Models of Set Theory I, Summer 2013

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Abstract

Transitive models of set theory; the relative consistency of the axiom of choice using the hereditarily ordinal definable sets; forcing conditions and generic filters; generic extensions; ZFC in generic extensions; the relative consistency of the continuum hypothesis and of the negation of the continuum hypothesis via forcing; possible behaviours of the function 2^κ; the relative consistency of the negation of the axiom of choice.

1 Introduction

Sets are axiomatized by the Zermelo-Fraenkel axiom system ZF. Following Jech [?] these axioms can be formulated in the first-order language with one binary relation symbol $\in$ as

- *Extensionality*: $\forall z(z\in x\leftrightarrow z\in y)\rightarrow x=y$
- *Pairing*: $\exists z\forall u(u\in z\leftrightarrow u=x\vee u=y)$
- *Union*: $\exists z\forall u(u\in z\leftrightarrow\exists y(u\in y\wedge y\in x))$
- *Power*: $\exists z\forall u(u\in z\leftrightarrow\forall v(v\in u\rightarrow u\in x))$
- *Infinity*: $\exists z(\exists x(x\in z\wedge\forall y\neg y\in x)\wedge\forall u(u\in z\rightarrow\exists v(v\in z\wedge\forall w(w\in v\leftrightarrow w\in u\vee w=u))))$
- *Separation*: for every $\in$-formula $\varphi(u,p)$ postulate $\exists z\forall u(u\in z\leftrightarrow u\in x\wedge\varphi(u,p))$
- *Replacement*: for every $\in$-formula $\varphi(u,v,p)$ postulate

$$\forall u,v,v'(\varphi(u,v,p)\wedge\varphi(u,v',p)\rightarrow v=v')\rightarrow\exists y\forall v(v\in y\leftrightarrow\exists u(u\in x\wedge\varphi(u,v,p)))$$

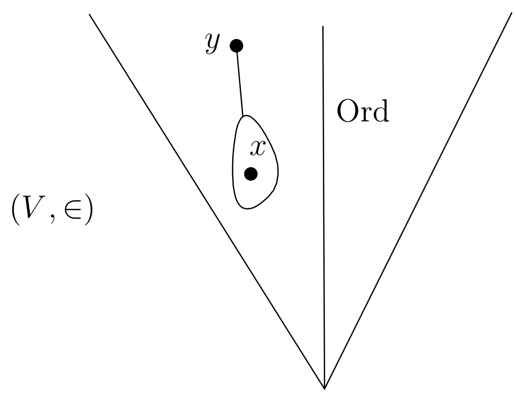
- *Foundation*: $\exists u\,u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))$

The axioms capture the basic intuitions of Cantorean set theory. They are strong enough to formalise all other mathematical fields. Usually the *Axiom of Choice* is also assumed

- *Choice* or AC: $\forall u,u'((u\in x\rightarrow\exists v\,v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in u')))\rightarrow\exists y\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))$.

ZFC is the system consisting of ZF and AC. ZF^- consists of all ZF-axioms except the powerset axiom.

We use the intuition of a standard model of set theory $(V, \in)$, the *universe* of all (mathematical) sets. This is often pictured like an upwards open triangle with the understanding that if $x \in y$ then x lies below y; x is in the *extension* of y. The ordinals are pictured by a central line, extending to infinity.



Although this picture gives some useful intuition, we can only know about sets by deduction from the ZF-axioms. On the other hand the axioms are incomplete in that they do not decide important properties of infinitary combinatorics. The most important examples that we shall also prove in this course are

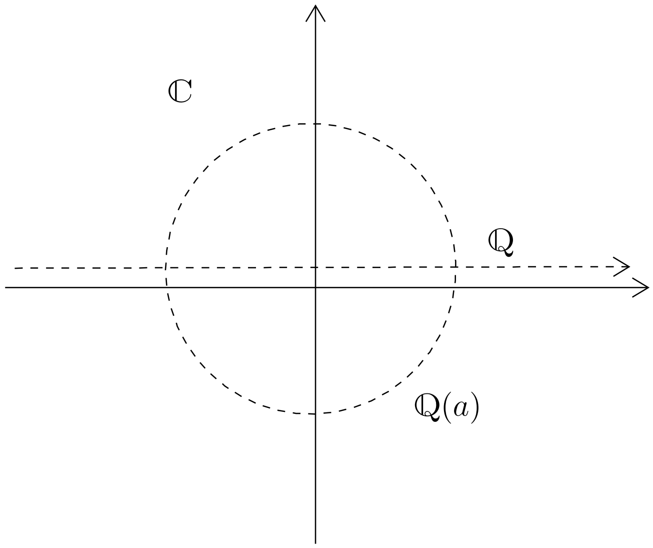
- the system ZF does not decide the axiom of choice AC: if ZF is a consistent theory, then so are $\mathrm{ZF} + \mathrm{AC}$ and $\mathrm{ZF} + \neg\mathrm{AC}$

- the system ZFC does not decide the continuum hypothesis: if ZFC is a consistent theory, then so are $\mathrm{ZFC} + \mathrm{CH}$ and $\mathrm{ZFC} + \neg\mathrm{CH}$

Here a theory is *consistent*, if it does not imply a contradiction like $x \neq x$.

We appeal to the following central fact from mathematical logic: a theory T is consistent iff it possesses a model. This allows to show consistency results by constructing models of ZF and of ZFC.

We motivate the construction methods by analogy with the construction of fields in algebra. The complex numbers $(\mathbb{C}, +, \cdot, 0, 1)$ form a standard field for many purposes.



$\mathbb{C}$ is an algebraically closed field. It contains (isomorphic copies of) many interesting fields, like the rationals $\mathbb{Q}$, or extensions of $\mathbb{Q}$ of finite degree (algebraic number fields), or extensions of $\mathbb{Q}$ of infinite degree by transcendental numbers. These subfields witness *consistency* results for the theory of fields:

- the field axioms do not decide the existence of $\sqrt{2}$: $\mathbb{Q}$ is a model of $\neg\exists x\, x\cdot x = 1+1$, whereas $\mathbb{Q}(\sqrt{2})$ is a model of $\exists x\, x\cdot x = 1+1$;

- by successively adjoining square roots one can form a field which satisfies $\forall y \exists x\, x\cdot x = y$ but which does not contain $\sqrt[3]{2}$. This is used to show that the doubling of the cube cannot be performed by ruler and compass.

Let us mention a few properties of field constructions which will have analogues in constructions of models of set theory

- interesting fields are (or can be) embedded into the standard field $\mathbb{C}$.

- the extension fields $k(a)$ can be described within the ground field k: a is either algebraic or transcendental over k; in the algebraic case one can treat a as a variable x which is a zero a certain polynomial in $k[x]$: $p(x) = 0$; in the transcendental case a corresponds to a variable x such that $p(x) \neq 0$ for all nontrivial $p \in k[x]$; calculations in $k(a)$ can be reduced to calculations in k.

- the ground field $\mathbb{Q}$ is countable. One can construct a transcendental real

$$a = 0, a_0 a_1 a_2 a_3 \ldots \in \mathbb{R}$$

by successively choosing decimals a_i so that $0,a_0a_1...a_m$ "forces" $p_n(a) \neq 0$, i.e.,

$$\forall b\,(b = 0, a_0a_1a_2a_3...a_mb_{m+1}b_{m+1}... \rightarrow p_n(b) \neq 0).$$

Here $(p_n)_{n<\omega}$ is some enumeration of $k[x]$. In view of the *forcing method* in set theory we can write this as

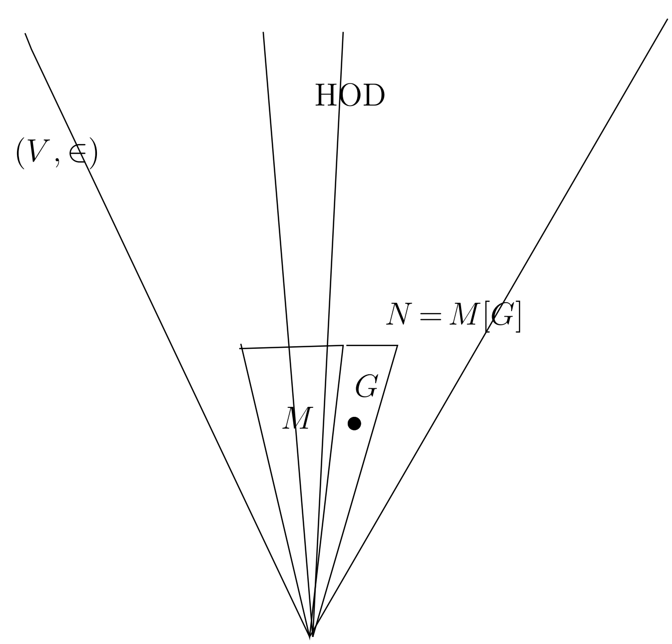
$$0, a_0a_1a_2a_3...a_m \Vdash p_n(\dot{x}) \neq 0$$

where $\dot{x}$ is a symbol or *name* for the transcendental or *generic* real to be constructed.

For models of set theory this translates into

- consider transitive submodels $(M, \in)$ of the standard universe $(V, \in)$.
- construct minimal submodels similar to the prime field $\mathbb{Q}$.
- construct *generic extensions* $N \supseteq M$ by adjoining *generic* sets G, corresponding to the transcendental numbers above: $N = M[G]$.
- G is describable in the countable ground model M by infinitely many formulas, it will be constructed by a countable recursion along countably many requirements which can be expressed inside M.

We shall consider the models HOD (*Hereditarily Ordinal Definable sets*), generic extensions $M[G]$, and *symmetric submodels* N of $M[G]$. This leads to a spectrum $\mathrm{HOD}, M, M[G], N...$ of models of set theory like



These models satisfy different extensions of the ZF-axioms: e.g., HOD $\vDash$ AC, $M[G]$ may satisfy CH or $\neg$CH, and symmetric submodels may satisfy $\neg$AC. This leads to the desired (relative) consistency results.

2 Transitive Models of Set Theory

Let W be a transitive class. We consider situations when W together with the $\in$-relation restricted to W is a model of axioms of set theory. So we are interested in the "model" $(W, \in)$ or $(W, \in\restriction W)$ where $\in\restriction W = \{(u, v)|u \in v \in W\}$. Considering W as a universe for set theory means that the quantifiers $\forall$ and $\exists$ in $\in$-formulas φ range over W instead over the full universe V. For simplicity we assume that $\in$-formulas are only formed by variables $v_0, v_1, \ldots$, the relations $=$ and $\in$, and logical signs $\neg$, $\vee$, $\exists$.

Definition 1. *Let W be a term and φ be an $\in$-formula which do not have common variables. The* relativisation *φ^W of φ to W is defined recursively along the structure of φ:*

- $(v_i \in v_j)^W \equiv (v_i \in v_j)$
- $(v_i = v_j)^W \equiv (v_i = v_j)$
- $(\neg\varphi)^W \equiv \neg(\varphi^W)$
- $(\varphi \vee \psi)^W \equiv ((\varphi^W) \vee (\psi^W))$
- $(\exists v_i\, \varphi)^W \equiv \exists v_i \in W\, (\varphi^W)$

If Φ is a collection of $\in$-formulas set $\Phi^W = \{\varphi^W | \varphi \in \Phi\}$. Instead of φ^W or Φ^W we also say "φ holds in W", "Φ holds in W", "W is a model of φ", etc.; we also write $W \vDash \varphi$ and $W \vDash \Phi$.

φ^W and Φ^W are obtained from φ and Φ by bounding all quantifiers by the class W. φ^W expresses that φ holds in the "model" $(W, \in)$. That $(W, \in)$, for $W \neq \emptyset$ behaves like a structure for 1-st order logic is expressed by

Lemma 2. *Let φ be a tautology in the language of set theory, i.e., φ is derivable in the sequent calculus. Let W be a non-empty term. Then φ^W.*

Proof. Mimick the correctness proof for the sequent calculus, proceeding by induction on the length of derivations. □

We prove criteria for set theoretic axioms to hold in W.

Lemma 3. *Assume* ZF*. Let W be a transitive class, $W \neq \emptyset$. Then*

a) $(\textit{Extensionality})^W$ *will always hold.*

b) $(\textit{Pairing})^W \leftrightarrow \forall x \in W \forall y \in W \ \{x, y\} \in W$.

c) $(\textit{Union})^W \leftrightarrow \forall x \in W \bigcup x \in W$.

d) $(\textit{Power})^W \leftrightarrow \forall x \in W \mathcal{P}(x) \cap W \in W$.

e) $(\textit{Infinity})^W \leftrightarrow \exists z \in W \ (\emptyset \in z \wedge \forall u \in z \ u + 1 \in z)$.

f) Let ψ be the instance of the Separation schema for the $\in$-formula $\varphi(x, \vec{w})$. Then

$$\psi^W \leftrightarrow \forall \vec{w} \in W \forall a \in W \ \{x \in a | \varphi^W(x, \vec{w})\} \in W.$$

g) Let ψ be the instance of the Replacement schema for the $\in$-formula $\varphi(x, y, \vec{w})$. Then ψ^W is equivalent to

$\forall \vec{w} \in W(\forall x, y, y' \in W(\varphi^W(x, y, \vec{w}) \wedge \varphi^W(x, y', \vec{w}) \rightarrow y = y') \rightarrow \forall a \in W\{y | \exists x \in a \varphi^W(x, y, \vec{w})\} \cap W \in W)$.

h) $(\textit{Foundation})^W$ *will always hold.*

i) $(\textit{Choice})^W \leftrightarrow \forall x \in W(\emptyset \notin x \wedge \forall u, u' \in x(u \neq u' \rightarrow u \cap u' = \emptyset) \rightarrow \exists y \in W \forall u \in x \exists v \ \{v\} = u \cap y)$.

Proof. *Bounded* quantications are not affected by relativisations to *transitive* classes:

(1) Let $x \in W$. Then $\forall y (y \in x \rightarrow \varphi) \leftrightarrow \forall y \in W(y \in x \rightarrow \varphi)$ and $\exists y (y \in x \wedge \varphi) \leftrightarrow \exists y \in W(y \in x \wedge \varphi)$.

Proof. Assume that $\forall y \in W(y \in x \rightarrow \varphi)$. To show $\forall y (y \in x \rightarrow \varphi)$ consider some $y \in x$. By the transitivity of W, $y \in W$. By assumption, φ holds. *qed*(1)

The following equivalences make use of (1).

a)

$$\begin{aligned}
(\text{Extensionality})^W &\leftrightarrow (\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y))^W \\
&\leftrightarrow \forall x \in W \forall y \in W [\forall z \in W (z \in x \leftrightarrow z \in y) \rightarrow x = y] \\
&\leftrightarrow \forall x \in W \forall y \in W [[\forall z \in W (z \in x \rightarrow z \in y) \wedge \forall z \in W (z \in y \rightarrow z \in x)] \rightarrow x = y] \\
&\leftrightarrow \forall x \in W \forall y \in W [[\forall z (z \in x \rightarrow z \in y) \wedge \forall z (z \in y \rightarrow z \in x)] \rightarrow x = y], \text{ by (1).}
\end{aligned}$$

The righthand side is a consequence of Extensionality in V.

b)

$$\begin{aligned}
(\text{Pairing})^W &\leftrightarrow (\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u = x \vee u = y))^W \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow u = x \vee u = y) \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W \forall u (u \in z \leftrightarrow u = x \vee u = y), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \forall y \in W \exists z \in W\, z = \{x, y\} \\
&\leftrightarrow \forall x \in W \forall y \in W\, \{x, y\} \in W.
\end{aligned}$$

c)

$$\begin{aligned}
(\text{Union})^W &\leftrightarrow (\forall x \exists z \forall u (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)))^W \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \exists y \in W (u \in y \wedge y \in x)) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u (u \in z \leftrightarrow \exists y (u \in y \wedge y \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W\, z = \bigcup x \\
&\leftrightarrow \forall x \in W \bigcup x \in W.
\end{aligned}$$

d)

$$\begin{aligned}
(\text{Power})^W &\leftrightarrow (\forall x \exists z \forall u (u \in z \leftrightarrow \forall v (v \in u \rightarrow u \in x)))^W \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \forall v \in W (v \in u \rightarrow u \in x)) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow \forall v (v \in u \rightarrow u \in x)), \text{ by (1)} \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u \in W (u \in z \leftrightarrow u \subseteq x) \\
&\leftrightarrow \forall x \in W \exists z \in W \forall u (u \in z \leftrightarrow u \in W \wedge u \subseteq x) \\
&\leftrightarrow \forall x \in W \exists z \in W\, z = \mathcal{P}(x) \cap W \\
&\leftrightarrow \forall x \in W\, \mathcal{P}(x) \cap W \in W.
\end{aligned}$$

e)

$$\begin{aligned}
(\text{Infinity})^W &\leftrightarrow (\exists z (\exists x (x \in z \wedge \forall y \neg y \in x) \wedge \forall u (u \in z \rightarrow \exists v (v \in z \wedge \forall w (w \in v \leftrightarrow w \in u \vee w = \\
&\qquad u)))))^W \\
&\leftrightarrow \exists z \in W (\exists x \in W (x \in z \wedge \forall y \in W \neg y \in x) \wedge \forall u \in W (u \in z \rightarrow \exists v \in W (v \in z \wedge \\
&\qquad \forall w \in W (w \in v \leftrightarrow w \in u \vee w = u)))) \\
&\leftrightarrow \exists z \in W (\exists x (x \in z \wedge \forall y \neg y \in x) \wedge \forall u (u \in z \rightarrow \exists v (v \in z \wedge \forall w (w \in v \leftrightarrow w \in u \vee \\
&\qquad w = u)))), \text{ by (1)} \\
&\leftrightarrow \exists z \in W (\emptyset \in z \wedge \forall u (u \in z \rightarrow u + 1 \in z)).
\end{aligned}$$

f) Separation:

$$
\begin{aligned}
(\forall\vec{w}\forall a\exists y\forall x(\,x\in y\leftrightarrow x\in a\wedge\varphi(x,\vec{w})))^W &\leftrightarrow \forall\vec{w}\in W\forall a\in W\exists y\in W\forall x\in W(\,x\in y\leftrightarrow x\in a\wedge \\
&\qquad \varphi^W(x,\vec{w})) \\
&\leftrightarrow \forall\vec{w}\in W\forall a\in W\exists y\in W\forall x(\,x\in y\leftrightarrow x\in a\wedge \\
&\qquad \varphi^W(x,\vec{w})),\text{ by (1)} \\
&\leftrightarrow \forall\vec{w}\in W\forall a\in W\exists y\in W\, y=\{x\in a|\varphi^W(x,\vec{w})\} \\
&\leftrightarrow \forall\vec{w}\in W\forall a\in W\ \{x\in a|\varphi^W(x,\vec{w})\}\in W
\end{aligned}
$$

g) Replacement:

$$
\begin{aligned}
\psi^W &= (\forall\vec{w}(\forall x,y,y'(\varphi(x,y,\vec{w})\wedge\varphi(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\exists z\forall y(y\in z\leftrightarrow\exists x(x\in a\wedge\varphi(x, \\
&\qquad y,\vec{w})))))^W \\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in W\forall y\in \\
&\qquad W(y\in z\leftrightarrow\exists x\in W(x\in a\wedge\varphi^W(x,y,\vec{w})))) \\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in \\
&\qquad W\forall y(y\in z\leftrightarrow(\exists x(x\in a\wedge\varphi^W(x,y,\vec{w}))\wedge y\in W)) \\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\exists z\in Wz= \\
&\qquad \{y|\exists x\in a\,\varphi^W(x,y,\vec{w}))\}\cap W) \\
&\leftrightarrow \forall\vec{w}\in W(\forall x,y,y'\in W(\varphi^W(x,y,\vec{w})\wedge\varphi^W(x,y',\vec{w})\rightarrow y=y')\rightarrow\forall a\in W\ \{y|\exists x\in \\
&\qquad a\,\varphi^W(x,y,\vec{w}))\}\cap W\in W).
\end{aligned}
$$

h)

$$
\begin{aligned}
(\text{Foundation})^W &\leftrightarrow (\forall x(\exists u\,u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))))^W \\
&\leftrightarrow \forall x\in W(\exists u\in W u\in x\rightarrow\exists u\in W(u\in x\wedge\forall v\in W(v\in u\rightarrow\neg v\in x))) \\
&\leftrightarrow \forall x\in W(\exists u\,u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))),\text{ by (1).} \\
&\leftarrow \forall x(\exists u\,u\in x\rightarrow\exists u(u\in x\wedge\forall v(v\in u\rightarrow\neg v\in x))) \\
&\leftrightarrow \text{Foundation in } V.
\end{aligned}
$$

i) Choice:

$$
\begin{aligned}
\mathrm{AC}^W &\leftrightarrow (\forall x(\forall u,u'((u\in x\rightarrow\exists v\,v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in \\
&\qquad u')))\rightarrow\exists y\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))))^W \\
&\leftrightarrow \forall x\in W(\forall u,u'\in W((u\in x\rightarrow\exists v\in W v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v\in \\
&\qquad W(v\in u\wedge v\in u')))\rightarrow\exists y\in W\forall u\in W(u\in x\rightarrow\exists v\in W(v\in u\wedge v\in y\wedge\forall v'\in W(v'\in \\
&\qquad u\wedge v'\in y\rightarrow v'=v))))) \\
&\leftrightarrow \forall x\in W(\forall u,u'((u\in x\rightarrow\exists v\,v\in u)\wedge(u\in x\wedge u'\in x\wedge u\neq u'\rightarrow\neg\exists v(v\in u\wedge v\in \\
&\qquad u')))\rightarrow\exists y\in W\forall u(u\in x\rightarrow\exists v(v\in u\wedge v\in y\wedge\forall v'(v'\in u\wedge v'\in y\rightarrow v'=v))))),\text{ by} \\
&\qquad \text{several applications of (1),} \\
&\leftrightarrow \forall x\in W(\emptyset\notin x\wedge\forall u,u'\in x(u\neq u'\rightarrow u\cap u'=\emptyset)\rightarrow\exists y\in W\forall u\in x\exists v\ \{v\}=u\cap y)
\end{aligned}
$$

□

By the Lemma there are many models of fragments of ZFC in the VON NEUMANN hierarchy $(V_\alpha)_{\alpha\in\mathrm{Ord}}$.

Theorem 4. *Assume* ZF. *Then*

a) $V_\alpha \vDash$ *Extensionality, Union, Separation, and Foundation;*

b) if α *is a limit ordinal then* $V_\alpha \vDash$ *Pairing and Powerset;*

c) if $\alpha > \omega$ *then* $V_\alpha \vDash$ *Infinity;*

d) if AC *holds then* $V_\alpha \vDash \mathrm{AC}$*;*

e) if AC *holds and if* α *is a regular limit ordinal and* $\forall\lambda < \alpha\, 2^\lambda < \alpha$ *, then* $V_\alpha \vDash$ *Replacement;*

f) $V_\omega \vDash$ *all axioms of* ZFC *except Infinity;*

g) if AC *holds and* α *is strongly inaccessible, i.e.* α *is a regular limit ordinal* $>\omega$ *and* $\forall\lambda<\alpha\, 2^\lambda<\alpha$ *then* $V_\alpha \vDash \mathrm{ZFC}$.

Proof. *e)* First prove by induction on $\xi\in[\omega,\alpha)$ that $\forall a\in V_\xi\ \mathrm{card}(a)<\alpha$. For the replacement criterion let $\vec{w}\in V_\alpha$ and assume that $\forall x,y,y'\in V_\alpha(\varphi^{V_\alpha}(x,y,\vec{w})\wedge\varphi^{V_\alpha}(x,y',\vec{w})\rightarrow y=y')$. Let $a\in V_\alpha$. Then

$$z=\{y|\exists x\in a\,\varphi^{V_\alpha}(x,y,\vec{w}))\}\cap V_\alpha$$

is a subset of $V_\alpha=\bigcup_{\beta<\alpha}V_\beta$ with $\mathrm{card}(z)\leqslant\mathrm{card}(a)<\alpha$. By the regularity of α , z us a subset of V_β for some $\beta<\alpha$. Hence $z\in V_\alpha$. □

Models of the form V_α can be used to show *relative consistencies*.

Theorem 5. *Let* ZF *be consistent. Then the theory consisting of all* ZFC*-axioms except Infinity together with the negation of Infinity is consistent.*

Proof. Assume that the latter theory is *inconsistent*, i.e. that it implies a contradiction like $\exists x\, x\not\equiv x$. ZF implies that the former theory holds in V_ω. So its implications hold in V_ω. Hence ZF implies $(\exists x\, x\not\equiv x)^{V_\omega}=\exists x\in V_\omega\, x\not\equiv x$. Thus ZF is inconsistent. □

The following lead ABRAHAM FRAENKEL to the introduction of the Replacement schema.

Theorem 6. *Let Z be the system of* Zermelo *set theory, consisting of the axioms of Extensionality, Pairing, Union, Power, Separation, Infinity, and Foundation. Then Z does not imply Replacement.*

Proof. (Sketch) $V_{\omega+\omega}$ is a model of Z but $V_{\omega+\omega}$ does not satisfy Replacement: define the map $F: \omega \rightarrow V_{\omega+\omega}$, $F(n) = V_{\omega+n}$. F is definable in $V_{\omega+\omega}$ by the $\in$-formula

$\varphi(x, y, \omega, V_\omega) = \exists f(f$ is a function $\wedge \operatorname{dom}(f) \in \omega \wedge x \in \operatorname{dom}(f) \wedge f(0) = V_\omega \wedge \forall n(n+1 \in \operatorname{dom}(f) \rightarrow \forall u(u \in f(n+1) \leftrightarrow u \subseteq f(n)))$.

