of subcompacts such that no set forcing extension satisfies PFA but there is a class forcing extension that satisfies PFA.

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Any large cardinal existing in V_κ when κ is subcompact is a quasi lower bound for PFA with respect to proper (forcing) extensions.

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Theorem (Viale, Weiß, 2011)

- *Suppose κ is inaccessible and PFA is forced by a standard iteration of length κ that collapses κ to ω_2. Then κ is strongly compact.*
- *Suppose κ is inaccessible and PFA is forced by a proper standard iteration of length κ that collapses κ to ω_2. Then κ is supercompact.*

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Thus Viale and Weiss obtain higher consistency quasi lower bounds, but only with respect to the narrower class of (proper) standard iterations (in fact, what they really need are certain covering and approximation properties, which are satisfied by standard iterations, to hold between the ground model and the forcing extensions).

Open Questions 1/2

All of the above make it very plausible that the consistency strength of PFA should be some global (=non-local) large cardinal property. Examples of such a property are:

Global large cardinal properties

- Strong cardinals
- Strongly compact cardinals
- Supercompact cardinals

Question

Are there any *nice* global large cardinal properties between Woodin and strongly compact cardinals?

Open Questions 2/2

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This basically comes down to the question whether it is possible to force stronger Condensation principles by sufficiently nice forcing.

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