φ formalises the definition of F by recursion on ω. Then $F[\omega] = \{V_{\omega+n} | n < \omega\} \notin V_{\omega+\omega}$, and so $V_{\omega+\omega}$ does not satisfy replacement for the formula φ. □

Exercise 1. Define $H_\kappa = \{x | \operatorname{card}(\operatorname{TC}(\{x\})) < \kappa\}$. Examine which ZFC-axiom hold in H_κ for various κ.

3 Absoluteness and Reflection

In the study of models of set theory one considers various models $(W, \in)$ and is interested in the truth values of formulas in the various structures. It is important that truth values many many basic formulas are invariant or *absolute*.

Definition 7. *Let W, W' be terms and let $\varphi(x_0, ..., x_{n-1})$ be an $\in$-formula which does not have common variables with W or W'. φ is W-W'-*absolute *if*

$$\forall x_0, ..., x_{n-1} \in W \cap W' \ (\varphi^W \leftrightarrow \varphi^{W'}).$$

*If $W' = V$ we call φ W-*absolute.

In the next section we shall give syntactic criteria for absoluteness

Theorem 8. (Levy reflection theorem) *Assume* ZF. *Let $(W_\alpha)_{\alpha \in \mathrm{Ord}}$ be a* continuous hierarchy*, i.e.*

$$\alpha < \beta \rightarrow W_\alpha \subseteq W_\beta, \text{ and if } \lambda \text{ is a limit ordinal then } W_\lambda = \bigcup_{\alpha < \lambda} W_\alpha.$$

Let $W = \bigcup_{\alpha \in \mathrm{Ord}} W_\alpha$ be the limit of the hierarchy. Let $\varphi_0(\vec{x}), ..., \varphi_{n-1}(\vec{x})$ be a finite list of $\in$-formulas. Let $\theta_0 \in \mathrm{Ord}$. Then there exists a limit ordinal $\theta > \theta_0$ such that $\varphi_0(\vec{x})$, ..., $\varphi_{n-1}(\vec{x})$ are W_θ-W-absolute.

Proof. We may assume that the $\in$-formulas φ_i are built using only $\neg$, $\wedge$, $\exists$ and that all subformulas of a formula φ_i occur in the initial part $\varphi_0(\vec{x})$, ..., $\varphi_{i-1}(\vec{x})$ of the list. By adding redundant variables one may also assume that all formuals in the list have the same vector $\vec{x}$ of free variables. Let r be the length of the vector $\vec{x}$. For $i < n$ define functions $F_i: W^r \rightarrow \mathrm{Ord}$ by

$$F_i(\vec{x}) = \begin{cases} \min\{\beta | \exists v \in W_\beta \, \psi^W(\vec{x})\}, \text{ if } \varphi_i = \exists v \psi \text{ and } \exists v \in W \psi^W(\vec{x}) \\ 0, \text{ else} \end{cases}$$

By the definition of F,

$$\forall\vec{x}\in W\,(\exists v\in W\,\psi^W(\vec{x}\,)\leftrightarrow\exists v\in W_{F_i(\vec{x}\,)}\,\psi^W(\vec{x}\,)). \tag{1}$$

Using the Replacement schema, recursively define an ω-sequence $(\theta_m)_{m<\omega}$ starting with the given θ_0 by

$$\theta_{m+1}=\bigcup\,\{F_i(\vec{x}\,)|i<n\wedge\vec{x}\in W_{\theta_m}\}\cup(\theta_m+1).$$

Define the limit ordinal $\theta=\bigcup_{m<\omega}\theta_m$. Then for $\varphi_i=\exists v\psi$ from the list and $\vec{x}\in W_\theta$

$$\exists v\in W_\theta\,\psi^W(\vec{x}\,)\leftrightarrow\exists v\in W_{F_i(\vec{x}\,)}\,\psi^W(\vec{x}\,). \tag{2}$$

Now we show by induction on $i<n$ that φ_i is W_θ-W-absolute. Let $\vec{x}\in W_\theta$.

Case 1. φ_i is atomic. Then φ_i is trivially absolute.

Case 2. $\varphi_i=\neg\varphi_j$ with $j<i$. Then $\varphi_i^{W_\theta}(\vec{x}\,)=\neg\varphi_j^{W_\theta}(\vec{x}\,)\leftrightarrow\neg\varphi_j^W(\vec{x}\,)=\varphi_i^W(\vec{x}\,)$, using the induction hypothesis.

Case 3. $\varphi_i=\varphi_j\vee\varphi_k$ with $j,k<i$. Then $\varphi_i^{W_\theta}(\vec{x}\,)=\varphi_j^{W_\theta}(\vec{x}\,)\vee\varphi_k^{W_\theta}(\vec{x}\,)\leftrightarrow\varphi_j^W(\vec{x}\,)\vee\varphi_k^W(\vec{x}\,)=\varphi_i^W(\vec{x}\,)$, using the induction hypothesis.

Case 4. $\varphi_i=\exists v\varphi_j$ with $j<i$. Then, using the induction hypothesis and (1) and (2)

$$\begin{aligned}\varphi_i^{W_\theta}(\vec{x}\,) &= \exists v\in W_\theta\,\varphi_j^{W_\theta}(\vec{x}\,)\\ &\leftrightarrow \exists v\in W_\theta\,\varphi_j^W(\vec{x}\,)\\ &\leftrightarrow \exists v\in W_{F_i(\vec{x}\,)}\,\varphi_j^W(\vec{x}\,)\\ &\leftrightarrow \exists v\in W\,\varphi_j^W(\vec{x}\,)\\ &= \varphi_i^W(\vec{x}\,).\end{aligned}$$

□

Theorem 9. *If* ZF *is consistent then* ZF *is not equivalent to a finite system of axioms.*

Proof. Work in ZF. Assume for a contradiction that ZF is equivalent to the list φ_0, ..., φ_{n-1} of formulas without free variables. By the reflection theorem, Theorem 8, there exists $\theta\in\mathrm{Ord}$ such that $\varphi_0^{V_\theta},\ldots,\varphi_{n-1}^{V_\theta}$. Thus ZF implies

$$\exists w(w\text{ is transitive}\wedge\varphi_0^w\wedge\ldots\wedge\varphi_{n-1}^w). \tag{3}$$

By Foundation take an $\in$-minimal such w_0. Since the φ_0, ..., φ_{n-1} imply all of ZF, they also imply (3). Therefore

$$(\exists w(w\text{ is transitive}\wedge\varphi_0^w\wedge\ldots\wedge\varphi_{n-1}^w))^{w_0}.$$

This is equivalent to

$$\exists w\in w_0((w \text{ is transitive})^{w_0}\wedge(\varphi_0^w)^{w_0}\wedge\ldots\wedge(\varphi_{n-1}^w)^{w_0}).$$

Let $w_1\in w_0$ be such a w. Since w_0 is transitive, $w_1\subseteq w_0$. Relativising to w_1 *and* to w_0 is equivalent to relativising to $w_1\cap w_0=w_1$:

$$(w_1 \text{ is transitive})^{w_0}\wedge\varphi_0^{w_1}\wedge\ldots\wedge\varphi_{n-1}^{w_1}.$$

Let “w_1 is transitive” be the formula

$$\forall u\in w_1\forall v\in u\, v\in w_1\,.$$

This is equivalent to

$$\forall u\in w_1\cap w_0\forall v\in u\cap w_0\ v\in w_1$$

and to

$$(\forall u\in w_1\forall v\in u\ v\in w_1)^{w_0}.$$

Hence

$$w_1 \text{ is transitive}\wedge\varphi_0^{w_1}\wedge\ldots\wedge\varphi_{n-1}^{w_1}.$$

This contradicts the $\in$-minimality of w_0. □

Similarly one gets

Theorem 10. *Let* Φ *be a collection of* $\in$*-formulas which is a consistent extension of the axiom system* ZF*. Then* Φ *is not finitely axiomatisable. So is* ZFC *is consistent it is not finitely axiomatisable.*

We can also use the reflection theorem to “justify” the assumption of transitive models of set theory.

Theorem 11. *Let* ZF *be consistent. Then the theory* ZF $+$ M *is transitive* $+\mathrm{ZF}^M$ *is consistent where* M *is a new variable.*

Proof. Assume that ZF + M is transitive+ZF^M is inconsistent. Then the inconsistency follows from finitely many formulas of that theory. Take ZF-axioms $\varphi_0,\ldots,\varphi_{n-1}$ such that

$$\varphi_0,\ldots,\varphi_{n-1},\varphi_0^M,\ldots,\varphi_{n-1}^M, M \text{ is transitive}$$

imply the inconsistent statement $x \neq x$. Work in ZF. By Reflection, Theorem 8, there is some V_θ such that $\varphi_0, \ldots, \varphi_{n-1}$ are V_θ-absolute. Then the following hold:

$$\varphi_0, \ldots, \varphi_{n-1}, \varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta}, V_\theta \text{ is transitive.}$$

But then the proof of $x \neq x$ can be carried out under the assignment $M \mapsto V_\theta$. This means that ZF is inconsistent. □

Similarly:

Theorem 12. *Let* ZFC *be consistent. Then the theory* ZFC + *M is transitive*+ZFC^M *is consistent where M is a new variable.*

4 Formalisation of Formal Languages

We want to construct Gödel's model HOD which abbreviates the class of **H**ereditarily **O**rdinal **D**efinable sets. HOD will be a model of the theory ZFC. The basic intuitions are:

- we want to define some "minimal" model of set theory which only contains "necessary" sets.

- a model of set theory must be closed under the formation of definable sets where definitions may contain parameters from that model.

- one might define the model as the collection of all sets definable from parameters out of some reasonable class.

- one could take the class Ord of ordinals as the class of parameters: the class OD of **O**rdinal **D**efinable sets is the collection of all sets of the form

  $$y = \{x | \varphi(x, \vec{\alpha})\}$$

  where φ is a formula of set theory and $\vec{\alpha} \in \mathrm{Ord}$.

- this may lead to a class which satisfies the axiom of choice since we can wellorder the collection of terms $\{x | \varphi(x, \vec{\alpha})\}$ by wellordering the countable set of formulas and the finite sequences of parameters.

- to get a transitive model we also need that elements $x \in y$ are also ordinal definable, that $u \in x \in y$ are ordinal definable etc., i.e. that y is *hereditarily* ordinal definable. That means $\mathrm{TC}(\{y\}) \subseteq \mathrm{OD}$.

So far we do not have a definition of HOD by a formula of set theory, since we are ranging over *all* formulas φ of set theory. This makes arguing about HOD in ZF difficult. GÖDEL's crucial observation is that HOD is, after all, definable by a single $\in$-formula which roughly is as follows:

$$z \in \mathrm{HOD} \leftrightarrow \mathrm{TC}(\{z\}) \subseteq \mathrm{OD}$$

and

$$y \in \mathrm{OD} \leftrightarrow \text{there exists an } \in\text{-formula } \varphi \text{ and } \vec{\alpha} \in \mathrm{Ord} \text{ such that } y = \{x | \varphi(x, \vec{\alpha})\}.$$

To turn the right-hand side into an $\in$-formula one has to formalise the collection of all $\in$-formulas in set theory and also the truth predicate $\varphi(x, \vec{\alpha})$ as a new formula in the variables φ (sic!), x, and $\vec{\alpha}$.

Consider the language of set theory formed by variables v_0, v_1, ..., the relations $\equiv$ and $\in$, and logical signs $\neg$, $\vee$, $\exists$. We code formulas φ of that language into sets $\ulcorner\varphi\urcorner$ by recursion on the structure of φ as follows.

Definition 13. *For a formula φ of set theory define the* Gödelisation $\ulcorner\varphi\urcorner$ *by recursion:*

- $\ulcorner v_i \equiv v_j \urcorner = (0, i, j)$
- $\ulcorner v_i \in v_j \urcorner = (1, i, j)$
- $\ulcorner \neg\varphi \urcorner = (2, \ulcorner\varphi\urcorner, \ulcorner\varphi\urcorner)$
- $\ulcorner \varphi \vee \psi \urcorner = (3, \ulcorner\varphi\urcorner, \ulcorner\psi\urcorner)$
- $\ulcorner \exists v_i \varphi \urcorner = (4, i, \ulcorner\varphi\urcorner)$

Note that $\ulcorner\varphi\urcorner \in V_\omega$ since V_ω contains all the natural numbers and is closed unter ordered triples.

This motivates a set theoretic formalisation of the language of set theory. First we formalise the operations employed in the above Gödelisation. The *set* of formulas is then the the set generated by these operations.

Definition 14. *For $i, j \in \omega$ and sets $x, y \in V$ define*

- $v_i \dot{\equiv} v_j := (0, i, j)$
- $v_i \dot{\in} v_j := (1, i, j)$
- $\dot{\neg} x := (2, x, x)$
- $x \dot{\vee} y := (3, x, y)$

– $\dot{\exists} v_i\, x := (4, i, x)$

By recursion on the wellfounded relation

$$yRx \leftrightarrow \exists u, v\ (x = (u, y, v) \vee x = (u, v, y))$$

define

$$\begin{aligned} x \in \mathrm{Fml} \ \leftrightarrow\ & \exists i, j < \omega\ x = v_i \dot{\equiv} v_j \\ & \vee \exists i, j < \omega\ x = v_i \dot{\in} v_j \\ & \vee \exists y\ (y \in \mathrm{Fml} \wedge x = \dot{\neg} y) \\ & \vee \exists y, z (y \in \mathrm{Fml} \wedge z \in \mathrm{Fml} \wedge x = y \dot{\vee} z) \\ & \vee \exists i < \omega \exists y (y \in \mathrm{Fml} \wedge x = \dot{\exists} v_i\, y). \end{aligned}$$

Fml is the *set* of formalised $\in$-formulas. We have: $\mathrm{Fml} \subseteq V_\omega$, and for every standard $\in$-formula φ:

$$\ulcorner \varphi \urcorner \in \mathrm{Fml}.$$

It is, however, possible that Fml contains *nonstandard* formulas which are not of the form $\ulcorner \varphi \urcorner$. One has to be very careful here since one is working in the vicinity of the GÖDEL incompleteness theorems. One can now prove that the set Fml satisfies the syntactic properties known from predicate logic, dealing with free and bound variables, substitution, etc.

We interpret elements of Fml in structures of the form (M, E) where E is a binary relation on the *set* M and in particular in models of the form $(M, \in)$ which is a short notation for the $\in$-relation restricted to M:

$$(M, \in) = (M, \{(u, v) | u \in M \wedge v \in M \wedge u \in v\}).$$

Definition 15. *Let* $\mathrm{Asn}(M) = {}^{<\omega}M = \{a | a : \mathrm{dom}(a) \to M, \exists n < \omega\ \mathrm{dom}(a) \subseteq n\}$ *be the set of* assignments *in M. We also denote the assignment a by $a(0)$, ..., $a(n-1)$ in case that* $\mathrm{dom}(a) = n$. *For $a \in \mathrm{Asn}(M)$, $x \in M$, and $i < \omega$ define the modified assigment $a\frac{x}{i}$ by*

$$a\frac{x}{i}(m) = \begin{cases} a(m), \text{ if } m \neq i \\ x, \text{ else} \end{cases}$$

Definition 16. *For a structure (M, E) with $M \in V$, $\varphi \in \mathrm{Fml}$, and a an assignment in M define the* satisfaction relation $(M, E) \vDash \varphi[a]$ *("(M, E) is* a model of φ *under the assignment a") by recursion on the complexity of φ:*

– $(M, E) \vDash v_i \dot{\equiv} v_j (0, i, j)[a]$ *iff* $a(i) = a(j)$

– $(M,E)\vDash v_i\dot{\in}v_j(1,i,j)[a]$ *iff* $a(i)Ea(j)$

– $(M,E)\vDash\dot{\neg}y[a]$ *iff not* $(M,E)\vDash y[a]$

– $(M,E)\vDash y\dot{\vee}z[a]$ *iff* $(M,E)\vDash y[a]$ *or* $(M,E)\vDash z[a]$

– $(M,E)\vDash\dot{\exists}v_i\,y[a]$ *iff there exists* $x\in M$*:* $(M,E)\vDash y[a\frac{x}{i}]$

If $\mathrm{dom}(a)=n$ *we also write* $(M,E)\vDash\varphi[a(0),\ldots,a(n-1)]$.

Note that the recursion requires that M is a *set* since in the last clause we recurse to $(M,E)\vDash y[a\frac{x}{i}]$ for $x\in M$ and we cannot in general recurse to a proper class of preconditions.

The semantics given by the satisfaction relation satisfies the usual semantic laws known from predicate logic. The satisfaction relation also agrees with the notion of "model" in terms of relativisations. A straightforward induction on the complexity of formulas shows:

Lemma 17. *Let* $\varphi(v_0,\ldots,v_{n-1})$ *an* $\in$*-formula. Then for any set* M *with* $a\in M$

$$\forall v_0,\ldots,v_{n-1}\in M((M,\in)\vDash\ulcorner\varphi\urcorner[v_0,\ldots,v_{n-1}]\leftrightarrow\varphi^M).$$

Exercise 2. Define a wellorder $<_{\mathrm{Fml}}$ of the set Fml in ordertype ω without using parameters.

Exercise 3. Show: for any $\varphi\in\mathrm{Fml}$ there is $n<\omega$ such that for any structure (M,E) and assignments b,b' in M:

$$\text{if } b\upharpoonright n=b'\upharpoonright n \text{ then } ((M,E)\vDash\varphi[b]\leftrightarrow(M,E)\vDash\varphi[b']).$$

5 Heriditarily Ordinal Definable Sets

We can now give an (official) definition of the class HOD.

Definition 18. *Define*

$$\mathrm{OD}=\{y|\exists\alpha\in\mathrm{Ord}\,\exists\varphi\in\mathrm{Fml}\,\exists a\in\mathrm{Asn}(\alpha)\;\; y=\{z\in V_\alpha|(V_\alpha,\in)\vDash\varphi[a\frac{z}{0}]\}\},$$

and

$$\mathrm{HOD}=\{x|\mathrm{TC}(\{x\})\subseteq\mathrm{OD}\}$$

We shall see that HOD is a model of ZFC.

Lemma 19. $\mathrm{Ord} \subseteq \mathrm{OD}$ *and* $\mathrm{Ord} \subseteq \mathrm{HOD}$.

Proof. Let $\xi \in \mathrm{Ord}$. Then

$$\begin{aligned} \xi &= \{z \in V_{\xi+1} | z \in \xi\} \\ &= \{z \in V_{\xi+1} | (z \in \xi)^{V_{\xi+1}}\} \\ &= \{z \in V_{\xi+1} | (V_{\xi+1}, \in) \vDash \ulcorner v_0 \in v_1 \urcorner [z, \xi]\} \\ &\in \mathrm{OD} \end{aligned}$$

If $\xi \in \mathrm{Ord}$ then $\mathrm{TC}(\{\xi\}) = \xi + 1 \subseteq \mathrm{OD}$ and so $\xi \in \mathrm{HOD}$. □

Lemma 20. HOD *is transitive.*

Proof. Let $x \in y \in \mathrm{HOD}$. Then $\mathrm{TC}(\{x\}) \subseteq \mathrm{TC}(\{y\}) \subseteq \mathrm{OD}$ and so $x \in \mathrm{HOD}$. □

An element $y = \{z \in V_\alpha | (V_\alpha, \in) \vDash \varphi[a\frac{z}{0}]\}$ of OD is determined or *named* by the tripel (V_α, φ, a).

Definition 21. *For* $x \in V$, $\varphi \in \mathrm{Fml}$, *and* $a \in \mathrm{Asn}(x)$ *define the* interpretation function

$$I(x, \varphi, a) = \{z \in x | (x, \in) \vDash \varphi[a\frac{z}{0}]\}.$$

We say that $I(x, \varphi, a)$ *is the* interpretation *of* (x, φ, a), *or that* (x, φ, a) *is a* name *for* $I(x, \varphi, a)$.

Lemma 22. *Let*

$$\mathrm{OD}^* = \{(V_\alpha, \varphi, a) | \alpha \in \mathrm{Ord}, \varphi \in \mathrm{Fml}, a \in \mathrm{Asn}(\alpha)\}$$

be the class of OD-names. *Then* $\mathrm{OD} = I[\mathrm{OD}^*]$. OD^* *has a wellorder* $<_{\mathrm{OD}^*}$ *of type* Ord *which is definable without parameters.*

Proof. Let $<_{\mathrm{Fml}}$ be a wellorder of Fml in ordertype ω which is definable without parameters (see Exercise 2).

Wellorder the class $\bigcup_{\alpha \in \mathrm{Ord}} \mathrm{Asn}(\alpha)$ of all relevant assignment by

$$\begin{aligned} a <_{\mathrm{Asn}} a' \leftrightarrow\ & \max(\mathrm{ran}(a)) < \max(\mathrm{ran}(a')) \\ & \vee (\max(\mathrm{ran}(a)) = \max(\mathrm{ran}(a')) \wedge \mathrm{dom}(a) < \mathrm{dom}(a')) \\ & \vee (\max(\mathrm{ran}(a)) = \max(\mathrm{ran}(a')) \wedge \mathrm{dom}(a) = \mathrm{dom}(a') \wedge \exists n \in \mathrm{dom}(a')(a \upharpoonright n = \\ & a' \upharpoonright n \wedge n \in \mathrm{dom}(a) \wedge a(n) < a'(n))) \end{aligned}$$

Wellorder OD^* in ordertype Ord by

$$\begin{aligned}(V_\alpha, \varphi, a) <_{\mathrm{OD}^*} (V_{\alpha'}, \varphi', a') \ \leftrightarrow \ & \alpha < \alpha' \\ & \vee(\alpha = \alpha' \wedge \varphi <_{\mathrm{Fml}} \varphi') \\ & \vee(\alpha = \alpha' \wedge \varphi = \varphi' \wedge a <_{\mathrm{Asn}} a').\end{aligned}$$

□

Lemma 23. OD *has a wellorder* $<_{\mathrm{OD}}$ *of type* Ord *which is definable without parameters.*

Proof. We let $<_{\mathrm{OD}}$ be the wellorder induced by $<_{\mathrm{OD}^*}$ via I:

$$\begin{aligned}x <_{\mathrm{OD}} x' \ \leftrightarrow \ & \exists (V_\alpha, \varphi, a) \in \mathrm{OD}^*(x = I(V_\alpha, \varphi, a) \wedge \forall (V_{\alpha'}, \varphi', a') \in \mathrm{OD}^*(x' = I(V_{\alpha'}, \varphi', a') \rightarrow \\ & (V_\alpha, \varphi, a) <_{\mathrm{OD}^*} (V_{\alpha'}, \varphi', a'))).\end{aligned}$$

□

Lemma 24. *Let z be definable from $x_1, \ldots, x_{n-1}$ by the $\in$-formula $\varphi(v_1, \ldots, v_n)$, i.e.,*

$$\forall v_n (v_n = z \leftrightarrow \varphi(x_1, \ldots, x_{n-1}, v_n)). \tag{4}$$

Let $x_1, \ldots, x_n \in \mathrm{OD}$ and $z \subseteq \mathrm{HOD}$. Then $z \in \mathrm{HOD}$.

Proof. $\mathrm{TC}(\{z\}) = \{z\} \cup \mathrm{TC}(z) \subseteq \{z\} \cup \mathrm{HOD}$. So it suffices to prove $z \in \mathrm{OD}$. Using the canonical wellorder $<_{\mathrm{OD}}$ from Lemma 23 every element x of OD is definable from one ordinal δ without further parameters: x is the δ-th element in the wellorder $<_{\mathrm{OD}}$. So we may simply assume that the parameters $x_1, \ldots, x_{n-1}$ *are* ordinals.

Let $z, x_1, \ldots, x_{n-1} \in V_{\theta_0}$. By Reflection take some $\theta > \theta_0$ such that φ is V_θ-absolute. Then

$$\begin{aligned} z &= \{u \in V_\theta | u \in z\} \\ &= \{u \in V_\theta | \exists v_n (\varphi(x_1, \ldots, x_{n-1}, v_n) \wedge u \in v_n)\} \\ &= \{u \in V_\theta | \exists v_n \in V_\theta (\varphi(x_1, \ldots, x_{n-1}, v_n)^{V_\theta} \wedge u \in v_n)\} \\ &= \{u \in V_\theta | (V_\theta, \in) \vDash \ulcorner \exists v_n (\varphi(v_1, \ldots, v_{n-1}, v_n) \wedge v_0 \in v_n) \urcorner [u, x_1, \ldots, x_{n-1}]\} \\ &\in \mathrm{OD}. \end{aligned}$$

□

The two previous Lemmas justify the notion "ordinal definable": if $z \in \mathrm{OD}$ it is definable as the δ-th element in $<_{\mathrm{OD}}$ for some ordinal δ. Conversely, if z is definable from ordinal parameters the preceding proof shows that $z \in \mathrm{OD}$.

Theorem 25. $\mathrm{ZF}^{\mathrm{HOD}}$.

Proof. Using the criteria of Theorem 3 we check certain closure properties of HOD.

a) Extensionality holds in HOD, since HOD is transitive.

b) Let $x, y \in \mathrm{HOD}$. Then $\{x, y\}$ is definable from x, y, and $\{x, y\} \subseteq \mathrm{HOD}$. By Lemma 24, $\{x, y\} \in \mathrm{HOD}$, i.e. HOD is closed with respect to unordered pairs. This implies Pairing in HOD.

c) Let $x \in \mathrm{HOD}$. Then $\bigcup x$ is definable from x, and $\bigcup x \subseteq \mathrm{TC}(\{x\}) \subseteq \mathrm{HOD}$. So $\bigcup x \in \mathrm{HOD}$, and so Union holds in HOD.

d) Let $x \in \mathrm{HOD}$. Then $\mathcal{P}(x) \cap \mathrm{HOD}$ is definable from x, and $\mathcal{P}(x) \cap \mathrm{HOD} \subseteq \mathrm{HOD}$. So $\mathcal{P}(x) \cap \mathrm{HOD} \in \mathrm{HOD}$ and Powerset holds in HOD.

e) $\omega \in \mathrm{HOD}$ implies that Infinity holds in HOD.

f) Let $\varphi(x, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}$. Then $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\}$ is a set by Separation in V, and it is definable from $\vec{w}, a$. Moreover $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\} \subseteq \mathrm{HOD}$. So $\{x \in a | \varphi^{\mathrm{HOD}}(x, \vec{w})\} \in \mathrm{HOD}$, and Separation for the formula φ holds in HOD.

g) Let $\varphi(x, y, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}$. Assume that

$$\forall x, y, y' \in \mathrm{HOD}(\varphi^{\mathrm{HOD}}(x, y, \vec{w}) \wedge \varphi^{\mathrm{HOD}}(x, y', \vec{w}) \rightarrow y = y').$$

Then $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD}$ is a set by Replacement and Separation in V. It is definable from $\vec{w}, a$. Moreover $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD} \subseteq \mathrm{HOD}$. So $\{y | \exists x \in a \varphi^{\mathrm{HOD}}(x, y, \vec{w})\} \cap \mathrm{HOD} \in \mathrm{HOD}$, and Replacement for φ holds in HOD.

h) Foundation holds in HOD since HOD is an $\in$-model. □

Hence HOD is an *inner model of set theory*, i.e. HOD is transitive, contains all ordinals, and is a model of ZF.

Theorem 26. $\mathrm{AC}^{\mathrm{HOD}}$.

Proof. We prove AC in HOD using Theorem 3. Consider $x \in \mathrm{HOD}$ with $\emptyset \notin x \wedge \forall u, u' \in x\, (u \neq u' \rightarrow u \cap u' = \emptyset)$. Define a choice set y for x by

$$y = \{v | \exists u \in x \colon v \text{ is the } <_{\mathrm{OD}}\text{-minimal element of } u\}.$$

Obviously y intersects every element of x in exactly one element. y is definable from $x \in \mathrm{HOD}$ and $y \subseteq \mathrm{HOD}$. By Lemma 24, $y \in \mathrm{HOD}$, as required. □

Theorem 27. (Kurt Gödel, 1938) *If* ZF *is consistent then* ZFC *is consistent. In other words: the Axiom of Choice is* relatively consistent *with the system* ZF.

Proof. Since ZF proves that HOD is a model for ZFC. □

Exercise 4. Extend the formal language by atomic formulas for "$x \in A$" where A is considered a unary predicate or relation. Define

$$\mathrm{OD}(A) = \{y | \exists \alpha \in \mathrm{Ord}\, \exists \varphi \in \mathrm{Fml}'\, \exists \beta : \omega \to A \cap V_\alpha\ y = \{z \in V_\alpha | (V_\alpha, A \cap V_\alpha, \in) \vDash \varphi[\beta \frac{z}{0}]\}\}$$

and the corresponding generalisation $\mathrm{HOD}(A)$ of HOD. Prove:

a) if A is transitive then $A \subseteq \mathrm{HOD}(A)$;

b) if A is moreover definable from some parameters $a_0, \ldots, a_{n-1} \in A$ then $\mathrm{ZF}^{\mathrm{HOD}(A)}$.

Note that AC does in general not hold in $\mathrm{HOD}(A)$.

6 Absolute and Definite Notions

For terms we define:

Definition 28. *Let W be a term, and $t(\vec{x}) = \{y | \varphi(y, \vec{x})\}$ be a term which has no common variables with W. Define the* relativisation

$$t^W(\vec{x}) = \{y \in W | \varphi^W(y, \vec{x})\}.$$

*Let W' be another term which has no common variables with t. Then t is W-W'-*absolute *if*

$$\forall \vec{x} \in W \cap W'((t^W(\vec{x}) \in W \leftrightarrow t^{W'}(\vec{x}) \in W') \wedge (t^W(\vec{x}) \in W \to t^W(\vec{x}) = t^{W'}(\vec{x}))).$$

*If $W' = V$ we call t W-*absolute*.*

Formulas and terms may be absolute for complicated reasons. In this section we want to study notions that are absolute between *all* transitive models of ZF^- simply due to their syntactical structure.

Definition 29. *Let $\psi(\vec{v})$ be an $\in$-formula and let $t(\vec{v})$ be a term, both in the free variables $\vec{v}$. Then*

a) ψ is definite *iff for every transitive ZF^--model $(M, \in)$*

$$\forall \vec{x} \in M\ (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x})).$$

b) t is definite *iff for every transitive ZF^--model $(M, \in)$*

$$\forall \vec{x} \in M\, t^M(\vec{x}) \in M \text{ and } \forall \vec{x} \in M\, t^M(\vec{x}) = t(\vec{x}).$$

Note that the notion of definiteness is in its generality not definable in set theory since it involves quantification over all transitive ZF^--models. We can, however, prove that most simple set-theoretical notions are definite. We shall work inductively: basic notions are definite and important set-theoretical operations lead from definite notions to definite notions.

The following lemma shows that the operations of relativisation and substitution of a term into a formula commute.

Lemma 30. *Let $\varphi(x, \vec{y})$ be a formula, $t(\vec{z})$ be a term, and M be a class. Assume that $\forall \vec{z} \in M\, t(\vec{z}) \in M$. Then*

$$\forall \vec{y}, \vec{z} \in M\ (\varphi(t(\vec{z}), \vec{y}))^M \leftrightarrow \varphi^M(t^M(\vec{z}), \vec{y})).$$

Proof. If $t = t(\vec{z})$ is of the form $t = z$ then there is nothing to show. Assume otherwise that t is of the form $t = \{u | \psi(u, \vec{z})\}$. We work by induction on the complexity of φ. Assume that $\varphi \equiv x = y$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(t(\vec{z}) = y)^M &\leftrightarrow (\{u | \psi(u, \vec{z})\} = y)^M \\
&\leftrightarrow (\forall u\ (\psi(u, \vec{z}) \leftrightarrow u \in y))^M \\
&\leftrightarrow \forall u \in M\ (\psi^M(u, \vec{z}) \leftrightarrow u \in y) \\
&\leftrightarrow \{u \in M | \psi^M(u, \vec{z})\} = y \\
&\leftrightarrow t^M(\vec{z}) = y \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

Assume that $\varphi \equiv y \in x$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(y \in t(\vec{z}))^M &\leftrightarrow \psi^M(\frac{y}{u}, \vec{z}) \\
&\leftrightarrow y \in \{u \in M | \psi^M(u, \vec{z})\} \\
&\leftrightarrow y \in t^M(\vec{z}) \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

Assume that $\varphi \equiv x \in y$ and $y, \vec{z} \in M$. Then

$$\begin{aligned}
(t(\vec{z}) \in y)^M &\leftrightarrow (\exists u\, (u = t(\vec{z}) \wedge u \in y)^M \\
&\leftrightarrow \exists u \in M\, ((u = t(\vec{z}))^M \wedge u \in y) \\
&\leftrightarrow \exists u \in M\, (u = t^M(\vec{z}) \wedge u \in y), \text{ by the first case,} \\
&\leftrightarrow \exists u\, (u = t^M(\vec{z}) \wedge u \in y), \text{ since } M \text{ is closed w.r.t. } t, \\
&\leftrightarrow t^M(\vec{z}) \in y \\
&\leftrightarrow \varphi^M(t^M(\vec{z}), y)
\end{aligned}$$

The induction steps are obvious since the terms t resp. t^M are only substituted into the atomic subformulas of φ. $\square$

Theorem 31.

a) *The formulas* $x = y$ *and* $x \in y$ *are definite.*

b) *If the formulas* φ *and* ψ *are definite then so are* $\neg\varphi$ *and* $\varphi \vee \psi$.

c) *Let the formula* $\varphi(x, \vec{y})$ *and the term* $t(\vec{z})$ *be definite. Then so are* $\varphi(t(\vec{z}), \vec{y})$ *and* $\exists x \in t(\vec{z})\ \varphi(x, \vec{y})$.

d) *The terms* x, $\emptyset$, $\{x, y\}$, *and* $\bigcup x$ *are definite.*

e) *Let the terms* $t(x, \vec{y})$ *and* $r(\vec{z})$ *be definite. Then so is* $t(r(\vec{z}), \vec{y})$.

f) *Let the formula* $\varphi(x, \vec{y})$ *be definite. Then so is the term* $\{x \in z \mid \varphi(x, \vec{y})\}$.

g) *Let the term* $t(x, \vec{y})$ *be definite. Then so is the term* $\{t(x, \vec{y}) \mid x \in z\}$.

h) *The formulas "R is a relation", "f is a function", "f is injective", and "f is surjective" are definite.*

i) *The formulas* $\mathrm{Trans}(x)$, $\mathrm{Ord}(x)$, $\mathrm{Succ}(x)$, *and* $\mathrm{Lim}(x)$ *are definite.*

j) *The term* ω *is definite.*

Proof. Let M be a transitive ZF^--model.
a) is obvious since $(x = y)^M \equiv (x = y)$ and $(x \in y)^M \equiv (x \in y)$.
b) Assume that φ and ψ are definite and that $(M, \in)$ is a transitive ZF^--model. Then $\forall \vec{x} \in M\ (\varphi^M(\vec{x}) \leftrightarrow \varphi(\vec{x}))$ and $\forall \vec{x} \in M\ (\psi^M(\vec{x}) \leftrightarrow \psi(\vec{x}))$. Thus

$$\forall \vec{x} \in M\ ((\varphi \vee \psi)^M(\vec{x}) \leftrightarrow (\varphi^M(\vec{x}) \vee \psi^M(\vec{x})) \leftrightarrow (\varphi(\vec{x}) \vee \psi(\vec{x})) \leftrightarrow (\varphi \vee \psi)(\vec{x}))$$

and

$$\forall \vec{x} \in M\ ((\neg\varphi(\vec{x}))^M \leftrightarrow \neg(\varphi^M(\vec{x})) \leftrightarrow \neg(\varphi(\vec{x})) \leftrightarrow (\neg\varphi)(\vec{x})).$$

c) Let $(M, \in)$ be a transitive ZF^--model. Let $\vec{y}, \vec{z} \in M$. $t(\vec{z}) \in M$ since t is definite. Then

$$\begin{aligned} (\varphi(t(\vec{z}), \vec{y}))^M &\leftrightarrow \varphi^M(t^M(\vec{z}), \vec{y}), \text{ by Lemma 30,} \\ &\leftrightarrow \varphi^M(t(\vec{z}), \vec{y}), \text{ since } t \text{ is definite,} \\ &\leftrightarrow \varphi(t(\vec{z}), \vec{y}), \text{ since } \varphi \text{ is definite.} \end{aligned}$$

Also

$$
\begin{aligned}
(\exists x \in t(\vec{z}\,)\ \varphi(x,\vec{y}\,))^M &\leftrightarrow (\forall x\,(x \in t(\vec{z}\,) \to \varphi(x,\vec{y}\,)))^M \\
&\leftrightarrow \forall x \in M\,((x \in t(\vec{z}\,))^M \to \varphi^M(x,\vec{y}\,)) \\
&\leftrightarrow \forall x \in M\,(x \in t^M(\vec{z}\,) \to \varphi^M(x,\vec{y}\,)) \\
&\leftrightarrow \forall x \in M\,(x \in t(\vec{z}\,) \to \varphi(x,\vec{y}\,)), \text{ since } t \text{ and } \varphi \text{ are definite,} \\
&\leftrightarrow \forall x\,(x \in t(\vec{z}\,) \to \varphi(x,\vec{y}\,)), \text{ since } t(\vec{z}\,) \subseteq M, \\
&\leftrightarrow \forall x \in t(\vec{z}\,)\ \varphi(x,\vec{y}\,)).
\end{aligned}
$$

d) A variable term x is trivially definite, since $x^M = x$.

Consider the term $\emptyset = \{u | u \neq u\}$. Since M is non-empty and transitive, $\emptyset \in M$. Also

$$\emptyset^M = \{u \in M | u \neq u\} = \emptyset.$$

Consider the term $\{x, y\}$. For $x, y \in M$:

$$\{x,y\}^M = \{u \in M\,|\,u = x \vee u = y\} = \{u | u = x \vee u = y\} = \{x, y\}.$$

The pairing axiom in M states that

$$(\forall x, y \exists z\ z = \{x, y\})^M.$$

This implies

$$\forall x, y \in M \exists z \in M\, z = \{x, y\}^M = \{x, y\}$$

and

$$\forall x, y \in M\ \ \{x, y\} \in M.$$

Consider the term $\bigcup x$. For $x \in M$:

$$(\bigcup x)^M = \{u \in M | (\exists v \in x\ u \in v)^M\} = \{u \in M | \exists v \in x \cap M\ u \in v\} = \{u | \exists v \in x\ u \in v\} = \bigcup x.$$

The union axiom in M states that

$$(\forall x \exists z\ z = \bigcup x)^M.$$

This implies

$$\forall x \in M \exists z \in M\ z = (\bigcup x)^M = \bigcup x$$

and

$$\forall x \in M \bigcup x \in M.$$

$e)$ is obvious.

$f)$ Let $\vec{y}, z \in M$. By the separation schema in M,

$$(\exists w\ w = \{x \in z \mid \varphi(x, \vec{y})\})^M,$$

i.e. $\{x \in z \mid \varphi(x, \vec{y})\}^M \in M$. Moreover by the definiteness of φ

$$\{x \in z \mid \varphi(x, \vec{y})\}^M = \{x \in M \mid x \in y \wedge \varphi^M(x, \vec{y})\} = \{x \mid x \in y \wedge \varphi(x, \vec{y})\} = \{x \in z \mid \varphi(x, \vec{y})\}.$$

$g)$ Since t is definite, $\forall x, \vec{y} \in M\, t^M(x, \vec{y}) \in M$. This implies

$$\forall x, \vec{y} \in M \exists w \in M\, w = t^M(x, \vec{y})$$

and $(\forall x, \vec{y} \exists w\ w = t(x, \vec{y}))^M$. Let $\vec{y}, z \in M$. By replacement in M,

$$(\exists a\ a = \{t(x, \vec{y}) | x \in z\})^M.$$

Hence $\{t(x, \vec{y}) | x \in z\}^M \in M$. Moreover

$$\begin{aligned}
\{t(x, \vec{y}) | x \in z\}^M &= \{w | \exists x \in z\ w = t(x, \vec{y})\}^M \\
&= \{w \in M \mid \exists x \in z\ w = t^M(x, \vec{y})\} \\
&= \{w \mid \exists x \in z\ w = t^M(x, \vec{y})\}, \text{ since } M \text{ is closed w.r.t. } t^M, \\
&= \{w \mid \exists x \in z\ w = t(x, \vec{y})\}, \text{ since } t \text{ is definite,} \\
&= \{t(x, \vec{y}) | x \in z\}.
\end{aligned}$$

$h)$ "R is a relation" is equivalent to

$$\forall z \in R \exists x, y \in (\bigcup \bigcup z)\ z = \{\{x\}, \{x, y\}\}.$$

This is definite, using $c), d), e)$. The other relational statements are definite for similar reasons.

$i)$

$$\begin{aligned}
\mathrm{Trans}(x) &\leftrightarrow \forall y \in x \forall z \in y\, z \in x \\
\mathrm{Ord}(x) &\leftrightarrow \mathrm{Trans}(x) \wedge \forall y \in x\ \mathrm{Trans}(y) \\
\mathrm{Succ}(x) &\leftrightarrow \mathrm{Ord}(x) \wedge \exists y \in x\, x = y \cup \{y\} \\
\mathrm{Lim}(x) &\leftrightarrow \mathrm{Ord}(x) \wedge \neg \mathrm{Succ}(x) \wedge x \neq \emptyset
\end{aligned}$$

$j)$ Consider the term $\omega = \bigcap \{x | x \text{ is inductive}\}$. Since M satisfies the axiom of infinity,

$$\exists x \in M\ (x = \omega)^M.$$

Take $x_0 \in M$ such that $(x_0 = \omega)^M$. Then $(\mathrm{Lim}(x_0))^M$, $(\forall y \in x_0 \neg \mathrm{Lim}(y))^M$. By definiteness, $\mathrm{Lim}(x_0)$, $\forall y \in x_0 \neg \mathrm{Lim}(y)$, i.e. x_0 is equal to the smallest limit ordinal ω. Hence $\omega \in M$. The formula "x is inductive" has the form

$$\emptyset \in x \wedge \forall y \in x \bigcup \{y, \{y\}\} \in x$$

and is definite by previous considerations. Now

$$\begin{aligned}
\omega^M &= (\bigcap \{x | x \text{ is inductive}\})^M \\
&= (\{y | \forall x\, (x \text{ is inductive} \rightarrow y \in x)\})^M \\
&= \{y \in M | \forall x \in M\, (x \text{ is inductive} \rightarrow y \in x)\}, \text{ since "}x\text{ is inductive" is definite,} \\
&= \bigcap \{x \in M | x \text{ is inductive}\} \\
&= \bigcap \{x \cap \omega | x \in M \text{ is inductive}\}, \text{ since } \omega \in M, \\
&= \bigcap \{\omega\}, \text{ since } \omega \text{ is the smallest inductive set,} \\
&= \omega.
\end{aligned}$$

□

We may view this theorem as a "definite" form of the ZF^--axioms: common notions and terms of set theory and mathematics are definite, and natural operations lead to further definite terms. Since the recursion principle is so important, we shall need a definite recursion schema:

Theorem 32. *Let $G(w, \vec{y})$ be a definite term, and let $F(\alpha, \vec{y})$ be the canonical term defined by $\in$-recursion with G:*

$$\forall x\, F(x, \vec{y}) = G(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}).$$

Then the term $F(x, \vec{y})$ is definite.

Proof. Let M be a transitive ZF^--model. By the recursion theorem, F is a total function in V and in M:

$$\forall x, \vec{y} \in M\, F^M(x, \vec{y}) \in M.$$

Assume that x were $\in$-minimal such that $F^M(x, \vec{y}) \neq F(x, \vec{y})$. Then we get a contradiction by

$$\begin{aligned}
F^M(x, \vec{y}) &= G^M(\{(z, F^M(z, \vec{y})) | z \in x\}, \vec{y}) \\
&= G^M(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}), \text{ by the minimality of } x, \\
&= G(\{(z, F(z, \vec{y})) | z \in x\}, \vec{y}), \text{ by the definiteness of } G, \\
&= F(x, \vec{y}).
\end{aligned}$$

□

Lemma 33. rank(x) *is a definite term.*

Proof. $\mathrm{rank}(x) = \bigcup \{\mathrm{rank}(y) + 1 \,|\, y \in x\} = G(\mathrm{rank} \upharpoonright x)$ with the definite recursion rule

$$G(f) = \{f(z) + 1 \,|\, z \in \mathrm{dom}(f)\}$$

□

Theorem 34. *Let $G(w, \vec{y}\,)$ be a definite term and let $R(z, x)$ be a strongly wellfounded relation such that the term $\{z|zRx\}$ is definite. Let $F(\alpha, \vec{y}\,)$ be the canonical term defined by R-recursion with G:*

$$\forall x\, F(x, \vec{y}\,) = G(\{(z, F(z, \vec{y}\,))|zRx\}, \vec{y}\,).$$

Then the term $F(x, \vec{y}\,)$ is definite.

Proof. Let M be a transitive ZF^--model. By the recursion theorem, F is a total function in V and in M:

$$\forall x, \vec{y} \in M\, F^M(x, \vec{y}\,) \in M.$$

Assume that x were R-minimal such that $F^M(x, \vec{y}\,) \neq F(x, \vec{y}\,)$. Then we get a contradiction by

$$\begin{aligned} F^M(x, \vec{y}\,) &= G^M(\{(z, F^M(z, \vec{y}\,))|(zRx)^M\}, \vec{y}\,) \\ &= G^M(\{(z, F^M(z, \vec{y}\,))|zRx\}, \vec{y}\,), \text{ by the assumptions on } R, \\ &= G^M(\{(z, F(z, \vec{y}\,))|zRx\}, \vec{y}\,), \text{ by the minimality of } x, \\ &= G(\{(z, F(z, \vec{y}\,))|zRx\}, \vec{y}\,), \text{ by the definiteness of } G, \\ &= F(x, \vec{y}\,). \end{aligned}$$

□

Also other kinds of recursions lead from definite recursion rules to definite functions.

Note that *not every* important notion is definite. For the powerset operation we have $\mathcal{P}^M(x) = \mathcal{P}(x) \cap M$. If M does not contain all subsets of x then $\mathcal{P}^M(x) \neq \mathcal{P}(x)$. We shall later produce countable transitive models M of ZF^- so that $\mathcal{P}^M(\omega) \neq \mathcal{P}(\omega)$, and we thus prove that $\mathcal{P}(x)$ is *not* definite. Obviously the construction of models of set theory is especially geared at exhibiting the *indefiniteness* of some interesting notions.

Exercise 5. Show that (x, y), $x \times y$, $f \upharpoonright x$ are definite terms.

Exercise 6. Show that $\mathrm{TC}(x)$ is a definite term.

Exercise 7. Show that the term V_n for $n < \omega$ is definite. Show that the term V_ω is definite.

Lemma 35. *The following modeltheoretic notions are definite:*

a) *the term* Fml *of all formalised* $\in$*-formulas;*

b) *the term* $\mathrm{Asn}(M)$*;*

c) *the formula "*$(M,E)\vDash\varphi[b]$*" in the variables* M,E,φ,b.

Proof. *a*) and *c*). Fml and $\vDash$are defined by recursion on the relation

$$yRx\leftrightarrow\exists u,v\ (x=(u,y,v)\vee x=(u,v,y)).$$

Then

$$\{y|yRx\}=\{y\in\mathrm{TC}(x)|\exists u,v\in\mathrm{TC}(x)\ (x=(u,y,v)\vee x=(u,v,y))\}$$

is definite. Therefore the characteristic function of Fml is definite as well as the term

$$\mathrm{Fml}=\{x\in V_\omega|x\in\mathrm{Fml}\}.$$

By Theorem 34 on definite recursions, Fml and $\vDash$ are definite.

b) Define by definite recursion $\mathrm{Asn}_0(M)=\{\emptyset\}$ and

$$\mathrm{Asn}_{n+1}(M)=\mathrm{Asn}_n(M)\cup\{a\frac{x}{n}\,|\,a\in\mathrm{Asn}_n(M)\wedge x\in M\}.$$

$\mathrm{Asn}_n(M)$ is a definite term, and $\mathrm{Asn}(M)=\bigcup\ \{\mathrm{Asn}_n(M)|n\in\omega\}$ is also definite. □

7 Skolem hulls

Theorem 36. (Downward Löwenheim-Skolem Theorem, ZFC) *Let* $X\subseteq M\neq\emptyset$ *be sets. Then there exists* $N\subseteq M$ *such that*

a) $X\subseteq N$ *and* $\mathrm{card}(N)\leqslant\mathrm{card}(X)+\aleph_0$*;*

b) *every* $\in$*-formula is* N*-*M*-absolute.*

Proof. Take a wellorder $\prec$ of M. Define a Skolem *function* $S\colon\mathrm{Fml}\times\mathrm{Asn}(M)$,

$$S(\varphi,a)=\begin{cases}\text{the }\prec\text{ smallest element of }I(M,\varphi,a)\text{, if this exists,}\\ m_0\text{, else,}\end{cases}$$

where m_0 is some fixed element of M. Intuitively, $S(\varphi, a_0, a_1, \ldots, a_{k-1})$ is the $\prec$-smallest element $z \in M$ such that $M \vDash \varphi(z, a_1, \ldots, a_{k-1})$, if such a z exists.

Define $N_0 = X$, $N_1, N_2, \ldots$ recursively:

$$N_{n+1} = N_n \cup S[\mathrm{Fml} \times \mathrm{Asn}(N_n)],$$

and let $N = \bigcup_{n<\omega} N_n$.

We show inductively that $\mathrm{card}(N_n) \leqslant \mathrm{card}(X) + \aleph_0$:

$$\begin{aligned}
\mathrm{card}(N_{n+1}) &\leqslant \mathrm{card}(N_n) + \mathrm{card}(\mathrm{Fml} \times \mathrm{Asn}(N_n)) \\
&\leqslant \mathrm{card}(N_n) + \mathrm{card}(\mathrm{Fml}) \cdot \mathrm{card}({}^{<\omega}N_n) \\
&\leqslant \mathrm{card}(N_n) + \aleph_0 \cdot \mathrm{card}(N_n)^{<\omega} \\
&\leqslant \mathrm{card}(X) + \aleph_0 + \aleph_0 \cdot (\mathrm{card}(X) + \aleph_0), \text{ by inductive assumption,} \\
&\leqslant \mathrm{card}(X) + \aleph_0 .
\end{aligned}$$

Hence

$$\mathrm{card}(N) \leqslant \sum_{n<\omega} \mathrm{card}(N_n) \leqslant \sum_{n<\omega} (\mathrm{card}(X) + \aleph_0) = \aleph_0 \cdot (\mathrm{card}(X) + \aleph_0) = \mathrm{card}(X) + \aleph_0 .$$

We prove the N-M-absoluteness of the $\in$-formula φ by induction on the complexity of φ. The cases $\varphi \equiv v_0 = v_1$ and $\varphi \equiv v_0 \in v_1$ are trivial. The induction steps for $\varphi \equiv \varphi_0 \vee \varphi_1$ and $\varphi \equiv \neg\varphi_0$ are easy. Finally consider the formula $\varphi \equiv \exists v_0\, \psi(v_0, v_1, \ldots, v_{k-1})$. Consider $a_1, \ldots, a_{k-1} \in N$. The Skolem value $u = S(\ulcorner\psi\urcorner, a_1, \ldots, a_{k-1})$ is an element of N. Then

$$\begin{aligned}
(\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^N &\rightarrow \exists v_0 \in N\, \psi^N(v_0, a_1, \ldots, a_{k-1}) \\
&\rightarrow \exists v_0 \in N \psi^M(v_0, a_1, \ldots, a_{k-1}), \text{ by the inductive assumption,} \\
&\rightarrow \exists v_0 \in M \psi^M(v_0, a_1, \ldots, a_{k-1}) \\
&\rightarrow (\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^M.
\end{aligned}$$

Conversely assume that $(\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^M$. Then $I(M, \ulcorner\psi\urcorner, a_1, \ldots, a_{k-1}) \neq \emptyset$ and $z = S(\ulcorner\psi\urcorner, a_1, \ldots, a_{k-1})$ is the $\prec$-smallest element of M such that $\psi^M(z, a_1, \ldots, a_{k-1})$. The construction of N implies that $z \in N$. By induction hypothesis, $\psi^N(z, a_1, \ldots, a_{k-1})$. Hence $\exists v_0 \in N\, \psi^N(z, a_1, \ldots, a_{k-1}) \equiv (\exists v_0\, \psi(v_0, a_1, \ldots, a_{k-1}))^N$. □

Note that this proof has some similarities with the proof of the Levy reflection principle. Putting $X = \emptyset$ the theorem implies that every formula that has some infinite model M has a *countable* model N. E.g., the formula "there is an uncountable set" has a countable model. This is the famous Skolem paradox. As a prepartation for the forcing method we also want the countable structure to be transitive.

Theorem 37. *Assume* $(Extensionality)^N$. *Then there is a transitive* $\bar{N}$ *and* $\pi: N \leftrightarrow \bar{N}$ *such that* π *is an* $\in$-isomorphism, *i.e.* $\forall x, y \in N\, (x \in y \leftrightarrow \pi(x) \in \pi(y))$. *Moreover,* $\bar{N}$ *and* π *are uniquely determined by* N. π *and* $\bar{N}$ *are called the* MOSTOWSKI transitivisation *or* collapse *of* N.

Proof. Define $\pi: N \rightarrow V$ recursively by

$$\pi(y) = \{\pi(x) | x \in y \cap N\}.$$

Set $\bar{N} = \pi[N]$.

(1) $\bar{N}$ is transitive.
Proof. Let $z \in \pi(y) \in \bar{N}$. Take $x \in y \cap N$ such that $z = \pi(x)$. Then $z \in \pi[N] = \bar{N}$. *qed*(1)

(2) $\pi: N \leftrightarrow \bar{N}$.
Proof. It suffices to show injectivity. Assume for a contradiction that $z \in \bar{N}$ is $\in$-minimal such that there are $y, y' \in N$, $y \neq y'$ with $z = \pi(y) = \pi(y')$. $(\text{Extensionality})^N$ implies $(\exists x(x \in y \leftrightarrow x \notin y'))^N$. Take $x \in N$ such that $x \in y \leftrightarrow x \notin y'$. We may assume that $x \in y$ and $x \notin y'$. Then $\pi(x) \in \pi(y) = \pi(y')$. According to the definition of π take $x' \in y' \cap N$ such that $\pi(x) = \pi(x')$. By the minimality of z, $x = x'$. But then $x = x' \in y'$, contradiction. *qed*(2)

(3) π is an $\in$-isomorphism.
Proof. Let $x, y \in N$. If $x \in y$ then $\pi(x) \in \pi(y)$ by the definition of π. Conversely assume that $\pi(x) \in \pi(y)$. By the definition of π take $x' \in y \cap N$ such that $\pi(x) = \pi(x')$. By (2), $x = x'$ and so $x \in y$. *qed*(3)

To show uniqueness assume that $\tilde{N}$ is transitive and $\tilde{\pi}: N \leftrightarrow \tilde{N}$ is an $\in$-isomorphism. Assume that $y \in N$ is $\in$-minimal such that $\pi(y) \neq \tilde{\pi}(y)$. We get a contradiction by showing that $\pi(y) = \tilde{\pi}(y)$. Consider $z \in \tilde{\pi}(y)$. The transitivity of $\tilde{N}$ implies $z \in \tilde{N}$. By the surjectivity of $\tilde{\pi}$ take $x \in N$ such that $z = \tilde{\pi}(x)$. Since $\tilde{\pi}$ is an $\in$-isomorphism, $x \in y$. And since π is an $\in$-isomorphism, $\pi(x) \in \pi(y)$. By the minimality of y, $\pi(x) = \tilde{\pi}(x)$. Hence $z = \tilde{\pi}(x) = \pi(x) \in \pi(y)$. Thus $\tilde{\pi}(y) \subseteq \pi(y)$. The converse can be shown analogously. Thus $\pi(y) = \tilde{\pi}(y)$, contradiction. □

It is easy to see that $\in$-isomorphisms preserve the truth of $\in$-formulas.

Lemma 38. *Let* $\pi: N \leftrightarrow \bar{N}$ *be an* $\in$*-isomorphism. Let* $\varphi(v_0, \ldots, v_{n-1})$ *be an* $\in$*-formula. Then*

$$\forall v_0, \ldots, v_{k-1} \in N\, (\varphi^N(v_0, \ldots, v_{k-1}) \leftrightarrow \varphi^{\bar{N}}(\pi(v_0), \ldots, \pi(v_{k-1}))).$$

Lemma 39. (ZFC) *Let* $\varphi_0, \ldots, \varphi_{n-1}$ *be* $\in$*-formulas without free variables with are true in* V*. Then there is a countable transitive set* $\bar{N}$ *such that* $\varphi_0^{\bar{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$.

Proof. We may assume that φ_0 is the extensionality axiom. By the Reflection Theorem 8 we can take $\theta \in \mathrm{Ord}$ such that $\varphi_0^{V_\theta}, \ldots, \varphi_{n-1}^{V_\theta}$. By Theorem 36 there is a countable N such that all $\in$-formulas as N-V_θ-absolute. In particular $\varphi_0^N, \ldots, \varphi_{n-1}^N$. By Theorem 37 there is transitive set $\bar{N}$ and an $\in$-isomorphism $\pi: N \leftrightarrow \bar{N}$. Then $\bar{N}$ is countable. By Lemma 38 $\varphi_0^{\bar{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$. □

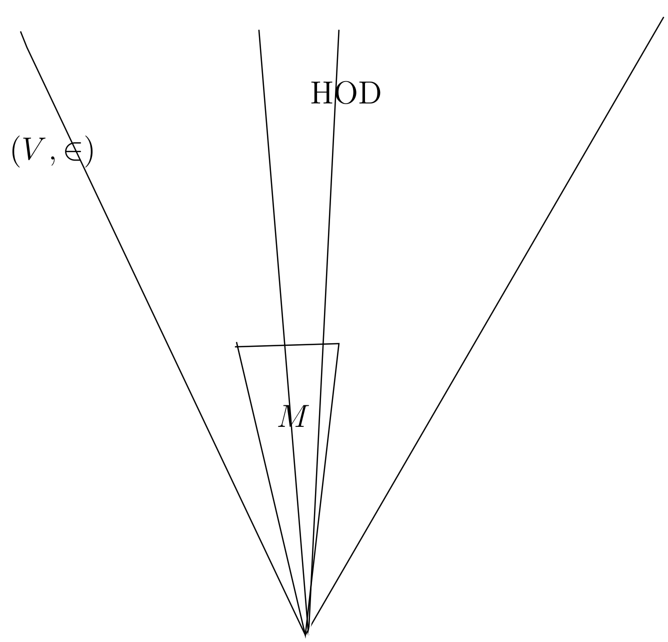
Theorem 40. *If* ZFC *is consistent then the following theory is also consistent:* ZFC + M *is countable and transitive* + ZFC^M*, where M is some variable.*

Proof. Assume that the theory ZFC + M is countable and transitive + ZFC^M is inconsistent. Then there is a finite sequence $\varphi_0, \ldots, \varphi_{n-1}$ of ZFC-axioms such that the theory

$$\varphi_0, \ldots, \varphi_{n-1}, M \text{ is countable and transitive}, \varphi_0^M, \ldots, \varphi_{n-1}^M$$

implies $x \neq x$. Work in ZFC. By Lemma 39 there is a countable transitive set $\bar{N}$ such that $\varphi_0^{\bar{N}}, \ldots, \varphi_{n-1}^{\bar{N}}$. Setting $M = \bar{N}$ we get the contradiction $x \neq x$. Hence ZFC is inconsistent. □

The considerations so far justify the following picture as a basis for further studies:



The argument of the Theorem can be extended to every $\in$-theory which extends ZFC, like ZFC + CH or ZFC + ¬CH.

Theorem 41. *Let T be a theory in the language of set theory which extends* ZFC*. Assume that T is consistent. Then the following theory is also consistent: $T + M$ is countable and transitive $+ T^M$, where M is some variable.*

8 Extensions of Models of Set Theory

So far we have constructed and studied the *inner model* HOD, i.e. a submodel of a given model of set theory. We shall now work towards *extending* models of set theory by the *forcing* method of Paul Cohen. Cohen introduced these techniques to show the independence of AC and CH from ZF.

We shall work in the situation justified by Theorem 40: assume ZFC and ZFC^M where M is countable and transitive. Such an $\in$-structure $(M, \in)$ is called a *ground model*. We intend to adjoin a *generic set* G to M so that the extension $M[G]$ is again a model of ZFC. Cohen proved the independence of CH by constructing a *generic extension*

$$M[G] \vDash \mathrm{ZFC} + \neg\mathrm{CH}.$$

As already said in the introduction the extension $M \subseteq M[G]$ has some similarities to a transcendental field extension $k \subseteq k(a)$. The transcendental element a can be described in the ground field k by a variable x; some properties of a can be described in k. That $k(a)$ is a field follows from the field axioms in k. The extension is generated by k and a: every intermediate field K with $k \subseteq K \subseteq k(a)$ and $a \in K$ satisfies $K = k(a)$.

The set-theoretic situation will be much more complicated than the algebraic analogue. Whereas there is up to isomorphism only one transcendental field extension of transcendence degree 1 we shall encounter a rich spectrum of generic extensions.

So fix the ground model M as above. We shall use sets G to determine extensions $M[G]$. G may be seen as the limit of a (countable) procedure in which more and more properties of $M[G]$ are being determined or *forced*. Limits are often described by filters. Our G will be a filter on a preordering $(P, \leqslant)$.

Definition 42. *A* partial order *or a* forcing *is a tripel $(P, \leqslant, 1_P)$ such that $(P, \leqslant)$ is a transitive and reflexive binary relation (a* preordering*) with a maximal element 1_P. The elements of P are called* (forcing) conditions*. We say that p is* stronger than *q iff $p \leqslant q$. Conditions $q_0, \ldots, q_{n-1}$ are* compatible *iff they have a common* extension *$p \leqslant q_0, \ldots, q_{n-1}$.*

An example of a forcing relation is Cohen forcing $(P, \leqslant, 1_P)$:

$$P = \mathrm{Fn}(\omega, 2, \aleph_0) = \{p | p\colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \omega \wedge \mathrm{card}(\mathrm{dom}(p)) < \aleph_0$$

consists of all *partial* functions from ω to 2. Cohen forcing will approximate a total function from ω to 2, i.e. a *real* number. The approximation of a total function is captured by the forcing relation: a condition p is *stronger* than q iff the function p extends the function q:

$$p \leqslant q \text{ iff } p \supseteq q.$$

Let $1_P = \emptyset$ be the function with the least information content. Two Cohen condition q_1, q_2 are compatible iff they are compatible as functions, i.e. if $q_1 \cup q_2$ is a function.

Fix some forcing relation $(P, \leqslant, 1_P) \in M$. It is important that the forcing relation is an element of the ground model so that the ZFC-properties of M may be applied to P.

Definition 43. *$G \subseteq P$ is a* filter *on P iff*

a) $1_P \in G$*;*

b) $\forall q \in G \forall p \geqslant q\, p \in G$*;*

c) $\forall p, q \in G \exists r \in G\, (r \leqslant p \wedge r \leqslant q)$*.*

In the case of Cohen forcing, a filter is a system of pairwise compatible partial function whose union is again a partial function from ω to 2. We shall later introduce *generic* filters which would make that union a total function.

Fix a filter G on P. We shall construct an extension $M[G]$ which will satisfy some axioms of ZFC. This will later be strengthened to *generic extensions* which satisfy all of ZFC. Elements $x \in M[G]$ will have *names* $\dot{x} \in M$ in the ground model; G allows to *interpret* $\dot{x}$ as $x : x = \dot{x}^G$. The crucial issue for computing the interpretation $\dot{x}^G$ is to decide when $\dot{y}^G \in \dot{x}^G$. This shall be decided by the filter G. So the important information about $\dot{x}$ is contained in the set

$$\{(\dot{y}, p) | p \text{ decides that } \dot{y} \in \dot{x}\}.$$

In the forcing method one identifies $\dot{x}$ with that set:

$$\dot{x} = \{(\dot{y}, p) | p \text{ decides that } \dot{y} \in \dot{x}\}.$$

This motivates the following interpretation function:

Definition 44. *Define the G-*interpretation *$\dot{x}^G$ of $\dot{x} \in M$ by recursion on the strongly wellfounded relation $\dot{y}\, R\, \dot{x}$ iff $\exists u\, (\dot{y}, u) \in \dot{x}$:*

$$\dot{x}^G = \{\dot{y}^G | \exists p \in G\ (\dot{y}, p) \in \dot{x}\}.$$

Let

$$M[G] = \{\dot{x}^G | \dot{x} \in M\}$$

be the extension *of M by P and G.*

We examine which set-theoretic axioms hold in $M[G]$.

Lemma 45. *$M[G]$ is transitive.*

Proof. Let $u \in \dot{x}^G \in M[G]$. Then $u \in \{\dot{y}^G | \exists p \in G\ (\dot{y}, p) \in \dot{x}\} \subseteq M[G]$. □

Lemma 46. $\forall \dot{x} \in M\ \mathrm{rank}(\dot{x}^G) \leqslant \mathrm{rank}(\dot{x})$.

Proof. By induction on the relation $\dot{y}\, R\, \dot{x}$ iff $\exists u\ (\dot{y}, u) \in \dot{x}$:

$$\begin{aligned}
\mathrm{rank}(\dot{x}^G) &= \bigcup \{\mathrm{rank}(\dot{y}^G) + 1 | \exists p \in G\ (\dot{y}, p) \in \dot{x}\} \\
&\leqslant \bigcup \{\mathrm{rank}(\dot{y}) + 1 | \exists p \in G\ (\dot{y}, p) \in \dot{x}\}, \text{ by inductive hypothesis,} \\
&\leqslant \bigcup \{\mathrm{rank}((\dot{y}, p)) + 1 | (\dot{y}, p) \in \dot{x}\} \\
&\leqslant \bigcup \{\mathrm{rank}(u) + 1 | u \in \dot{x}\} \\
&= \mathrm{rank}(\dot{x}).
\end{aligned}$$

□

To show that $M[G] \supseteq M$ we define names for elements of M.

Definition 47. *Define by $\in$-recursion the* canonical name *for $x \in M$:*

$$\check{x} = \{(\check{y}, 1_P) \,|\, y \in x\}.$$

Lemma 48. *For $x \in M$ holds $\check{x}^G = x$. Hence $M \subseteq M[G]$.*

Proof. By $\in$-induction.

$$\begin{aligned}
\check{x}^G &= \{\dot{y}^G | \exists p \in G\ (\dot{y}, p) \in \dot{x}\} \\
&= \{\check{y}^G | y \in x\}, \text{ by the definition of } \check{x} \text{ and since } 1_P \in G, \\
&= \{y | y \in x\}, \text{ by inductive hypothesis,} \\
&= x.
\end{aligned}$$

□

Lemma 49. $M[G] \cap \mathrm{Ord} = M \cap \mathrm{Ord}$.

Proof. Let $\alpha \in M[G] \cap \mathrm{Ord}$. Take $\dot{x} \in M$ such that $\dot{x}^G = \alpha$. By Lemma 33, $\mathrm{rank}(u)$ is a definite term. Hence $\mathrm{rank}(\dot{x}) \in M \cap \mathrm{Ord}$. Hence

$$\alpha = \mathrm{rank}(\alpha) = \mathrm{rank}(\dot{x}^G) \leqslant \mathrm{rank}(\dot{x}) \in M \cap \mathrm{Ord}.$$ □

To check that $G \in M[G]$ we need a name for G.

Definition 50. $\dot{G} = \{(\check{p}, p) | p \in P\}$ *is the* canonical name *for the filter on* P.

Lemma 51. $\dot{G} \in M$ *and* $\dot{G}^H = H$ *for any filter* H *on* P.

Proof. The term $\check{x}$ in the variable x is definite since it is defined by a definite $\in$-recursion. So $(\check{x}, x)$ and $\{(\check{p}, p) | p \in P\}$ are definite terms in the variables x and P resp. Then $P \in M$ implies that $\dot{G} \in M$. Moreover

$$\dot{G}^H = \{\check{p}^H | p \in H\} = \{p | p \in H\} = H.$$

□

Theorem 52. *$M[G]$ is a model of Extensionality, Pairing, Union, Infinity, and Foundation.*

Proof. We employ the criteria of Theorem 3. Extensionality and Choice hold since $M[G]$ is a transitive $\in$-model.

Pairing: Let $x, y \in M[G]$. Take names $\dot{x}, \dot{y} \in M$ such that $x = \dot{x}^G$, $y = \dot{y}^G$. Set

$$\dot{z} = \{(\dot{x}, 1_P), (\dot{y}, 1_P)\}.$$

Then

$$\{x, y\} = \{\dot{x}^G, \dot{y}^G\} = \dot{z}^G \in M[G].$$

Union: Let $x \in M[G]$ and $x = \dot{x}^G$, $\dot{x} \in M$. Set

$$\dot{z} = \{(\dot{u}, r) | \exists p, q \in P \exists \dot{v} (r \leqslant p \wedge r \leqslant q \wedge (\dot{u}, p) \in \dot{v} \wedge (\dot{v}, q) \in \dot{x}\}.$$

The right-hand side is a definite term in the variables $P, \leqslant, \dot{x} \in M$, hence $\dot{z} \in M$. We show that $\bigcup x = \dot{z}^G$.

Let $u \in \bigcup x$. Take $v \in x$ such that $u \in v \in x = \dot{x}^G$. Take $\dot{v} \in M$ and $q \in G$ such that $(\dot{v}, q) \in \dot{x}$ and $\dot{v}^G = v$. Take $\dot{u} \in M$ and $p \in G$ such that $(\dot{u}, p) \in \dot{v}$ and $\dot{u}^G = u$. Take $r \in G$ such that $r \leqslant p, q$. By the definition of $\dot{z}$, $(\dot{u}, r) \in \dot{z}$, and $u = \dot{u}^G \in \dot{z}^G$ since $r \in G$.

Conversely let $u \in \dot{z}^G$. Take $r \in G$ and $\dot{u} \in M$ such that $(\dot{u}, r) \in \dot{z}$ and $u = \dot{u}^G$. By the definition of $\dot{z}$, take $p, q \in P$ and $\dot{v} \in M$ such that

$$r \leqslant p \wedge r \leqslant q \wedge (\dot{u}, p) \in \dot{v} \wedge (\dot{v}, q) \in \dot{x}.$$

Then $p, q \in G$ and $u = \dot{u}^G \in \dot{v}^G \in \dot{x}^G = x$. Hence $u \in \bigcup x$.

Infinity holds in $M[G]$ since $\omega \in M \subseteq M[G]$. □

Problem 1. Do Powerset and Choice hold in $M[G]$?

9 Generic Filters and the Forcing Relation

If $(\dot{y}, p) \in \dot{x}$ then $p \in H \rightarrow \dot{y}^H \in \dot{x}^H$; so regardless of other aspects p "forces" that $\dot{y} \in \dot{x}$. And if $\dot{y}^H \in \dot{x}^H$ this is (leaving some technical issues aside) forced by some $p \in H$. We want to generalise this phenomenon from the most fundamental of formulas, $v_0 \in v_1$, to all $\in$-formulas: consider a formula $\varphi(v_0, \ldots, v_{n-1})$ and names $\dot{x}_0, \ldots, \dot{x}_{n-1}$. We want a relation

$$p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$$

such that

a) $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ implies that $M[H] \vDash \varphi(\dot{x}_0^H, \ldots, \dot{x}_{n-1}^H)$ for every appropriate filter H on P with $p \in H$

b) if $M[H] \vDash \varphi(\dot{x}_0^H, \ldots, \dot{x}_{n-1}^H)$ for some appropriate filter H on P with $p \in H$ then there is $p \in H$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

Let us continue the discussion with the vague notion of "appropriate filter". By b), an appropriate filter H has to decide every φ. There is $r \in H$ such that $r \Vdash \varphi$ or $r \Vdash \neg\varphi$:

$$\{r \in P | r \Vdash \varphi \text{ or } r \Vdash \neg\varphi\} \cap H \neq \emptyset;$$

We argue that the set $D = \{r \in P | r \Vdash \varphi \text{ or } r \Vdash \neg\varphi\}$ is a *dense* set in P. Let $p \in P$. Take an appropriate filter H on P with $p \in H$. Suppose that $M[H] \vDash \varphi$. By b) take some $q \in H$ such that $q \Vdash \varphi$. By the compatibility of filter elements take $r \in H$ such that $r \leqslant p, q$. Then $r \Vdash \varphi$ and $r \in D$. In case $M[H] \vDash \neg\varphi$ we similarly find $r \leqslant p$, $r \in D$.

It will turn out that the set D will be definable inside the ground model, thus $D \in M$. Accordingly, a filter H on P will be appropriate if it intersects *every* $D \in M$ which is a dense subset of P. We now give rigorous definitions of appropriate filters and of the forcing relation.

Definition 53. *Let* $(P, \leqslant, 1_P)$ *be a forcing.*

a) $D \subseteq P$ *is* dense *in* P *iff* $\forall p \in P \exists q \in D\, q \leqslant p$.

b) A filter G *on* P *is* M-generic *iff* $D \cap G \neq \emptyset$ *for every* $D \in M$ *which is dense in* P.

If $M[G]$ *is an extension of* M *by an* M*-generic filter we call* $M[G]$ *a* generic extension.

For *countable* ground models we have

Theorem 54. *Let $(P, \leqslant, 1_P)$ be a partial order, let M be a countable ground model, and let $p \in P$. Then there is an M-generic filter G on P with $p \in G$.*

Proof. Take a wellorder $\prec$ of M in ordertype ω. Let $(D_n | n < \omega)$ be an enumeration of all $D \in M$ which are dense in P. Define an ω-sequence $p = p_0 \geqslant p_1 \geqslant p_2 \geqslant \ldots$ recursively:

$$p_{n+1} \text{ is the } \prec\text{-smallest element of } M \text{ such that } p_{n+1} \leqslant p_n \text{ and } p_{n+1} \in D_n .$$

Then $G = \{p \in P | \exists n < \omega\, p_n \leqslant p\}$ is as desired. □

Fix a ground model M and a partial order $(P, \leqslant, 1_P) \in M$.

Definition 55. *Let $\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ be a* sentence of the forcing language*, i.e. $\varphi(v_0, \ldots, v_{n-1})$ is an $\in$-formulas and $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$. For $p \in P$ define $p \Vdash_P^M \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, p* forces *$\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, iff for all M-generic filters G on M with $p \in G$:*

$$\varphi^{M[G]}(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G).$$

If M or P are obvious from the context we also write $\Vdash_P$ or $\Vdash$ instead of $\Vdash_P^M$.

We shall state several properties of $\Vdash$. Some of the properties amount to a definition of $\Vdash\varphi$ by recursion on the complexity of φ which can be carried out inside the ground model M.

Lemma 56.

a) If $p \Vdash \varphi$ and $q \leqslant p$ then $q \Vdash \varphi$.

b) If $p \Vdash \varphi$ and φ implies ψ then $p \Vdash \psi$.

c) If $(\dot{y}, p) \in \dot{x}$ and $p \in P$ then $p \Vdash \dot{y} \in \dot{x}$.

Proof. *a*) Let $G \ni q$ be M-generic on P. Then $p \in G$. Hence $M[G] \vDash \varphi$.

b) Let $G \ni p$ be M-generic on P. Then $M[G] \vDash \varphi$. Since φ implies ψ, also $M[G] \vDash \psi$.

c) Let $G \ni p$ be M-generic on P. Then

$$\dot{y}^G \in \{\dot{u}^G | \exists q \in G\ (\dot{u}, q) \in \dot{x}\} = \dot{x}^G.$$ □

For simplicity we assume that $\in$-formulas are only built from the connectives $\wedge, \neg, \forall$. We want to show (recursively) that every $\in$-formula has the following property:

Definition 57. *The* $\in$*-formula* $\varphi(v_0, \ldots, v_{n-1})$ satisfies the forcing theorem *iff the following hold:*

a) *The class*

$$\text{Force}_\varphi = \{(p, \dot{x}_0, \ldots, \dot{x}_{n-1}) \,|\, p \in P \wedge \dot{x}_0, \ldots, \dot{x}_{n-1} \in M \wedge p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})\}$$

is definable in M;

b) *if* $M[G]$ *is a generic extension and* $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$ *with* $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ *then there is* $p \in G$ *such that* $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$*.*

Lemma 58. *Let* $\varphi(v_0, \ldots, v_{n-1})$ *and* $\psi(v_0, \ldots, v_{n-1})$ *be* $\in$*-formulas satisfying the forcing theorem. Then we have for all names* $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$

a) $p \Vdash (\varphi \wedge \psi)(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *iff* $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *and* $p \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$*.*

b) $p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ *iff* $\forall q \leqslant p \, \neg q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$*.*

c) $p \Vdash \forall v_0 \varphi(v_0, \dot{x}_1, \ldots, \dot{x}_{n-1})$ *iff* $\forall \dot{x}_0 \in M \, p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$*.*

d) *The formulas* $(\varphi \wedge \psi)$*,* $\neg\varphi$*, and* $\forall v_0 \, \varphi$ *satisfy the forcing theorem.*

Proof. *a*) is immediate.

b) For the implication from left to right assume $p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ and let $q \leqslant p$. If $q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ then $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take an M-generic $G \ni p$. Then $M[G] \vDash \neg \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ and $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Contradiction.

For the converse assume $\neg p \Vdash \neg\varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. By the definition of $\Vdash$ take an M-generic $G \ni p$ such that $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Since φ satisfies the forcing theorem take $r \in G$ with $r \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take $q \in G$ such that $q \leqslant p, r$. Then $q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, and the right-hand side of the equivalence is false.

c) is similar to the case *a*). The implication from left to right is immediate. For the converse assume $\forall \dot{x}_0 \in M \, p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. et $G \ni p$ be M-generic on P. Then $\forall \dot{x}_0 \in M \, M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Then $M[G] \vDash \forall v_0 \varphi(v_0, \dot{x}_1^G, \ldots, \dot{x}_{n-1}^G)$. Thus $p \Vdash \forall v_0 \varphi(v_0, \dot{x}_1, \ldots, \dot{x}_{n-1})$.

d) The cases *a*) $-$ *c*) contain definitions of $\text{Force}_{\varphi \wedge \psi}$, $\text{Force}_{\neg\varphi}$, and $\text{Force}_{\forall v_0 \varphi}$ on the basis of definitions of Force_φ and Force_ψ. We now show *b*) of Definition 57 for $\varphi \wedge \psi$, $\neg\varphi$, and $\forall v_0 \, \varphi$. So let $M[G]$ be a generic extension.

$\varphi \wedge \psi$: Assume $M[G] \vDash (\varphi \wedge \psi)(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Then $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ and $M[G] \vDash \psi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$. Since φ and ψ satisfy the forcing theorem, take $p, q \in G$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ and $q \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$. Take $r \in G$ with $r \leqslant p$, q. Then $r \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, $r \Vdash \psi(\dot{x}_0, \ldots, \dot{x}_{n-1})$, and $r \Vdash (\varphi \wedge \psi)(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

$\neg\varphi$: Assume $M[G] \vDash \neg\varphi(\dot{x}_0^G, ..., \dot{x}_{n-1}^G)$. Define

$$D = \{p \in P \,|\, p \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1}) \text{ or } \forall q \leqslant p \;\; \neg q \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})\}.$$

Since Force_φ is definable in M, we get $D \in M$. It is easy to see that D is dense in P. By the genericity of G take $p \in G \cap D$. We cannot have $p \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})$ because $M[G] \vDash \neg \varphi(\dot{x}_0^G, ..., \dot{x}_{n-1}^G)$. Hence $\forall q \leqslant p \;\; \neg q \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})$. Then *b*) implies that $p \Vdash \neg\varphi(\dot{x}_0, ..., \dot{x}_{n-1})$.

$\forall v_0 \, \varphi$: Assume $M[G] \vDash \forall v_0 \, \varphi(v_0, \dot{x}_1^G, ..., \dot{x}_{n-1}^G)$. Define

$$D = \{p \in P \,|\forall \dot{x}_0 \in M \, p \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1}) \text{ or } \exists \dot{x}_0 \in M \, p \Vdash \neg\varphi(\dot{x}_0, \dot{x}_1 ..., \dot{x}_{n-1})\}.$$

Then $D \in M$ since Force_φ and $\text{Force}_{\neg\varphi}$ are definable in M.

(1) D is dense in P.
Proof. Consider $r \in P$. If $\forall \dot{x}_0 \in M \, r \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})$ then $r \in D$. Otherwise take $\dot{x}_0 \in M$ with $\neg r \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})$. Take an M-generic filter $H \ni r$ such that $M[H] \vDash \neg\varphi(\dot{x}_0^G, ..., \dot{x}_{n-1}^G)$. Since $\neg\varphi$ satisfies the forcing theorem, take $s \in H$ with $s \Vdash \neg\varphi(\dot{x}_0, ..., \dot{x}_{n-1})$. Take $p \in H$ such that $p \leqslant r, s$. Then $p \Vdash \neg\varphi(\dot{x}_0, ..., \dot{x}_{n-1})$ and $p \in D$. *qed*(1)

By the genericity of G take $p \in G \cap D$. Assume for a contradiction that $\exists \dot{x}_0 \in M p \Vdash \neg\varphi(\dot{x}_0, \dot{x}_1 ..., \dot{x}_{n-1})$. Take $\dot{x}_0 \in M$ such that $p \Vdash \neg\varphi(\dot{x}_0, \dot{x}_1 ..., \dot{x}_{n-1})$. Since $p \in G$, $M[G] \vDash \neg\varphi(\dot{x}_0^G, \dot{x}_1^G, ..., \dot{x}_{n-1}^G)$, contradicting the assumption of the quantifier case. So p is in the "other half' of D, i.e. $\forall \dot{x}_0 \in M \, p \Vdash \varphi(\dot{x}_0, ..., \dot{x}_{n-1})$. By *c*), $p \Vdash \forall v_0 \, \varphi(v_0, \dot{x}_1, ..., \dot{x}_{n-1})$. □

10 The Atomic Case

The atomic case of the forcing theorem turns out more complicated than the cases that we have considered so far. This is due to the hierarchical structure of sets. We treat the equality case $v_1 = v_2$ as two inclusions $v_1 \subseteq v_2$ and $v_2 \subseteq v_1$. The relation $x_1^G \subseteq x_2^G$ is equivalent to

$$\{y_1^G | \exists s_1 \in G \; (y_1, s_1) \in x_1\} \subseteq \{y_2^G | \exists s_2 \in G \; (y_2, s_2) \in x_2\}.$$

Lemma 59.

a) *$p \Vdash x_1 \subseteq x_2$ iff $\forall (y_1, s_1) \in x_1 \, (s_1 \in P \rightarrow$
$D(y_1, s_1, x_2) := \{q \in P | q \leqslant s_1 \rightarrow \exists (y_2, s_2) \in x_2 (s_2 \in P \wedge q \leqslant s_2 \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1)\}$
is dense in P below p).*

b) $\text{Force}_{v_1 \subseteq v_2}$ *is definable in M.*

c) *If $x_1^G \subseteq x_2^G$ then there is $p \in G$ such that $p \Vdash x_1 \subseteq x_2$.*

Here we say that a set $D \subseteq P$ is dense in P below p *iff* $\forall p' \leqslant p \exists q \leqslant q\, q \in D$.

Proof. Consider the relation

$$(q, y_1, y_2)\, R\, (p, x_1, x_2) \leftrightarrow (y_1 \in \mathrm{dom}(x_1) \vee y_1 \in \mathrm{dom}(x_2)) \wedge (y_2 \in \mathrm{dom}(x_1) \vee y_2 \in \mathrm{dom}(x_2)).$$

(1) R is strongly wellfounded.
Proof. If $(q, y_1, y_2)\, R\, (p, x_1, x_2)$ then

$$(\mathrm{rg}(y_1) < \mathrm{rg}(x_1) \vee \mathrm{rg}(y_1) < \mathrm{rg}(x_2)) \wedge (\mathrm{rg}(y_2) < \mathrm{rg}(x_1) \vee \mathrm{rg}(y_2) < \mathrm{rg}(x_2)),$$

and so $\max(\mathrm{rg}(y_1), \mathrm{rg}(y_2)) < \max(\mathrm{rg}(x_1), \mathrm{rg}(x_2))$. Hence an infinite decreasing sequence in R leads to an infinite decreasing sequence in Ord. *qed*(1)

By recursion on R define

$$\begin{aligned} S(p, x_1, x_2) \ \leftrightarrow\ & \forall (y_1, s_1) \in x_1\, (s_1 \in P \rightarrow \\ & \{q \in P \,|\, q \leqslant s_1 \rightarrow \exists (y_2, s_2) \in x_2 (s_2 \in P \wedge q \leqslant s_2 \wedge S(q, y_1, y_2) \wedge S(q, y_2, y_1)\} \\ & \text{is dense in } P \text{ below } p). \end{aligned}$$

By a simultaneous induction on R we prove that $(p \Vdash x_1 \subseteq x_2) \leftrightarrow S(p, x_1, x_2)$ and properties *a*) and *c*). This also proves *b*).

a) Assume $p \Vdash x_1 \subseteq x_2$. Let $(y_1, s_1) \in x_1$ and $s_1 \in P$. To show that $D(y_1, s_1, x_2)$ is dense in P below p consider $p' \leqslant p$. It suffices to find $q \leqslant p'$ with $q \in D(y_1, s_1, x_2)$. Let $G \ni p'$ be M-generic on P.

If $\neg p' \leqslant s_1$ then $p' \in D(y_1, s_1, x_2)$ and we can take $q = p'$.

So assume that $p' \leqslant s_1$. Then $s_1, p \in G$ and

$$y_1^G \in x_1^G \subseteq x_2^G = \{y_2^G \,|\, \exists s_2 \in G\ (y_2, s_2) \in x_2\}.$$

Take $(y_2, s_2) \in x_2$ such that $s_2 \in G$ and $y_1^G = y_2^G$. Then $y_1^G \subseteq y_2^G$ and $y_2^G \subseteq y_1^G$. By the inductive assumption c) take $p'', p''' \in G$ such that $p'' \Vdash y_1 \subseteq y_2$ and $p''' \Vdash y_2 \subseteq y_1$. Take $q \in G$ such that $q \leqslant p', s_2, p'', p'''$. Then $q \leqslant p' \leqslant s_1$, $q \leqslant s_2$, $q \Vdash y_1 \subseteq y_2$, and $q \Vdash y_2 \subseteq y_1$. Hence $q \in D(y_1, s_1, x_2)$.

Conversely assume the right-hand side of *a*). Let $G \ni p$ be M-generic on P. We have show that $x_1^G \subseteq x_2^G$, i.e. $\{y_1^G | \exists s_1 \in G\ (y_1, s_1) \in x_1\} \subseteq \{y_2^G | \exists s_2 \in G\ (y_2, s_2) \in x_2\}$. So let $y_1^G \in x_1^G$. Take $s_1 \in G$ such that $(y_1, s_1) \in x_1$. Take $p' \in G$, $p' \leqslant p, s_1$. The right-hand side of a) implies that $D(y_1, s_1, x_2)$ is dense in P below p and thus below p'. By the inductive assumption, $D(y_1, s_1, x_2) \in M$. By the genericity of G, take $q \in G$, $q \leqslant p'$, $q \in D(y_1, s_1, x_2)$. By the definition of $D(y_1, s_1, x_2)$ take $(y_2, s_2) \in x_2$ such that

$$s_2 \in P \wedge q \leqslant s_2 \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1.$$

Since $q, s_2 \in G$ this implies $y_1^G \subseteq y_2^G$, $y_2^G \subseteq y_1^G$, and so

$$y_1^G = y_2^G \in x_2^G.$$

Ror $(q,\ y_1,\ y_2)\ R\ (p,\ x_1,\ x_2)$ the induction hypothesis implies that $S(q,\ y_1,\ y_2)$ and $S(q,\ y_2, y_1)$ agree with $q \Vdash y_1 \subseteq y_2$ and $q \Vdash y_2 \subseteq y_1$ respectively. Now $a)$ and the recursive definition of $S(p, x_1, x_2)$ agree and yield that

$$(p \Vdash x_1 \subseteq x_2) \leftrightarrow S(p, x_1, x_2).$$

$c)$ Let $M[G]$ be a generic extension such that $M[G] \vDash x_1^G \subseteq x_2^G$. Set

$$\begin{aligned} D = \{p \in P \ \mid \ & p \Vdash x_1 \subseteq x_2 \\ & \vee \exists (y_1, s_1) \in x_1 \, (s_1 \in P \wedge \forall q \leqslant p \\ & \ (q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2((s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1) \rightarrow \neg q \leqslant s_2)))\}. \end{aligned}$$

$D \in M$ since by the inductive assumption we may replace $\Vdash$ in the definition of D by the predicate S which is definable in M.

(2) D is dense in P.
Proof. Let $r \in P$. If $r \Vdash x_1 \subseteq x_2$ we are done. So assume $\neg r \Vdash x_1 \subseteq x_2$. By the equivalence in $a)$ take $(y_1, s_1) \in x_1$ such that $s_1 \in P$ and $D(y_1, s_1, x_2)$ is *not* dense in P below r. Take $p \leqslant r$ such that $\forall q \leqslant p \, q \notin D(y_1, s_1, y_2)$. $q \notin D(y_1, s_1, y_2)$ is equivalent to

$$q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2(s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1 \rightarrow \neg q \leqslant s_2).$$

Hence $p \leqslant r$ is an element of D. *qed*(2)

By the M-genericity take $p \in G \cap D$. We claim that $p \Vdash x_1 \subseteq x_2$. If not then the alternative in the definition of D holds: take $(y_1, s_1) \in x_1$ such that $s_1 \in P$ and

$$\forall q \leqslant p \ (q \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2((s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1) \rightarrow \neg q \leqslant s_2)). \tag{5}$$

In particular for $q = p$ we have

$$p \leqslant s_1 \wedge \forall (y_2, s_2) \in x_2((s_2 \in P \wedge p \Vdash y_1 \subseteq y_2 \wedge p \Vdash y_2 \subseteq y_1) \rightarrow \neg p \leqslant s_2).$$

Then $s_1 \in G$ and $y_1^G \in x_1^G \subseteq x_2^G = \{y_2^G | \exists s_2 \in G \ (y_2, s_2) \in x_2\}$. Take $(y_2, s_2) \in x_2$ such that $s_2 \in G$ and $y_1^G = y_2^G$. Then $y_1^G \subseteq y_2^G$ and $y_2^G \subseteq y_1^G$. Since $c)$ holds at R-smaller triples, there are $q', q'' \in G$ such that $q' \Vdash y_1 \subseteq y_2$ and $q'' \Vdash y_2 \subseteq y_1$. Take $q \in G$ such that $q \leqslant p, s_2, q', q''$. Then (y_2, s_2) satisfies

$$s_2 \in P \wedge q \Vdash y_1 \subseteq y_2 \wedge q \Vdash y_2 \subseteq y_1 \wedge q \leqslant s_2.$$

But this contradicts (5). Hence $p \Vdash x_1 \subseteq x_2$. □

We can now deal with the other atomic cases:

Lemma 60.

a) $x = y$ *satisfies the forcing theorem.*

b) $x \in y$ *satisfies the forcing theorem.*

Proof. For *a*) observe that $p \Vdash x = y$ iff $p \Vdash x \subseteq y$ and $p \Vdash y \subseteq x$.

b) We claim that $p \Vdash x \in y$ iff $D = \{q \leqslant p | \exists (u, r) \in y \ (q \leqslant r \wedge q \Vdash x = u)\}$ is dense in P below p.

Assume that $p \Vdash x \in y$. To prove the density of D consider $s \leqslant p$. Take an M-generic filter G on P with $s \in G$. $s \Vdash x \in y$ and so $x^G \in y^G = \{u^G | \exists r \in G \, (u, r) \in y\}$. Take $(u, r) \in y$ such that $x^G = u^G$ and $r \in G$. By the forcing theorem for equalities take $t \in G$ such that $t \Vdash x = u$. Take $q \in G$ such that $q \leqslant s, r, t$. Then $q \leqslant p$, $q \leqslant r$, and $q \Vdash x = u$. Hence $q \in D$.

Conversely let D be dense in P below p. To show that $p \Vdash x \in y$ let G be an M-generic filter on P with $p \in G$. By the genericity there is $q \leqslant p$ such that $q \in G \cap D$. Take $(u, r) \in y$ such that $q \leqslant r \wedge q \Vdash x = u$. Then $r \in G$ and $x^G = u^G \in y^G$.

Finally assume that $x^G \in y^G$. $y^G = \{u^G | \exists r \in G \ (u, r) \in y\}$. Take some $(u, r) \in y$ such that $r \in G$ and $x^G = u^G$. By *a*) take $s \in G$ such that $s \Vdash x = u$. Take $p \in G$ such that $p \leqslant r, s$. Then $p \Vdash x = u$ and $p \Vdash u \in y$. Hence $p \Vdash x \in y$. $\square$

So we have proved the *forcing theorem*:

Theorem 61. *For every $\in$-formula $\varphi(v_0, \ldots, v_{n-1})$ the following hold:*

a) The property $p \Vdash_P^M \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$ is definable in M;

b) if $M[G] \vDash \varphi(\dot{x}_0^G, \ldots, \dot{x}_{n-1}^G)$ in a generic extension $M[G]$ then there is $p \in G$ such that $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1})$.

11 ZFC in $M[G]$

Let $M[G]$ be a generic extension of the ground model M by the generic filter G on $P \in M$. We know already that $M[G]$ is a model of Extensionality, Pairing, Union, Infinity, and Foundation.

Theorem 62. $M[G] \vDash$ *Separation*

Proof. Consider an $\in$-formula $\varphi(x, \vec{w})$ and sets $\vec{w}, a \in M[G]$. It suffices to prove that $\{x \in a | \varphi^{M[G]}(x, \vec{w})\} \in M[G]$. Take names $\dot{\vec{w}}, \dot{a} \in M$ such that $\dot{\vec{w}}^G = \vec{w}$ and $\dot{a}^G = a$. Define the name

$$\dot{z} = \{(\dot{x}, p) | \dot{x} \in \mathrm{dom}(\dot{a}) \wedge p \Vdash (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{\vec{w}}))\}.$$

By the forcing theorem, $\dot{z} \in M$. It suffices to show that $\dot{z}^G = \{x \in a | \varphi^{M[G]}(x, \vec{w})\}$.

Consider $x \in \dot{z}^G$. Take $(\dot{x}, p) \in \dot{z}$ such that $p \in G$ and $x = \dot{x}^G$. Then $p \Vdash (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{\vec{w}}))$. By the definition of $\Vdash$, $x = \dot{x}^G \in \dot{a}^G = a$, $\varphi^{M[G]}(\dot{x}^G, \dot{\vec{w}}^G)$, and $\varphi^{M[G]}(x, \vec{w})$. Hence $x \in \{x \in a | \varphi^{M[G]}(x, \vec{w})\}$.

Conversely let $x \in \{x \in a | \varphi^{M[G]}(x, \vec{w})\}$. Take $(\dot{x}, p') \in \dot{a}$ such that $p' \in G$ and $x = \dot{x}^G$. Then $\varphi^{M[G]}(\dot{x}^G, \dot{\vec{w}}^G)$. By the forcing theorem, take $p'' \in G$ such that $p'' \Vdash \varphi(\dot{x}, \dot{\vec{w}})$. Take $p \in G$ such that $p \leqslant p'$ and $p \leqslant p''$. Then

$$\dot{x} \in \mathrm{dom}(\dot{a}) \wedge p \Vdash (\dot{x} \in \dot{a} \wedge \varphi(\dot{x}, \dot{\vec{w}}))$$

and

$$(\dot{x}, p) \in \dot{z}.$$

Hence $x = \dot{x}^G \in \dot{z}^G$. $\square$

Theorem 63. $M[G] \vDash$ *Power.*

Proof. Let $x \in M[G]$. It suffices to show that $\mathcal{P}(x) \cap M[G] \in M[G]$. Take a name $\dot{x} \in M$ such that $\dot{x}^G = x$. Define the name

$$\dot{z} = \{(\dot{y}, p) | \dot{y} \subseteq \mathrm{dom}(\dot{x}) \times P \wedge p \Vdash \dot{y} \subseteq \dot{x}\}.$$

By the forcing theorem and by the powerset axiom in M, $\dot{z} \in M$. It suffices to show that $\dot{z}^G = \mathcal{P}(x) \cap M[G]$.

Consider $y \in \dot{z}^G$. Take $(\dot{y}, p) \in \dot{z}$ such that $y = \dot{y}^G$ and $p \in G$. Then $p \Vdash \dot{y} \subseteq \dot{x}$ and so $y = \dot{y}^G \subseteq \dot{x}^G = x$. Hence $y \in \mathcal{P}(x) \cap M[G]$.

Conversely let $v \in \mathcal{P}(x) \cap M[G]$. Take $\dot{v} \in M$ such that $v = \dot{v}^G$ and take $q \in G$ such that $q \Vdash \dot{v} \subseteq \dot{x}$. Define

$$\dot{y} = \{(\dot{u}, r) \in \mathrm{dom}(\dot{x}) \times P | r \leqslant q \wedge q \Vdash \dot{u} \in \dot{v}\}.$$

(1) $\dot{y}^G = \dot{v}^G$.
Proof. Let $\dot{u}^G \in \dot{y}^G$, $(\dot{u}, r) \in \dot{y}$ and $r \in G$. Then $r \Vdash \dot{u} \in \dot{v}$, and $\dot{u}^G \in \dot{v}^G$. Conversely let $u \in \dot{v}^G \subseteq \dot{x}^G$. Then $u = \dot{u}^G$ for some $\dot{u} \in \mathrm{dom}(\dot{x})$. Since $\dot{u}^G \in \dot{v}^G$ take $r \in G$ such that $r \leqslant q$ and $r \Vdash \dot{u} \in \dot{v}$. Then $(\dot{u}, r) \in \dot{y}$ and $u = \dot{u}^G \in \dot{y}^G$. *qed*(1)

Take $p \in G$ such that $p \Vdash \dot{y} = \dot{v}$ and $p \Vdash \dot{v} \subseteq \dot{x}$. Then $p \Vdash \dot{y} \subseteq \dot{x}$ and $(\dot{y}, p) \in \dot{z}$. Hence $v = \dot{v}^G = \dot{y}^G \in \dot{z}^G$. $\square$

Theorem 64. $M[G] \vDash$ *Replacement.*

Proof. Consider an $\in$-formula $\varphi(x, y, \vec{w})$ and sets $\vec{w}, a \in M[G]$. Suppose that $\varphi(x, y, \vec{w})$ is functional in $M[G]$, i.e.,

$$M[G] \vDash \forall x, y, y' (\varphi(x, y, \vec{w}) \wedge \varphi(x, y', \vec{w}) \rightarrow y = y').$$

Take $p \in G$ such that $p \Vdash \forall x, y, y' (\varphi(x, y, \vec{w}) \wedge \varphi(x, y', \vec{w}) \rightarrow y = y')$.

It suffices to prove that

$$\{y | \exists x \in a \varphi^{M[G]}(x, y, \vec{w})\} \cap M[G] \in M[G].$$

Take names $\dot{\vec{w}}, \dot{a} \in M$ such that $\dot{\vec{w}}^G = \vec{w}$ and $\dot{a}^G = a$. Using replacement in the ground model M take a set $B \in M$ such that

$$\forall \dot{x} \in \operatorname{dom}(\dot{a}) \forall s \in P (\exists \dot{y} \in M s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\vec{w}}) \rightarrow \exists \dot{y} \in B s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\vec{w}})).$$

Define the name

$$\dot{z} = \{(\dot{y}, p) | \dot{y} \in B\} \in M.$$

We claim that

$$\{y | \exists x \in a \varphi^{M[G]}(x, y, \vec{w})\} \cap M[G] \subseteq \dot{z}^G.$$

Let $x \in a$ such that $\varphi^{M[G]}(x, y, \vec{w})$. Take $\dot{x} \in \operatorname{dom}(\dot{a})$ such that $x = \dot{x}^G$ and take $\dot{y} \in M$ such that $y = \dot{y}^G$. Take $s \in G$ such that $s \Vdash \varphi(\dot{x}, \dot{y}, \dot{\vec{w}})$. We may assume that $s \leqslant p$. By the choice of B there is $\dot{u} \in B$ such that $s \Vdash \varphi(\dot{x}, \dot{u}, \dot{\vec{w}})$. Since s forces the functionality of φ we have $s \Vdash \dot{y} = \dot{u}$. Hence $y = \dot{y}^G = \dot{u}^G \in \dot{z}^G$.

But then separation in $M[G]$ implies that

$$\{y | \exists x \in a \varphi^{M[G]}(x, y, \vec{w})\} \cap M[G] = \{y | \exists x \in a \varphi^{M[G]}(x, y, \vec{w})\} \cap \dot{z}^G \in M[G]$$

as required. □

Theorem 65. $M[G] \vDash \mathrm{AC}$.

Proof. Let $x \in M[G]$. It suffices to find a surjection $f: \alpha \rightarrow x' \supseteq x$, $f \in M[G]$, $\alpha \in \mathrm{Ord}$. Take a name $\dot{x} \in M$ such that $x = \dot{x}^G$. Since AC holds in M, take a surjective $g: \alpha \rightarrow \operatorname{dom}(\dot{x})$, $g \in M$, $\alpha \in \mathrm{Ord}$. Define $f: \alpha \rightarrow \operatorname{range}(f)$ by

$$f(\xi) = g(\xi)^G.$$

Note that the interpretation function $\dot{z} \mapsto \dot{z}^G$ is defined by a definite recursion in the parameter G. Hence it is definable within $M[G]$ and $f \in M[G]$, using replacement in $M[G]$. To show that $x \subseteq \text{range}(f)$ consider $y \in x$. Take $\dot{y} \in \text{dom}(\dot{x})$ such that $y = \dot{y}^G$. Take $\xi < \alpha$ such that $g(\xi) = \dot{y}$. Then

$$y = \dot{y}^G = g(\xi)^G = f(\xi) \in \text{range}(f).$$

□

Theorem 66. *$M[G]$ is the $\subseteq$-minimal transitive model of* ZF^- *such that $M \cup \{G\} \subseteq M[G]$.*

Proof. Assume that N is a transitive model of ZF^- such that $M \cup \{G\} \subseteq N$. Since the interpretation function $\dot{z} \mapsto \dot{z}^G$ is defined by a definite recursion in the parameter G, the model N is closed under the interpretation function. This means that

$$M[G] = \{\dot{z}^G | \dot{z} \in M\} \subseteq N.$$

□

12 Adding a Cohen real

To exclude trivial generic extensions of the form $M[G] = M$ we define

Definition 67. *A forcing $(P, \leqslant, 1)$ is* separative *if $\forall p \in P \exists q, r \leqslant p\ q \perp r$ where $q \perp r$ denotes that q and r are* incompatible.

Most forcings are separative. As an example consider the COHEN forcing $(P, \leqslant, 1)$

$$P = \text{Fn}(\omega, 2, \aleph_0) = \{p | p: \text{dom}(p) \rightarrow 2 \wedge \text{dom}(p) \subseteq \omega \wedge \text{card}(\text{dom}(p)) < \aleph_0\}$$

with $\leqslant = \supseteq$ and $1 = \emptyset$. Given $p \in P$ take $i \in \omega \setminus \text{dom}(p)$. Then

$$p \cup \{(i, 0)\} \leqslant p,\ p \cup \{(i, 1)\} \leqslant p, \text{ and } p \cup \{(i, 0)\} \perp p \cup \{(i, 1)\}.$$

Lemma 68. *Let G be M-generic on the separative forcing $(P, \leqslant, 1) \in M$. Then $G \notin M$ and $M[G] \neq M$.*

Proof. Assume that $G \in M$. Define

$$D = \{q \in P | q \notin G\} \in M.$$

(1) D is dense in P.
Proof. Let $p\in P$. By separativity take $q,r\leqslant p$ such that $q\perp r$. Since elements of the filter G are pairwise compatible, $q\notin G$ or $r\notin G$. Hence $q\in D$ or $r\in D$. *qed*(1)

By the genericity of G, $G\cap D\neq\emptyset$. But by the definition of D, $G\cap D=\emptyset$. Contradiction. ☐

Now consider Cohen forcing $P=\mathrm{Fn}(\omega,2,\aleph_0)$ over the ground model M. Let G be M-generic on P.

Lemma 69. *The set*

$$c=\bigcup G$$

is a Cohen *real over* M*, i.e.,*

a) $c:\omega\to 2$*;*

b) the generic G *can be defined from* c *as* $G=\{p\in P\,|\,p\subseteq c\}$*;*

c) $c\notin M$*.*

Proof. *a)* We show that c is functional. Let $(n,i),(n,j)\in c$. Take $p,q\in G$ such that $(n,i)\in p$ and $(n,j)\in q$. Since G is a filter there is $r\in G$ such that $r\leqslant p,q$. Then $(n,i),(n,j)\in r$. Since r is a function, $i=j$.

Hence $c:\mathrm{dom}(c)\to 2$ where $\mathrm{dom}(c)\subseteq\omega$.

(1) $\mathrm{dom}(c)=\omega$.
Proof. We use a density argument which is typical for the analysis of forcing extensions. Let $n\in\omega$. Define

$$D_n=\{p\in P|n\in\mathrm{dom}(p)\}.$$

Obviously $D_n\in M$. We show that D_n is dense in P. Let $q\in P$. If $n\in\mathrm{dom}(q)$ then $q\in D_n$. Otherwise $q\cup\{(n,0)\}\leqslant q$ and $q\cup\{(n,0)\}\in D_n$.

Since G is M-generic on P, $G\cap D_n\neq\emptyset$. Take $p\in G\cap D_n$. Then

$$n\in\mathrm{dom}(p)\subseteq\mathrm{dom}(c).$$

b) If $q\in G$ then $q\subseteq\bigcup G=c$. Conversely consider $q\in P$, $q\subseteq c$. For every assignment $(n,i)\in q$ take $p_n\in G$ such that $p_n(n)=i$. Since $\mathrm{dom}(q)$ is finite and since G is a filter we can take $p\in G$ such that $\forall n\in\mathrm{dom}(q)\ p\leqslant p_n$. Then $\forall n\in\mathrm{dom}(q)\ p(n)=p_n(n)=q(n)$ and $p\leqslant q$. Since G is upwards closed, $q\in G$.

c) follows from b) since $G\notin M$. But we want to give another density argument. Consider a real $x\in M$, $x{:}\,\omega\to 2$. Define

$$D_x=\{p\in P|p\nsubseteq x\}.$$

$D_x\in M$ since this is a definite definition from the parameter $x\in M$. We show that D_x is dense in P. Let $q\in P$. Since $\mathrm{dom}(q)$ is finite, take $n\in\omega\setminus\mathrm{dom}(q)$. Then $q\cup\{(n,1-x(n))\}\leqslant q$ and $q\cup\{(n,1-x(n))\}\in D_x$.

Since G is M-generic on P, $G\cap D_x\neq\emptyset$. Take $p\in G\cap D_x$. Then $p\nsubseteq x$. Take $n\in\mathrm{dom}(p)$ such that $p(n)\neq x(n)$. Then $c(n)=p(n)\neq x(n)$ and so $c\neq x$. Since c is different from every ground model real x we have that $c\notin M$. □

By b) the extension $M[G]$ is also the $\subseteq$-smallest ZF^--extension of M which contains c. So one can also write $M[c]$ instead of $M[G]$ and call it a Cohen extension.

Cohen reals are a very common component of forcing extensions. In the next section we shall adjoin a great number of Cohen reals thereby violating the continuum hypothesis. As a preparation we shall study some more properties of the present generic extension $M[G]$. A lot of information can be decoded from (the bits of) a Cohen real and we shall see an example now.

If we identify reals with characteristic functions on ω, $\mathbb{R}={}^{\omega}2$, then $\mathbb{R}\cap M[G]\supseteq\mathbb{R}\cap M$ is a proper transcendental field extension. We shall see that the adjunction of c makes the set of ground model reals very small.

Definition 70. *A set $X\subseteq\mathbb{R}$ has* measure zero *if for every $\varepsilon>0$ there exists sequence $(I_n|n<\omega)$ of intervals in $\mathbb{R}$ such that $X\subseteq\bigcup_{n<\omega}I_n$ and $\sum_{n<\omega}\mathrm{length}(I_n)\leqslant\varepsilon$.*

Lemma 71. *In the Cohen extension $M[c]$ the set $\mathbb{R}\cap M$ of ground model reals has measure zero.*

Proof. For our purposes we define real *intervals* as follows: for $s\in{}^{<\omega}2=\{t|t{:}\,\mathrm{dom}(t)\to 2\wedge\mathrm{dom}(t)\in\omega\}$ define the *interval*

$$I_s=\{x\in\mathbb{R}|s\subseteq x\}\subseteq\mathbb{R}$$

and define $\mathrm{length}(I_s)=2^{-\mathrm{dom}(s)}$. Note that $I_s=I_{s\cup\{(\mathrm{dom}(s),0)\}}\cup I_{s\cup\{(\mathrm{dom}(s),1)\}}$, $\mathrm{length}(\mathbb{R})=I_\emptyset=2^{-0}=1$, and $\mathrm{length}(I_{s\cup\{(\mathrm{dom}(s),0)\}})=\mathrm{length}(I_{s\cup\{(\mathrm{dom}(s),1)\}})=\frac{1}{2}\mathrm{length}(I_s)$.

Let $\varepsilon>0$ be given. We may assume that $\varepsilon=2^{-i}$. We shall extract intervals $I_0, I_1, I_2, \ldots$ of lengths $2^{-i-1}, 2^{-i-2}, 2^{-i-3}, \ldots$ from the Cohen real c. For $n<\omega$ define $s_n{:}\,i+n+1\to 2$ by

$$s_n(l)=c(n+l).$$

Let $I_n = I_{s_n}$. Then

$$\sum_{n<\omega} \operatorname{length}(I_n) = \sum_{n<\omega} 2^{-i-n-1} = 2^{-i} = \varepsilon.$$

We show by a density argument that $\mathbb{R} \cap M \subseteq \bigcup_{n<\omega} I_n$. Let $x \in \mathbb{R} \cap M$. Define

$$D_x = \{p \in P \mid \exists n < \omega \, \forall l < i + n + 1 \, (n + l \in \operatorname{dom}(p) \wedge p(n + l) = x(l))\} \in M.$$

(1) D_x is dense in P.
Proof. Let $q \in P$. Take $n < \omega$ such that $\operatorname{dom}(q) \subseteq n$. Set

$$p = q \cup \{(n + l, x(l)) \mid l < i + n + 1\}.$$

Then $p \leqslant q$ and $p \in D_x$. *qed*(1)

By the genericity of G take $p \in G \cap D_x$. Take $n < \omega$ such that

$$\forall l < i + n + 1 \, (n + l \in \operatorname{dom}(p) \wedge p(n + l) = x(l)).$$

Then

$$\forall l < i + n + 1 \; c(n + l) = x(l)$$

and

$$\forall l < i + n + 1 \; s_n(l) = x(l).$$

Hence $s_n \subseteq x$ and $x \in I_n \subseteq \bigcup_{n<\omega} I_n$. □

It is conceivable that $\mathbb{R} \cap M$ becomes small in $M[G]$ for trivial reasons, namely that $\mathbb{R} \cap M$ is countable in $M[G]$. We shall however show that cardinalities are absolute between M and $M[G]$. In particular $\mathbb{R} \cap M$ is uncountable in $M[G]$.

Definition 72. *A forcing P* preserves cardinals *if for every generic extension $M[G] \supseteq M$ by P the following holds: every cardinal in M is a cardinal in $M[G]$.*

The following simple arguments will be generalised later.

Lemma 73. *Let $M[G]$ be a generic extension by* Cohen *forcing $P = \operatorname{Fn}(\omega, 2, \aleph_0)$ and let $f: \alpha \to \beta$, $f \in M[G]$. Then there is a function $F: \alpha \to M$, $F \in M$ such that*

$$\forall \xi < \alpha \, (f(\xi) \in F(\xi) \wedge \operatorname{card}^M(F(\xi)) \leqslant \aleph_0).$$

Proof. Take a name $\dot{f} \in M$ such that $f = \dot{f}^G$. Take $p \in G$ such that $p \Vdash \dot{f} : \check{\alpha} \rightarrow \check{\beta}$. In M, define $F : \alpha \rightarrow M$ by

$$F(\xi) = \{\zeta < \beta | \exists q \leqslant p \ q \Vdash \dot{f}(\check{\xi}) = \check{\zeta}\}.$$

Let $\xi < \alpha$.

(1) $f(\xi) \in F(\xi)$.
Proof. Let $\zeta = f(\xi)$. Take $q \in G$ such that $q \Vdash \dot{f}(\check{\xi}) = \check{\zeta}$. We may assume that $q \leqslant p$. Then $\zeta \in F(\xi)$. *qed*(1)

(2) $\text{card}^M(F(\xi)) \leqslant \aleph_0$.
Proof. For $\zeta \in F(\xi)$ choose $q_\zeta \leqslant p$ such that $q_\zeta \Vdash \dot{f}(\check{\xi}) = \check{\zeta}$. If $\zeta, \zeta' \in F(\xi)$ and $q_\zeta = q_{\zeta'}$ then $q_\zeta \Vdash \check{\zeta} = \dot{f}(\check{\xi}) = \check{\zeta}'$. Since $\check{\zeta}$ and $\check{\zeta}'$ are canonical names this implies $\zeta = \zeta'$. Thus the function $\zeta \mapsto q_\zeta$ is an injection of $F(\xi)$ into the countable set P. $\square$

Theorem 74. Cohen *forcing* $P = \text{Fn}(\omega, 2, \aleph_0)$ *preserves cardinals.*

Proof. Let $M[G]$ be a generic extension by Cohen forcing. Assume that $\kappa > \omega$ is not a cardinal in $M[G]$. Take a surjective function $f : \alpha \rightarrow \kappa$, $f \in M[G]$ with $\alpha < \kappa$. By the previous Lemma take a function $F : \alpha \rightarrow M$, $F \in M$ such that

$$\forall \xi < \alpha \ (f(\xi) \in F(\xi) \wedge \text{card}^M(F(\xi)) \leqslant \aleph_0).$$

Then $\kappa \subseteq \bigcup_{\xi < \alpha} F(\xi)$ and

$$\text{card}^M(\kappa) \leqslant \text{card}^M(\bigcup_{\xi < \alpha} F(\xi)) \leqslant \sum_{\xi < \alpha} \text{card}^M(F(\xi)) \leqslant \sum_{\xi < \alpha} \aleph_0 = \text{card}^M(\alpha) \cdot \aleph_0 = \text{card}^M(\alpha) < \kappa.$$

So κ is not a cardinal in M. $\square$

Problem 2. Is $\mathbb{R} \cap M$ *meager* in $M[G]$, i.e., is it a union of countably many nowhere dense sets?

13 Models for ¬CH

We shall obtain ¬CH by adjoining λ Cohen reals to a ground model M where $\lambda \in [\omega_2^M, \text{Ord} \cap M)$. So define λ-fold Cohen forcing $P = (P, \leqslant, 1) \in M$ by $P = \text{Fn}(\lambda \times \omega, 2, \aleph_0)$, $\leqslant = \supseteq$, and $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G$. Like Lemma 69*a*) one can show

(1) $F : \lambda \times \omega \rightarrow 2$.

We extract a sequence $(c_\alpha|\alpha<\lambda)$ of reals $c_\alpha:\omega\rightarrow 2$ from F by:

$$c_\alpha(n)=F(\alpha,n).$$

(2) $\alpha<\beta<\lambda\rightarrow c_\alpha\neq c_\beta$.
Proof. Define

$$D_{\alpha\beta}=\{p\in P|\exists n<\omega\,((\alpha,n)\in\mathrm{dom}(p)\wedge(\beta,n)\in\mathrm{dom}(p)\wedge p(\alpha,n)\neq p(\beta,n))\}\in M.$$

To prove that $D_{\alpha\beta}$ is dense in P consider $q\in P$. Since q is finite take $n<\omega$ such that $(\alpha, n)\notin\mathrm{dom}(q)$ and $(\beta,n)\notin\mathrm{dom}(q)$. Then

$$p=q\cup\{((\alpha,n),0),((\beta,n),1)\}\leqslant q$$

and $p\in D_{\alpha\beta}$. By the M-genericity of G take $p\in G\cap D_{\alpha\beta}$. Take $n<\omega$ such that

$$(\alpha,n)\in\mathrm{dom}(p)\wedge(\beta,n)\in\mathrm{dom}(p)\wedge p(\alpha,n)\neq p(\beta,n).$$

Then $c_\alpha\neq c_\beta$ since

$$c_\alpha(n)=F(\alpha,n)=p(\alpha,n)\neq p(\beta,n)=F(\beta,n)=c_\beta(n).$$

qed(2)

So in $M[G]$ there is an injection $\alpha\mapsto c_\alpha$ of λ into $\mathbb{R}$ and

$$\mathrm{card}^{M[G]}(\mathbb{R})\geqslant\mathrm{card}^{M[G]}(\lambda).$$

If we can show that cardinals are absolute between M and $M[G]$ then this would yield $\neg$ CH in $M[G]$ by

$$\mathrm{card}^{M[G]}(\mathbb{R})\geqslant\mathrm{card}^{M[G]}(\lambda)\geqslant\mathrm{card}^{M[G]}(\omega_2^M)=\mathrm{card}^{M[G]}(\omega_2^{M[G]})=\omega_2^{M[G]}.$$

The proof of the absoluteness of cardinals is modeled after the proof of cardinal preservation for simple COHEN forcing. The countability of simple COHEN forcing is replaced by the following combinatorial property of forcings:

Definition 75. *Let* $Q=(Q,\leqslant,1)$ *be a forcing.* $A\subseteq Q$ *is an* antichain *in* Q *if* $\forall p,q\in A\,(p\neq q\rightarrow p\perp q)$. Q *has the* countable chain condition (ccc) *if every antichain in* Q *is at most countable.*

To prove that $\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$ has the ccc we use the following.

Theorem 76. *Let $(a_i|i<\aleph_1)$ be a sequence of finite sets. Then there are $Z\subseteq\aleph_1$, $\mathrm{card}(Z)=\aleph_1$ and a finite set b such that $(a_i|i<Z)$ is a* Δ-system *with* root b*, i.e.,*

$$\forall i,j\in Z\,(i\neq j\rightarrow a_i\cap a_j=b).$$

Proof. By the regularity of $\aleph_1$ there is $Z_0\subseteq\aleph_1$, $\mathrm{card}(Z_0)=\aleph_1$ and a finite $n<\omega$ such that

$$\forall i\in Z_0\ \mathrm{card}(a_i)=n.$$

Take $m\leqslant n$ maximal such that there is a set b with $\mathrm{card}(b)=m$ and $Z_1\subseteq Z_0$, $\mathrm{card}(Z_1)=\aleph_1$ such that

$$\forall i\in Z_1\ b\subseteq a_i.$$

Such an m exists, since trivially $\forall i\in Z_0\ \emptyset\subseteq a_i$.

(1) For all $u\notin b$ there is $i(u)<\aleph_1$ such that $\forall i\in Z_1\,(i>i(u)\rightarrow u\notin a_i)$, because otherwise $b\cup\{u\}$ would contradict the maximality of m.

Define a strictly increasing sequence $(i_\xi|\xi<\aleph_1)$ of ordinals $i_\xi\in Z_1$ by recursion: let i_ξ be the minimal $i\in Z_1$ such that $\forall\zeta<\xi\,\forall u\in a_\zeta\setminus b\ i>i(u)$.

(2) $Z=\{a_{i_\xi}|\xi<\aleph_1\}$ is a Δ-system with root b.
Proof. Let $\zeta,\xi\in Z$, $\zeta<\xi$. By the choice of Z_1 we have $b\subseteq a_{i_\zeta}$, $b\subseteq a_{i_\xi}$, and so $a_{i_\zeta}\cap a_{i_\xi}\supseteq b$. For the converse consider $u\in a_{i_\zeta}\cap a_{i_\xi}$. Assume for a contradiction that $u\notin b$. By construction $i_\xi>i(u)$. Then (1) implies that $u\notin a_{i_\xi}$. Contradiction, and so $u\in b$. □

Theorem 77. $\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$ *has the ccc.*

Proof. Assume for a contradiction that $\{p_i|i<\aleph_1\}$ is an antichain in $\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$ consisting of pairwise distinct conditions. $(\mathrm{dom}(p_i)|i<\aleph_1)$ is a sequence of finite sets and by the Δ-system theorem one can take a finite set b and $Z\subseteq\aleph_1$, $\mathrm{card}(Z)=\aleph_1$ such that $(\mathrm{dom}(p_i)|i<Z)$ forms a Δ-system with root b. Since there are only finitely many 0-1-valued functions on the finite set b take a function $q{:}\,b\rightarrow 2$ and $Z_1\subseteq Z$, $\mathrm{card}(Z_1)=\aleph_1$ such that

$$\forall i\in Z_1\ p_i\upharpoonright b=q.$$

Take $i,j\in Z_1$, $i\neq j$. Then $\mathrm{dom}(p_i)\cap\mathrm{dom}(p_j)=b$ and $p_i\upharpoonright b=p_j\upharpoonright b=q$. Then p_i and p_j are compatible in $\mathrm{Fn}(\lambda\times\omega,2,\aleph_0)$, contradiction. □

We extend the proof of Lemma 73 from countable forcing to ccc forcing.

Lemma 78. *Let $M[G]$ be a generic extension by some ccc forcing $Q=(Q,\leqslant,1)$ and let f: $\alpha\rightarrow\beta$, $f\in M[G]$. Then there is a function $F:\alpha\rightarrow M$, $F\in M$ such that*

$$\forall\xi<\alpha\,(f(\xi)\in F(\xi)\wedge\operatorname{card}^M(F(\xi))\leqslant\aleph_0).$$

Proof. Take a name $\dot{f}\in M$ such that $f=\dot{f}^G$. Take $p\in G$ such that $p\Vdash\dot{f}:\check{\alpha}\rightarrow\check{\beta}$. In M, define $F:\alpha\rightarrow M$ by

$$F(\xi)=\{\zeta<\beta|\exists q\leqslant p\ q\Vdash\dot{f}(\check{\xi})=\check{\zeta}\}.$$

Let $\xi<\alpha$. As before we see that

(1) $f(\xi)\in F(\xi)$.

(2) $\operatorname{card}^M(F(\xi))\leqslant\aleph_0$.
Proof. For $\zeta\in F(\xi)$ choose $q_\zeta\leqslant p$ such that $q_\zeta\Vdash\dot{f}(\check{\xi})=\check{\zeta}$. Consider $\zeta,\zeta'\in F(\xi)$, $\zeta\neq\zeta'$ and assume for a contradiction that q_ζ and $q_{\zeta'}$ are compatible in Q. Take $q\in Q$, $q\leqslant q_\zeta$, $q_{\zeta'}$. Then $q\Vdash\check{\zeta}=\dot{f}(\check{\xi})=\check{\zeta}'$. Since $\check{\zeta}$ and $\check{\zeta}'$ are canonical names this implies $\zeta=\zeta'$, which is a contradiction. So q_ζ and $q_{\zeta'}$ are incompatible in Q. Thus the function $\zeta\mapsto q_\zeta$ is an injection of $F(\xi)$ into to antichain $\{q_\zeta|\zeta\in F(\xi)\}$. By the ccc, $\{q_\zeta|\zeta\in F(\xi)\}$ is at most countable, and so $F(\xi)$ is at most countable. □

This covering property implies immediately that the ccc forcing Q preserves cardinals. Hence $M[G]\vDash\neg$CH, and we have

Theorem 79. *Assume that* ZFC *is consistent. Then so is* ZFC + ¬CH.

Proof. The construction above showed the consistency of ZFC + ¬CH. □

The violation of CH means that $2^{\aleph_0}>\aleph_1$. We now want to arrange that the value of $2^{\aleph_0}$ is exactly equal to κ.

Lemma 80. *Let $M\subseteq M[G]$ be a generic extension by some partial order $P\in M$. Let $\beta\in\mathrm{Ord}^M$ and $x\in M[G]$, $x\subseteq\beta$. Then there is a name $\dot{x}\in M$, $\dot{x}^G=x$ of the form*

$$\dot{x}=\{(\check{\alpha},q)|\alpha<\beta\wedge q\in A_\alpha\},$$

where every A_α is an antichain in P.

Proof. Take some name $\tilde{x}\in M$, $\tilde{x}^G=x$. Take $p\in G$ such that $p\Vdash\tilde{x}\subseteq\check{\beta}$. Work in M. Consider $\alpha<\beta$ and let $F_\alpha=\{q\in P|q\Vdash\check{\alpha}\in\tilde{x}\}$. The set

$$Z=\{A\subseteq F_\alpha|A\text{ is an antichain in }P\}$$

is partially ordered by $\subseteq$. Let $C \subseteq Z$ be a chain in $(Z, \subseteq)$, i.e.

$$A, A' \in Z \rightarrow A \subseteq A' \vee A' \subseteq A.$$

Then $\bigcup C \subseteq F_\alpha$ is also an antichain in P. Thus $(Z, \subseteq)$ is an inductive partial order. By ZORN's lemma, choose a maximal element $A_\alpha \in Z$. Then define

$$\dot{x} = \{(\check{\alpha}, q) | \alpha < \beta \wedge q \in A_\alpha\} \in M.$$

We show that $\dot{x}^G = x$. Let $\alpha \in \dot{x}^G$. Take $q \in G$ such that $(\check{\alpha}, q) \in \dot{x}$. Then $q \in A_\alpha \subseteq F_\alpha$ and $q \Vdash \check{\alpha} \in \tilde{x}$. Hence $\alpha \in \tilde{x}^G = x$.

Conversely let $\alpha \in x = \tilde{x}^G$. Take $r \in G$ such that $r \Vdash \check{\alpha} \in \tilde{x}$. We may assume that $r \leqslant p$ so that $r \Vdash \tilde{x} \subseteq \check{\beta}$. Then $r \Vdash \check{\alpha} < \check{\beta}$ and so $\alpha < \beta$. Set

$$D = \{s \in P | \exists q \in A_\alpha \, s \leqslant q\} \in M.$$

D is dense in P below r: Let $r' \leqslant r$. Then $r' \Vdash \check{\alpha} \in \tilde{x}$ and $r' \in F_\alpha$. If $r' \in A_\alpha$ then $r' \in D$. Otherwise $r' \notin F_\alpha$ and by the maximality of A_α we have that $A_\alpha \cup \{r'\}$ is not antichain. So take $q \in A_\alpha$ such that q and r' are compatible. Take $s \leqslant q, r'$. Then $s \in D$.

Since $G \ni r$ is M-generic there is $s \in G \cap D$. Take $q \in A_\alpha$ such that $s \leqslant q$. Then $q \in G$ and $(\check{\alpha}, q) \in \dot{x}$. Hence $\alpha = \check{\alpha}^G \in \dot{x}^G$. □

Lemma 81. *Let $M \subseteq M[G]$ be a generic extension by some ccc partial order $P \in M$. Let $\beta \in \mathrm{Ord}^M$, $\beta \geqslant \omega$. Then*

$$\mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\beta)) \leqslant (\mathrm{card}(P)^{\mathrm{card}(\beta)})^M.$$

Proof. In $M[G]$ define a map $F: ({}^{\beta \times \omega}P)^M \rightarrow \mathcal{P}^{M[G]}(\beta)$ by

$$f \mapsto \{(\alpha, f(\alpha, n)) | \alpha < \beta \wedge n < \omega\}^G.$$

By the previous lemma, F is a surjection. Hence

$$\mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\beta)) \leqslant \mathrm{card}^{M[G]}(({}^{\beta \times \omega}P)^M) \leqslant \mathrm{card}^M(({}^{\beta \times \omega}P)^M) \leqslant (\mathrm{card}(P)^{\mathrm{card}(\beta)})^M.$$ □

Let us reconsider the forcing extension of the ground model M by λ COHEN reals. Assume GCH in M and take $\lambda \in \mathrm{Card}^M$ with $M \vDash \mathrm{cof}(\lambda) \geqslant \omega_1$. Let G be M-generic on $P = \mathrm{Fn}(\lambda \times \omega, 2, \aleph_0) \in M$. Cardinals are absolute between M and $M[G]$.

Lemma 82. $M[G] \vDash 2^{\aleph_0} = \lambda$.

Proof.

$$\begin{aligned}\lambda &\leqslant \operatorname{card}^{M[G]}(\mathcal{P}^{M[G]}(\omega))\text{, by our first results about this forcing,}\\ &\leqslant (\operatorname{card}(P)^{\aleph_0})^M\text{, by the previous lemma,}\\ &\leqslant (\lambda^{\aleph_0})^M\text{, by GCH in } M,\\ &= \lambda\text{, by GCH in } M.\end{aligned}$$

□

So $2^{\aleph_0}$ can be any cardinal λ of uncountable cofinality. Note that $\operatorname{cof}(2^{\aleph_0}) > \omega$ by KÖNIG's lemma. For cardinals $\lambda = \aleph_2, \aleph_3, \ldots, \aleph_{\omega+1}, \ldots, \aleph_{\omega_1}, \ldots$ in M we obtain relative consistency results of the following type.

Theorem 83. *Assume that* ZFC + GCH *is consistent. Then the following theories are consistent:*

a) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_2$ *;*

b) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_3$ *;*

c) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_{\omega+1}$ *;*

d) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_{\omega_1}$ *.*

14 Models for CH

We shall obtain CH by adjoining a surjection from $\aleph_1^M$ onto $\mathcal{P}(\omega)^M$.

In M, let $P = \mathrm{Fn}(\aleph_1, \mathcal{P}(\omega), \aleph_1)$, $\leqslant = \supseteq$, and $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G$. Like Lemma 69*a*) one can show

(1) $F: \aleph_1^M \to \mathcal{P}(\omega)^M$.

(2) $F: \aleph_1^M \to \mathcal{P}(\omega)^M$ is surjective.

Proof. Let $a \in \mathcal{P}(\omega)^M = \mathcal{P}(\omega) \cap M$. Define, in M,

$$D_a = \{p \in P \mid a \in \operatorname{rng}(p)\}.$$

$D_a \in M$ is dense in P. By genericity take $p \in D_a \cap G$. Then $a \in \operatorname{rng}(p) \subseteq \operatorname{rng}(F)$. *qed*.

To show CH in $M[G]$ it suffices to show $\aleph_1^M = \aleph_1^{M[G]}$ and $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$.

For this we use another combinatorial property of partial orders.

Definition 84. *Let* $Q=(Q,\leqslant,1)$ *be a forcing. A subset* $C\subseteq Q$ *is a* chain in Q *if* $(C,\leqslant)$ *is a linear order.* Q *is* θ-closed *if every chain in* Q *of cardinality* $<\theta$ *has a lower bound in* Q*, i.e.,*

$$\forall C(C \text{ is a chain in } Q \wedge \operatorname{card}(C) < \theta \to \exists p \in Q \forall q \in C\colon p \leqslant q).$$

Lemma 85. $P=\operatorname{Fn}(\aleph_1,\mathcal{P}(\omega),\aleph_1)$ *is* $\aleph_1$*-closed (in* M*).*

Proof. Let $C\subseteq P$, $\operatorname{card}(C)<\aleph_1$. Then $p=\bigcup C\in P$ is a lower bound for C. □

Lemma 86. *Let* $M[G]$ *be a generic extension by* $Q=(Q,\leqslant,1)\in M$ *where* Q *is* θ*-closed in* M*. Let* $f\colon\alpha\to\beta$*,* $f\in M[G]$*,* $\alpha<\theta$*. Then* $f\in M$*.*

Proof. Take a name $\dot{f}\in M$ and $p\in G$ such that $f=\dot{f}^G$ and $p\Vdash\dot{f}\colon\check{\alpha}\to\check{\beta}$. Define

$$D=\{q\in Q|\exists g\in M\colon q\Vdash\dot{f}=\check{g}\}\in M.$$

(1) D is dense in Q below p.
Proof. Work in M. Let $p'\leqslant p$. Define a descending sequence $(p_i|i<\alpha)$ of conditions $\leqslant p'$ and a function $g\colon\alpha\to\beta$ by recursion. Let $(p_i|i<j)$ and $g\restriction j$ be defined. $\{p_i|i<j\}\cup\{p'\}$ is a chain in Q of cardinality $\leqslant\operatorname{card}(j)\leqslant\operatorname{card}(\alpha)<\theta$. By θ-closure choose a lower bound p_j' of $\{p_i|i<j\}\cup\{p'\}$. Then choose $p_j\leqslant p_j'$ and $g(j)<\beta$ such that $p_j\Vdash\dot{f}(\check{j})=\widecheck{g(j)}$.

$\{p_i|i<\alpha\}$ is a chain in Q of cardinality $\operatorname{card}(\alpha)<\theta$. By θ-closure choose a lower bound q of $\{p_i|i<\alpha\}$. Then

$$q\Vdash\dot{f}\colon\check{\alpha}\to\check{\beta}\wedge\check{g}\colon\check{\alpha}\to\check{\beta}\wedge\forall i<\vec{\alpha}\colon\dot{f}(\check{j})=\widecheck{g(j)}.$$

Hence $q\Vdash\dot{f}=\check{g}$ and $q\in D$. *qed*(1)

By genericity take $q\in G\cap D$. Take $g\in M$ such that $q\Vdash\dot{f}=\check{g}$. Then $f=\dot{f}^G=g\in M$. □

Lemma 87. *Let* $M[G]$ *be a generic extension by* $Q=(Q,\leqslant,1)\in M$ *where* Q *is* θ*-closed in* M*. Then*

a) $\forall\beta<\theta\ \mathcal{P}(\beta)^M=\mathcal{P}(\beta)^{M[G]}$.

b) $\forall\beta\in\operatorname{Card}^M(\beta\leqslant\theta\to\beta\in\operatorname{Card}^{M[G]})$

Proof. *a)* By the previous lemma, $({}^{\beta}2)^M=({}^{\beta}2)^{M[G]}$. This is equivalent to $\mathcal{P}(\beta)^M=\mathcal{P}(\beta)^{M[G]}$.

b) Let $\beta\leqslant\theta$ and $\beta\notin\operatorname{Card}^{M[G]}$. Take some surjective function $f\in M[G]$, $f\colon\alpha\to\beta$, with $\alpha<\beta$. By the previous lemma $f\in M$. Hence $\beta\notin\operatorname{Card}^M$. □

So forcing with $P = \mathrm{Fn}(\aleph_1, \mathcal{P}(\omega), \aleph_1)$ preserves $\aleph_1$ and $\mathcal{P}(\omega)$, i.e., $\aleph_1^M = \aleph_1^{M[G]}$ and $\mathcal{P}(\omega)^M = \mathcal{P}(\omega)^{M[G]}$. Hence $M[G] \vDash \mathrm{CH}$, and we have

Theorem 88. *Assume that* ZFC *is consistent. Then so is* ZFC + CH.

Theorem 89. *Let n be a natural number $\geqslant 1$. Assume that* ZFC *is consistent. Then so is* $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_n$.

Proof. The case $n = 1$ is the previous theorem. So let $n > 1$. We can assume the relative consistency of CH, and force over a ground model of ZFC + CH. By the Hausdorff recursion formula

$$\aleph_n^{\aleph_0} = \aleph_n \cdot \aleph_{n-1}^{\aleph_0}$$

and so, by induction $\aleph_n^{\aleph_0} = \aleph_n$ in M. Adjoin $\aleph_n$ Cohen reals with $\mathrm{Fn}(\aleph_n \times \omega, 2, \aleph_0)$. Using the estimates from Theorem XXX we get

$$\aleph_n^{M[G]} \leqslant (2^{\aleph_0})^{M[G]} = \mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\omega)) \leqslant (\mathrm{card}(P)^{\aleph_0})^M \leqslant (\aleph_n^{\aleph_0})^M = \aleph_n^M = \aleph_n^{M[G]}.$$

□

15 Changing the value of 2^κ

We have seen that the value of $2^{\aleph_0}$ is highly independent of the ZFC-axioms. We now extend our techniques from $\aleph_0$ to an arbitrary regular cardinal κ. So fix a ground model M with $M \vDash$ ZFC and let κ be a regular cardinal in M. Also let λ be a cardinal in M with $\mathrm{cof}(\lambda) > \kappa$. We want to construct a generic extension with $2^\kappa = \lambda$.

In M, define the forcing $P = (P, \leqslant, 1) \in M$ for adding λ Cohen subsets of κ by

$$P = \mathrm{Fn}(\lambda \times \kappa, 2, \kappa) = \{p \in M \mid p\colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \lambda \times \kappa \wedge \mathrm{card}^M(\mathrm{dom}(p)) < \kappa\} \in M,$$

and $\leqslant = \supseteq$, $1 = \emptyset$. Let G be M-generic on P. Let $F = \bigcup G$. Like Lemma 69*a*) one can show

(1) $F\colon \lambda \times \kappa \to 2$.

We extract a sequence $(c_\alpha \mid \alpha < \lambda)$ of functions $c_\alpha\colon \kappa \to 2$ from F by:

$$c_\alpha(n) = F(\alpha, n).$$

The density argument used for the case $\kappa=\aleph_0$ also shows

(2) $\alpha<\beta<\lambda\rightarrow c_\alpha\neq c_\beta$.

So in $M[G]$ there is an injection $\alpha\mapsto c_\alpha$ of λ into ${}^\kappa 2$ and

$$(2^\kappa)^{M[G]}=\mathrm{card}^{M[G]}({}^\kappa 2)\geqslant\mathrm{card}^{M[G]}(\lambda).$$

To study the cardinality situation in $M[G]$ we extend the ccc-techniques developed previously.

Definition 90. *Let $Q=(Q,\leqslant,1)$ be a forcing. Q has the* θ-chain condition (θ-cc) *if every antichain in Q has cardinality $<\theta$.*

Note that the $\aleph_1$-chain condition is the countable chain condition.

Theorem 91. *Assuming the* GCH, *$P=\mathrm{Fn}(\lambda\times\kappa,2,\kappa)$ has the κ^+-cc.*

Proof. Consider an antichain A in P. By simultaneous recursion we define ascending sequences $(A_i|i\leqslant\kappa)$ of subsets of A and $(D_i|i\leqslant\kappa)$ where every A_i and D_i has cardinality $\leqslant\kappa$. Set $A_0=\{p_0\}$ where p_0 is some fixed element of A. If A_i is defined let $D_i=\bigcup\{\mathrm{dom}(p)|p\in A_i\}$. For limit $l\leqslant\kappa$ let $A_l=\bigcup_{i<l}A_i$ and $D_l=\bigcup_{i<l}D_i$. Then $\mathrm{card}(A_l)\leqslant\kappa\cdot\kappa=\kappa$ and $\mathrm{card}(D_l)\leqslant\kappa\cdot\kappa=\kappa$

It remains to define A_{i+1} from A_i and D_i . For h: $\mathrm{dom}(h)\rightarrow 2$ with $\mathrm{dom}(h)\subseteq D_i$ and $\mathrm{card}(\mathrm{dom}(h))<\kappa$ choose $p_{h,i}\in A$ with $p_{h,i}\upharpoonright D_i=h$, if possible; if such a $p_{h,i}$ does not exist set $p_{h,i}=p_0$. Then let

$$A_{i+1}=A_i\cup\{p_{h,i}|h:\mathrm{dom}(h)\rightarrow 2\wedge\mathrm{dom}(h)\subseteq D_i\wedge\mathrm{card}(\mathrm{dom}(h))<\kappa\}.$$

When we view the relevant functions h as defined on a bounded subset of $\kappa\geqslant\mathrm{card}(D_i)$ then their number is $\leqslant\mathrm{card}({}^{<\kappa}\kappa)=\kappa^{<\kappa}=\kappa$; the last equality follows from GCH and the regularity of κ. Hence $\mathrm{card}(A_i)\leqslant\mathrm{card}(A_i)+\kappa=\kappa$.

To show that $\mathrm{card}(A)\leqslant\kappa$ it suffices to see that $A=A_\kappa$. So let $p\in A$. Since $\mathrm{card}(\mathrm{dom}(p))<\kappa$ there is some $i<\kappa$ such that $\mathrm{dom}(p)\cap D_\kappa=\mathrm{dom}(p)\cap D_i$. Set $h=p\upharpoonright D_i$. Then in the recursive construction we chose some $p_{h,i}\in A$ with $p_{h,i}\upharpoonright D_i=h$. $\mathrm{dom}(p_{h,i})\subseteq D_{i+1}$.

$$\begin{aligned}\mathrm{dom}(p)\cap\mathrm{dom}(p_{h,i}) &\subseteq \mathrm{dom}(p)\cap D_{i+1}\text{, since }\mathrm{dom}(p_{h,i})\subseteq D_{i+1}\\ &\subseteq \mathrm{dom}(p)\cap D_i\text{, since }\mathrm{dom}(p)\cap D_\kappa=\mathrm{dom}(p)\cap D_i\\ &\subseteq D_i\end{aligned}$$

But $p\upharpoonright D_i=h=p_{h,i}\upharpoonright D_i$, and so p and $p_{h,i}$ are compatible. Since A is an antichain we have $p=p_{h,i}\in A_\kappa$. □

We generalise a "covering" lemma from the ccc case.

Lemma 92. *Let $M[G]$ be a generic extension by some θ-cc forcing $Q = (Q, \leqslant, 1) \in M$ where θ is a cardinal in M. Let $f: \alpha \to \beta$, $f \in M[G]$. Then there is a function $F: \alpha \to M$, $F \in M$ such that*

$$\forall \xi < \alpha \, (f(\xi) \in F(\xi) \wedge \mathrm{card}^M(F(\xi)) < \theta).$$

Proof. Take a name $\dot{f} \in M$ such that $f = \dot{f}^G$. Take $p \in G$ such that $p \Vdash \dot{f}: \check{\alpha} \to \check{\beta}$. In M, define $F: \alpha \to M$ by

$$F(\xi) = \{\zeta < \beta | \exists q \leqslant p \; q \Vdash \dot{f}(\check{\xi}) = \check{\zeta}\}.$$

Let $\xi < \alpha$. As before we can show that

(1) $f(\xi) \in F(\xi)$.

(2) $\mathrm{card}^M(F(\xi)) < \theta$. ☐

θ-cc forcing preserves cardinals $\geqslant \theta$:

Lemma 93. *Let $M[G]$ be a generic extension by some θ-cc forcing $Q = (Q, \leqslant, 1) \in M$ where θ is a cardinal in M. Then*

$$\forall \beta \in \mathrm{Card}^M (\beta \geqslant \theta \to \beta \in \mathrm{Card}^{M[G]}).$$

Proof. Let $\beta \geqslant \theta$ and $\beta \notin \mathrm{Card}^{M[G]}$. We may assume that β is a limit ordinal. Take some surjective function $f \in M[G]$, $f: \alpha \to \beta$, with $\alpha < \beta$. By the previous lemma take a function $F: \alpha \to M$, $F \in M$ such that

$$\forall \xi < \alpha \, (f(\xi) \in F(\xi) \wedge \mathrm{card}^M(F(\xi)) < \theta).$$

Then $\beta \subseteq \bigcup_{\xi < \alpha} F(\xi)$ and

$$\mathrm{card}^M(\beta) \leqslant \mathrm{card}^M(\bigcup_{\xi < \alpha} F(\xi)) \leqslant \sum_{\xi < \alpha} \mathrm{card}^M(F(\xi)) \leqslant \sum_{\xi < \alpha} \aleph_0 = \mathrm{card}^M(\alpha) \cdot \aleph_0 = \mathrm{card}^M(\alpha) < \kappa.$$

So κ is not a cardinal in M. ☐

The previous lemmas show that forcing by $P = \mathrm{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves cardinals $\geqslant \kappa^+$. To show the preservation of cardinals $\leqslant \kappa$ we use another combinatorial property of the partial order.

Definition 94. *Let* $Q = (Q, \leqslant, 1)$ *be a forcing. A subset* $C \subseteq Q$ *is a* chain in Q *if* $(C, \leqslant)$ *is a linear order.* Q *is* θ-closed *if every chain in* Q *of cardinality* $<\theta$ *has a lower bound in* Q*, i.e.,*

$$\forall C(C \text{ is a chain in } Q \wedge \operatorname{card}(C) < \theta \rightarrow \exists p \in Q \forall q \in C \colon p \leqslant q).$$

Lemma 95. $P = \operatorname{Fn}(\lambda \times \kappa, 2, \kappa)$ *is* κ*-closed.*

Proof. Let $C \subseteq P$, $\operatorname{card}(C) < \kappa$. Then $p = \bigcup C \in P$ is a lower bound for C. □

Lemma 96. *Let* $M[G]$ *be a generic extension by* $Q = (Q, \leqslant, 1) \in M$ *where* Q *is* θ*-closed in* M*. Let* $f \colon \alpha \rightarrow \beta$*,* $f \in M[G]$*,* $\alpha < \theta$*. Then* $f \in M$*.*

Proof. Take a name $\dot{f} \in M$ and $p \in G$ such that $f = \dot{f}^G$ and $p \Vdash \dot{f} \colon \check{\alpha} \rightarrow \check{\beta}$. Define

$$D = \{q \in Q | \exists g \in M \colon q \Vdash \dot{f} = \check{g}\} \in M.$$

(1) D is dense in Q below p.
Proof. Work in M. Let $p' \leqslant p$. Define a descending sequence $(p_i | i < \alpha)$ of conditions $\leqslant p'$ and a function $g \colon \alpha \rightarrow \beta$ by recursion. Let $(p_i | i < j)$ and $g \upharpoonright j$ be defined. $\{p_i | i < j\} \cup \{p'\}$ is a chain in Q of cardinality $\leqslant \operatorname{card}(j) \leqslant \operatorname{card}(\alpha) < \theta$. By θ-closure choose a lower bound p_j' of $\{p_i | i < j\} \cup \{p'\}$. Then choose $p_j \leqslant p_j'$ and $g(j) < \beta$ such that $p_j \Vdash \dot{f}(\check{j}) = \overset{\smile}{g(j)}$.

$\{p_i | i < \alpha\}$ is a chain in Q of cardinality $\operatorname{card}(\alpha) < \theta$. By θ-closure choose a lower bound q of $\{p_i | i < \alpha\}$. Then

$$q \Vdash \dot{f} \colon \check{\alpha} \rightarrow \check{\beta} \wedge \check{g} \colon \check{\alpha} \rightarrow \check{\beta} \wedge \forall i < \vec{\alpha} \colon \dot{f}(\check{j}) = \overset{\smile}{g(j)}.$$

Hence $q \Vdash \dot{f} = \check{g}$ and $q \in D$. *qed*(1)

By genericity take $q \in G \cap D$. Take $g \in M$ such that $q \Vdash \dot{f} = \check{g}$. Then $f = \dot{f}^G = g \in M$. □

Lemma 97. *Let* $M[G]$ *be a generic extension by* $Q = (Q, \leqslant, 1) \in M$ *where* Q *is* θ*-closed in* M*. Then*

$$\forall \beta \in \operatorname{Card}^M (\beta < \theta \rightarrow \beta \in \operatorname{Card}^{M[G]}).$$

Proof. Let $\beta < \theta$ and $\beta \notin \operatorname{Card}^{M[G]}$. Take some surjective function $f \in M[G]$, $f \colon \alpha \rightarrow \beta$, with $\alpha < \beta$. By the previous lemma $f \in M$. Hence $\beta \notin \operatorname{Card}^M$. □

So forcing with $P = \operatorname{Fn}(\lambda \times \kappa, 2, \kappa)$ preserves all cardinals. To see that it makes $2^\kappa = \lambda$ we use again names of a canonical form for subsets of κ.

Lemma 98. *Let $M \subseteq M[G]$ be a generic extension by $P = \mathrm{Fn}(\lambda \times \kappa, 2, \kappa)^M$. Let $\beta \in \mathrm{Ord}^M$, $\beta \geqslant \omega$. Then*

$$\mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\beta)) \leqslant (\lambda^{\mathrm{card}(\beta)})^M.$$

Proof. We first show $\mathrm{card}(P) = \lambda$ in M:

$$\begin{aligned}
\mathrm{card}(\mathrm{Fn}(\lambda \times \kappa, 2, \kappa)) &= \mathrm{card}(\{p | p\colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \lambda \times \kappa \wedge \mathrm{card}(p) < \kappa\}) \\
&\leqslant \mathrm{card}(\{p | \exists \bar{\lambda} < \lambda \, (p\colon \mathrm{dom}(p) \to 2 \wedge \mathrm{dom}(p) \subseteq \bar{\lambda} \times \kappa \}) \\
&\quad\ \text{since } \mathrm{cof}(\lambda) > \kappa, \\
&\leqslant \sum_{\bar{\lambda} < \lambda} 2^{\mathrm{card}(\bar{\lambda} \times \kappa)} \\
&\leqslant \sum_{\bar{\lambda} < \lambda} \lambda \text{, by GCH,} \\
&= \lambda
\end{aligned}$$

In $M[G]$ define a map $F\colon ({}^{\beta \times \kappa}P)^M \to \mathcal{P}^{M[G]}(\beta)$ by

$$f \mapsto \{(\check{\alpha}, f(\alpha, i)) | \alpha < \beta \wedge i < \kappa\}^G.$$

By Lemma 80, F is a surjection. Hence

$$\begin{aligned}
\mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\beta)) &\leqslant \mathrm{card}^{M[G]}(({}^{\beta \times \kappa}P)^M) \\
&\leqslant \mathrm{card}^M(({}^{\beta \times \kappa}P)^M) \\
&= (\lambda^{\kappa \cdot \mathrm{card}(\beta)})^M \\
&= ((\lambda^\kappa)^{\mathrm{card}(\beta)})^M \\
&= (\lambda^{\mathrm{card}(\beta)})^M \text{, by GCH and } (\lambda^\kappa)^M = \lambda.
\end{aligned}$$

□

Theorem 99. *Let M be a ground model satisfying* GCH. *Let κ, λ be cardinals in M where κ is regular in M and $\mathrm{cof}^M(\lambda) > \kappa$. Let $M \subseteq M[G]$ be a generic extension by $P = \mathrm{Fn}(\lambda \times \kappa, 2, \kappa)^M$. Then*

a) $M[G] \vDash \forall \mu < \kappa \, 2^\mu = \mu^+$;

b) $M[G] \vDash 2^\kappa = \lambda$.

Proof. In M, the forcing $P = \mathrm{Fn}(\lambda \times \kappa, 2, \kappa)^M$ is κ-closed and satisfies κ^+-cc.

a) Let $\mu < \kappa$. By Lemma 96, $\mathcal{P}^{M[G]}(\mu) = \mathcal{P}^M(\mu)$. Using GCH in M we get in $M[G]$

$$\mu^+ \leqslant 2^\mu = \mathrm{card}(\mathcal{P}(\mu)) = \mathrm{card}(\mathcal{P}^M(\mu)) \leqslant \mathrm{card}^M(\mathcal{P}^M(\mu)) = (\mu^+)^M = \mu^+.$$

b) Combining a first observation about $M[G]$ with the preservation of cardinals we get

$$(2^{\kappa})^{M[G]} = \mathrm{card}^{M[G]}({}^{\kappa}2) \geqslant \mathrm{card}^{M[G]}(\lambda) = \lambda.$$

For the converse

$$(2^{\kappa})^{M[G]} = \mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\kappa)) \leqslant (\lambda^{\kappa})^{M} = \lambda.$$

□

With more effort, the behaviour of 2^{μ} above κ can also be exactly determined. We obtain relative consistency results of the form

Theorem 100. *Assume that* ZFC + GCH *is consistent. Then the following theories are consistent:*

a) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_3$ *;*

b) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_{\omega+1}$ *;*

c) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} = \aleph_{\omega_2}$ *;*

d) $\mathrm{ZFC} + 2^{\aleph_0} = \aleph_3 + 2^{\aleph_1} = \aleph_4 + 2^{\aleph_2} = \aleph_{\omega+1}$ *.*

Proof. To prove results like (b), first apply our construction to $\kappa_1 = \aleph_2$, $\lambda_1 = \aleph_{\omega+1}$. Then apply the construction to $\kappa_2 = \aleph_1$, $\lambda_2 = \aleph_4$ in the generic extension $M[G]$. Finally apply it in the next generic extension to $\kappa_3 = \aleph_0$, $\lambda_3 = \aleph_3$. □

Note, that it is important to proceed backwards, dealing with the largest cardinal first. This is necessary because the first step preserves GCH below κ_1 which is used in the second step to preserve cardinals. Since we have to proceed backwards, we can change the value of the continuum function only at finitely many places in this way. Can we also do it for infinitely many places? Yes, but then we have to do it all in one step. This is done by *product forcing*.

16 Product forcing

Definition 101. *Let* $P = (P, \leq_P, 1_P)$ *and* $Q = (Q, \leq_Q, 1_Q)$ *be forcings. Then the* product forcing

$$P \times Q = (P \times Q, \leq, 1) = (P, \leq_P, 1_P) \times (Q, \leq_Q, 1_Q)$$

is defined by

$$(p_1, q_1) \leq (p_2, q_2) \text{ iff } p_1 \leq_P p_2 \wedge q_1 \leq_Q q_2\,,$$

and $1 = (1_p, 1_Q)$.

Let M be a ground model. Let G be an M-generic filter on $P \times Q$. Define

$$G_1 = \{p \in P | \exists q \in Q (p, q) \in G\}$$

and

$$G_1 = \{q \in Q | \exists p \in P (p, q) \in G\}.$$

Then it is easily seen that G_1 and G_2 are M-generic on P and Q respectively, and $G = G_1 \times G_2$. The exact relationship is given by the next lemma.

Theorem 102. (The Product Lemma) *$G \subseteq P \times Q$ is an M-generic filter if and only if $G = G_1 \times G_2$ for some M-generic filter G_1 on P and some $M[G_1]$-generic filter G_2 on Q. Moreover, $M[G] = M[G_1][G_2]$.*

Proof. For the direction from left to right, let G be the M-generic filter on $P \times Q$. Define G_1 and G_2 as above. It is easily checked that G_1 and G_2 are filters and that $G \subseteq G_1 \times G_2$. For $G_1 \times G_2 \subseteq G$, let $(p_1, p_2) \in G_1 \times G_2$. Then there are $p_1' \in P$ and $p_2' \in Q$ such that $(p_1', p_2) \in G$ and $(p_1, p_2') \in G$. Since G is a filter, there exists $(p, q) \leq (p_1', p_2), (p_1, p_2')$ with $(p, q) \in G$. But $(p, q) \leq (p_1, p_2)$. So $(p_1, p_2) \in G$ since G is a filter. This proves $G = G_1 \times G_2$.

It is easy to see that G_1 is generic over M: If $D_1 \in M$ is dense in P, then $D_1 \times Q$ is dense in $P \times Q$; and since $(D_1 \times Q) \cap G \neq \emptyset$, we have $D_1 \cap G_1 \neq \emptyset$.

To show that G_2 is generic over $M[G_1]$, let $D_2 \in M[G_1]$ be dense in Q. Let $\dot{D}_2$ be a P-name such that $\dot{D}_2^{G_1} = D_2$ and $p_1 \in G_1$ such that

$$p_1 \Vdash_P (\dot{D}_2 \text{ is dense in } Q).$$

Let $D' = \{(r_1, r_2) | \, r_1 \leq p_1 \text{ and } r_1 \Vdash_P \check{r}_2 \in \dot{D}_2\}$. We first show that D' is dense below $(p_1, 1_q)$. Fix $(q_1, q_2) \leq (p_1, 1_Q)$. We have $q_1 \leq p_1$, so

$$q_1 \Vdash_P \exists x \in \check{Q} (x \in \dot{D}_2 \wedge x \leq q_2).$$

So there is $r_2 \in Q$ and a $r_1 \leq q_1$ such that

$$r_1 \Vdash_P (\check{r}_2 \in \dot{D}_2 \wedge \check{r}_2 \leq \check{q}_2);$$

thus $(r_1, r_2) \leq (q_1, q_2)$ and $(r_1, r_2) \in D'$. Since $(p_1, 1_Q) \in G_1 \times G_2$, there is some $(r_1, r_2) \in (G_1 \times G_2) \cap D'$. Hence $r_1 \Vdash_P \check{r}_2 \in \dot{D}_2$, i.e. $r_2 = \check{r}_2^{G_1} \in \dot{D}_2^{G_1} = D_2$. So $r_2 \in G_2 \cap D_2$.
For the direction from right to left, let $G_1 \subseteq P$ be M-generic and let $G_2 \subseteq Q$ be $M[G_1]$-generic. Set $G = G_1 \times G_2$. Clearly G is a filter on $P \times Q$. To show that G is M-generic, let $D \in M$ be dense in $P \times Q$. Let

$$D_2 = \{p_2 \in Q | \, (p_1, p_2) \in D \text{ for some } p_1 \in G_1\}.$$

The set D_2 is in $M[G_1]$. We will show that D_2 is dense in Q and thus $D \cap (G_1 \times G_2) \neq \emptyset$. Let $q_2 \in Q$ be arbitrary. Since D is dense in $P \times Q$, it follows that the set $D_1 = \{p_1 \in P | \exists p_2 \leq q_2 (p_1, p_2) \in D\}$ is dense in P. Hence there is $p_1 \in G_1 \cap D_1$ and so D_2 is dense in Q.

The fact that $M[G_1 \times G_2] = M[G_1][G_2]$ follows from the fact that generic extensions are minimal ZFC models containing M as a subset and the generic filter as element. Thus $M \subseteq M[G_1][G_2]$ and $G_1 \times G_2 \in M[G_1][G_2]$ implies $M[G_1 \times G_2] \subseteq M[G_1][G_2]$. Conversely, $M \subseteq M[G_1 \times G_2]$ and $G_1 \in M[G_1 \times G_2]$, so $M[G_1] \subseteq M[G_1 \times G_2]$; but also $G_2 \in M[G_1 \times G_2]$, so $M[G_1][G_2] \subseteq M[G_1 \times G_2]$. □

17 Forcing initial segments of GCH

Theorem 103. (Easton) *Let M be a ground model and $\mu \in \mathrm{Ord}^M$. Then there exists a generic extension $M[G]$ of M such that*

$$M[G] \vDash \forall \kappa < \mu \ 2^\kappa = \kappa^+.$$

Recall the recursively defined $\beth$-sequence of cardinals:

$$\beth_0 = \aleph_0, \ \beth_{\alpha+1} = 2^{\beth_\alpha}, \text{ and } \beth_\lambda = \bigcup_{\alpha < \lambda} \beth_\alpha \text{ for limit ordinals } \lambda.$$

The GCH is equivalent to the property

$$\beth_{\alpha+1} = \beth_\alpha^+$$

It is thus natural to require from the forcing that the ground model sequence

$$\aleph_0 < (\beth_0^+)^M < (\beth_1^+)^M < \ldots < \beth_\omega^M < \ldots < (\beth_{\alpha+1}^+)^M < (\beth_{\alpha+2}^+)^M < \ldots < \mu$$

becomes the sequence

$$\aleph_0 < \aleph_1^{M[G]} < \aleph_2^{M[G]} < \ldots < \aleph_\omega^{M[G]} < \ldots < \aleph_\alpha^{M[G]} < \aleph_{\alpha+1}^{M[G]} < \ldots < \mu$$

in the generic extension. This will be achieved by adjoining surjections from $(\beth_\alpha^+)^M$ onto $\beth_{\alpha+1}^M$ for $(\beth_\alpha^+)^M < \mu$.

Work in M. Let $A = \{\alpha | \beth_\alpha^+ < \mu\}$. For each $\alpha \in A$, let $P_\alpha = \mathrm{Fn}(\beth_\alpha^+, \beth_{\alpha+1}, \beth_\alpha^+)$, i.e. the set of all $p: \mathrm{dom}(p) \to \beth_{\alpha+1}$ such that

$$\mathrm{dom}(p) \subseteq \beth_\alpha^+ \quad \text{and} \quad |\mathrm{dom}(p)| < \beth_\alpha^+.$$

Let $(P, \leq, 1)$ be the *Easton product* of P_α, $\alpha \in A$: P consists of all functions $p = (p_\alpha | \alpha \in A) \in \Pi_{a \in A} P_\alpha$ such that

$$|\{\alpha \in A | p_\alpha \neq \emptyset\} \cap \gamma| < \gamma$$

for every strongly inaccessible γ. $s(p) := \{\alpha \in A | p_\alpha \neq \emptyset\}$ is called the *support* of p. Set $p \leq q$ iff

$$p(\alpha) \leq q(\alpha) \quad \text{for all } \alpha \in A.$$

Also let $1 = (\emptyset \mid \alpha \in A)$.

Set

$$P_{<\beta} = \{p \in P | s(p) \subseteq \beta\}$$

and

$$P_{\geqslant \beta} = \{p \in P | s(p) \subseteq \mathrm{Ord} \setminus \beta\}.$$

There is a canonical isomorphism

$$P \cong P_{\geqslant \beta} \times P_{<\beta}.$$

Hence we can for any β view an extension by P as an extension obtained in two steps, first by $P_{\geqslant \beta}$ and then by $P_{<\beta}$. This allows us to do for infinitely many regular cardinals what we could do before only for finitely many.

Let $\beta \in \mathrm{Ord}$. The regularity of $\beth_\beta^+$ implies

Lemma 104. *$P_{\geqslant \beta}$ is $\beth_\beta^+$-closed.*

In particular $P = P_{\geqslant 0}$ is $\beth_0^+ = \aleph_1$-closed, i.e., countably closed.

Lemma 105. $\mathrm{card}(P_{<\beta}) \leqslant \beth_{\beta+1}$

Proof. For $P_\alpha = \mathrm{Fn}(\beth_\alpha^+, \beth_{\alpha+1}, \beth_\alpha^+)$ we have

$$\mathrm{card}(P_\alpha) \leqslant \mathrm{card}({}^{\beth_\alpha}(\beth_\alpha^+ \times \beth_{\alpha+1})) \leqslant \mathrm{card}({}^{\beth_\alpha}\beth_{\alpha+1}) = \beth_{\alpha+1}^{\beth_\alpha} = \beth_{\alpha+1}\,.$$

Then

$$\begin{aligned}\mathrm{card}(P_{<\beta}) &\leqslant \mathrm{card}\left(\prod_{\alpha<\beta} P_\alpha\right)\\ &\leqslant \prod_{\alpha<\beta} \beth_{\alpha+1}\\ &\leqslant \prod_{\alpha<\beta} \beth_\beta\\ &\leqslant \beth_{\beta+1}\end{aligned}$$

□

Lemma 106. *If* $\beth_\beta$ *is regular or if* β *is a successor ordinal then* $\mathrm{card}(P_{<\beta}) \leqslant \beth_\beta$

Proof. *Case 1*. $\beta = 0$ is trivial, since $P_{<0} = \{1\}$.

Case 2. β is a successor ordinal, say $\beta = \gamma + 1$. Then

$$\mathrm{card}(P_{<\beta}) \leqslant \mathrm{card}\left(\prod_{i\leqslant\gamma} P_i\right) \leqslant \mathrm{card}({}^\gamma(P_\gamma)) \leqslant \left(\beth_{\gamma+1}^{\beth_\gamma}\right)^{\mathrm{card}(\gamma)} \leqslant ((2^{\beth_\gamma})^{\beth_\gamma})^{\beth_\gamma} = \beth_{\gamma+1} = \beth_\beta$$

Case 3. β is a limit ordinal. Then $\beth_\beta$ is regular. By regularity, $\beth_\beta = \beta$ is strongly inaccessible. In the Easton product, the supports of conditions in $P_{<\beta}$ are bounded below β . Therefore

$$P_{<\beta} = \bigcup_{\gamma<\beta} P_{<\gamma}\,.$$

Using Case 2,

$$\mathrm{card}(P_{<\beta}) \leqslant \mathrm{card}\left(\bigcup_{\gamma<\beta} P_{<\gamma}\right) \leqslant \sum_{\gamma<\beta} \mathrm{card}(P_{<\gamma}) \leqslant \beth_\beta \cdot \beth_\beta = \beth_\beta$$

□

Now let G be M-generic for P. For each $\alpha \in A$, the set

$$G_\alpha = \{p(\alpha) |\, p \in G\} \in M[G]$$

is an M-generic filter on P_α. As before, $\bigcup G_\alpha$ is a surjection from $(\beth_\alpha^+)^M$ onto $\beth_{\alpha+1}^M$ for $(\beth_\alpha^+)^M < \mu$.

Lemma 107. *If* $(\beth_\alpha^+)^M < \mu$ *then* $(\beth_\alpha^+)^M$ *is a cardinal in* $M[G]$.

Proof. As before $P \cong P_{\geqslant\alpha} \times P_{<\alpha}$. Let $G_{\geqslant\alpha}$ and $G_{<\alpha}$ be the corresponding projections of G. Then

$$M[G] = M[G_{\geqslant\alpha}][G_{<\alpha}].$$

Since $P_{\geqslant\alpha}$ is $\beth_\alpha^+$-closed in M, cardinals $\leqslant(\beth_\alpha^+)^M$ are preserved between M and $M[G_{\geqslant\alpha}]$. We want to show that the cardinal-ness of $(\beth_\alpha^+)^M$ is preserved by the further forcing with $P_{<\alpha}$.

Case 1. $\beth_\alpha$ is regular in M or α is a successor ordinal. By Lemma 106, we have that $\mathrm{card}^{M[G_{\geqslant\alpha}]}(P_{<\alpha}) \leqslant \mathrm{card}^M(P_{<\alpha}) < (\beth_\alpha^+)^M$. So $P_{<\alpha}$ has the $\beth_\alpha^+$-chain condition in $M[G_{\geqslant\alpha}]$, and so $(\beth_\alpha^+)^M = (\beth_\alpha^+)^{M[G_{\geqslant\alpha}]}$ remains a cardinal by the extension from $M[G_{\geqslant\alpha}]$ to $M[G]$.

Case 2. $\beth_\alpha$ is singular in M and α is a limit ordinal. Then $\beth_\alpha^M$ is a limit of cardinals of the form $(\beth_\gamma^+)^M$ where $\beth_\gamma^M$ is regular in M; by Case 1 these $(\beth_\gamma^+)^M$ are preserved as cardinals between M and $M[G]$. Hence $\beth_\alpha^M$ is a (singular) cardinal in $M[G]$.

Assume for a contradiction that $(\beth_\alpha^+)^M$ is *not* a cardinal in $M[G]$. Then

$$\mathrm{cof}^{M[G]}((\beth_\alpha^+)^M) \leqslant \beth_\alpha^M,$$

and since $(\beth_\alpha)^M$ is singular in $M[G]$

$$\mathrm{cof}^{M[G]}((\beth_\alpha^+)^M) < \beth_\alpha^M.$$

Take some $\beta < \alpha$ such that

$$\mathrm{cof}^{M[G]}((\beth_\alpha^+)^M) < \beth_\beta^M$$

and such that $\beth_\beta^M$ is regular in M.

Take $f \in M[G]$ such that $f\colon \kappa \to (\beth_\alpha^+)^M$ cofinally and $\kappa \leqslant \beth_\beta^M$. The extension $M[G_{\geqslant\beta}] \subseteq M[G_{\geqslant\beta}][G_{<\beta}]$ is an extension by the forcing $P_{<\beta}$ which has size $\leqslant(\beth_\beta)^M$ in $M[G_{\geqslant\beta}]$. By the "covering lemma" for forcings with chain conditions there is a function $F\colon \kappa \to \mathcal{P}((\beth_\alpha^+)^M)$, $F \in M[G_{\geqslant\beta}]$ such that for all $i < \kappa$

$$f(i) \in F(i) \text{ and } \mathrm{card}^{M[G_{\geqslant\beta}]}(F(i)) \leqslant (\beth_\beta)^M.$$

In $M[G_{\geqslant\beta}]$ F can be transformed into a cofinal function $f'\colon (\beth_\beta)^M \to (\beth_\alpha^+)^M$, $f' \in M[G_{\geqslant\beta}]$. The extension $M \subseteq M[G_{\geqslant\beta}]$ is by the forcing $P_{\geqslant\beta}$ which is $\beth_\beta^+$-closed in M. By another lemma, M and $M[G_{\geqslant\beta}]$ have the same $\beth_\beta^M$-sequences of ordinal. Hence $f' \in M$ which means that $(\beth_\alpha^+)^M$ is singular in M, contradicting the regularity of successor cardinals. □

This lemma together with the surjections $\bigcup G_\alpha$ from $(\beth_\alpha^+)^M$ onto $\beth_{\alpha+1}^M$ for $(\beth_\alpha^+)^M < \mu$ shows that below μ the sequence of infinite cardinals in $M[G]$ consists of cardinals $(\beth_\alpha^+)^M$ and their limits:

$$\begin{aligned}\aleph_0^{M[G]} &= \aleph_0\\ \aleph_{\alpha+1}^{M[G]} &= (\beth_\alpha^+)^M \text{ , for } (\beth_\alpha^+)^M < \mu\\ \aleph_\alpha^{M[G]} &= \beth_\alpha^M \text{ for limit ordinals } \alpha \text{ such that } \beth_\alpha^M < \mu\end{aligned}$$

We now prove the GCH for cardinals $<\mu$ in $M[G]$. Consider some $\kappa < \mu$ which is a cardinal in $M[G]$.

Case 1. $\kappa = \aleph_0$. Note that P is a countably closed forcing which does not adjoin subsets of $\aleph_0$. It also adjoins a surjection from $\aleph_1^M$ onto $\beth_1^M$. Hence $(2^{\aleph_0})^{M[G]} = \aleph_1^M = \aleph_1^{M[G]}$.

Case 2. $\kappa = \aleph_{\alpha+1}^{M[G]} = (\beth_\alpha^+)^M$. View $M[G]$ as the two-step extension $M[G_{\geqslant\alpha+1}][G_{<\alpha+1}]$. For any subset $x \subseteq \kappa$, $x \in M[G]$ there is a "canonical" name $\dot{x} \in M[G_{\geqslant\alpha+1}]$ which is basically of the form

$$\dot{x} \subseteq \kappa \times P_{<\alpha+1}$$

By Lemma 106 $\mathrm{card}^M(P_{<\alpha+1}) \leqslant \beth_{\alpha+1}^M$. So $\dot{x}$ is basically a subset of $\beth_{\alpha+1}^M$. The forcing $P_{\geqslant\alpha+1}$ is $\beth_{\alpha+1}^+$-closed in M and does not adjoin new subsets of $\beth_{\alpha+1}^M$. Hence $\dot{x} \in M$. Since $M[G]$ contains a surjection from $(\beth_{\alpha+1}^+)^M$ onto $\beth_{\alpha+2}^M$:

$$(2^{\aleph_{\alpha+1}})^{M[G]} = \mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\kappa)) \leqslant \mathrm{card}^{M[G]}(\mathcal{P}^M(\beth_{\alpha+1}^M)) \leqslant \mathrm{card}^{M[G]}(\beth_{\alpha+2}^M) \leqslant (\beth_{\alpha+1}^+)^M = \aleph_{\alpha+2}^{M[G]}$$

Case 3. $\kappa = \aleph_\alpha^{M[G]} = \beth_\alpha^M$ where α is a *regular* limit ordinal. View $M[G]$ as the two-step extension $M[G_{\geqslant\alpha}][G_{<\alpha}]$. For any subset $x \subseteq \kappa$, $x \in M[G]$ there is a "canonical" name $\dot{x} \in M[G_{\geqslant\alpha}]$ which is basically of the form

$$\dot{x} \subseteq \kappa \times P_{<\alpha}$$

By Lemma 106 $\mathrm{card}^M(P_{<\alpha}) \leqslant \beth_\alpha^M$. So $\dot{x}$ is basically a subset of $\beth_\alpha^M$. The forcing $P_{\geqslant\alpha}$ is $\beth_\alpha^+$-closed in M and does not adjoin new subsets of $\beth_\alpha^M$. Hence $\dot{x} \in M$. Since $M[G]$ contains a surjection from $(\beth_\alpha^+)^M$ onto $\beth_{\alpha+1}^M$:

$$(2^{\aleph_\alpha})^{M[G]} = \mathrm{card}^{M[G]}(\mathcal{P}^{M[G]}(\kappa)) \leqslant \mathrm{card}^{M[G]}(\mathcal{P}^M(\beth_\alpha^M)) \leqslant \mathrm{card}^{M[G]}(\beth_{\alpha+1}^M) \leqslant (\beth_\alpha^+)^M = \aleph_{\alpha+1}^{M[G]}$$

Case 4. $\kappa = \aleph_\alpha^{M[G]} = \beth_\alpha^M$ where α is a *singular* limit ordinal. We first show

(1) For $\rho < \kappa$, $M[G] \vDash \kappa^\rho \leqslant \kappa^+$.

Proof. Take some successor ordinal $\beta < \alpha$ such that $\rho < \beth_\beta^M$. View $M[G]$ as the two-step extension $M[G_{\geqslant\beta}][G_{<\beta}]$. Any map $f: \rho \to \kappa$, $f \in M[G]$ has a "canonical" name $\dot{f} \in M[G_{\geqslant\beta}]$ which is a function defined on ρ such that

$$\dot{f}(i) = \left\{ (j, p) \in \kappa \times P_{<\beta} \,\middle|\, p \Vdash \dot{f}(\check{i}) = \check{j} \right\}$$

Since $\text{card}^M(P_{<\beta}) \leqslant \beth_\beta^M$, $\dot{f}$ is basically a function from $\beth_\beta^M$ into κ. The forcing $P_{\geqslant\beta}$ is $\beth_\beta^+$-closed in M and does not adjoin new function from $\beth_\beta^M$ into κ. Hence $\dot{f} \in M$. Since $M[G]$ contains a surjection from $(\beth_\alpha^+)^M$ onto $\beth_{\alpha+1}^M$:

$$(\kappa^\rho)^{M[G]} = \text{card}^{M[G]}({}^\rho\kappa) \leqslant \text{card}^{M[G]}\big((\beth_\alpha^{\beth_\beta})^M\big) \leqslant \text{card}^{M[G]}(\beth_{\alpha+1}^M) \leqslant (\beth_\alpha^+)^M = \aleph_{\alpha+1}^{M[G]}$$

qed(1)

Now GCH at κ in $M[G]$ follows from cardinal arithmetic. Work in $M[G]$. Let $c\colon \text{cof}(\alpha) \to \alpha$ be cofinal. The map

$$x \mapsto (x \cap c(i) \mid i < \text{cof}(\alpha))$$

injects $\mathcal{P}(\kappa)$ into $\prod_{i<\text{cof}(\alpha)} \mathcal{P}(c(i))$. Hence

$$2^\kappa \leqslant \prod_{i<\text{cof}(\alpha)} 2^{\aleph_{c(i)}} \leqslant \prod_{i<\text{cof}(\alpha)} \kappa \leqslant \kappa^{\text{cof}(\alpha)} \leqslant \kappa^+.$$

This concludes the proof of Theorem 103.

Forcing with *sets* $(P, \leqslant, 1) \in M$ can be generalized to *classes* P, $\leqslant$ which are definable over the ground model M. If $\leqslant$ satisfies some properties, the resulting generic extension $M[G]$ is a model of ZFC. The above Theorem can then be extended to:

Theorem 108. (Easton) *Let M be a ground model. Then there exists a generic extension $M[G]$ of M by class forcing such that $M[G] \vDash \text{GCH}$.*

Hence

Theorem 109. (Gödel) *If* ZFC *is consistent then* ZFC + GCH *is consistent.*

18 Forcing arbitrary values of 2^κ at regular cardinals

Theorem 110. (Easton) *Let $M \vDash \text{GCH}$ be a ground model. In M, let $F \in M$ be a function whose arguments are regular cardinals and whose values are cardinals, such that for all κ, $\lambda \in \text{dom}(F)$*

a) $F(\kappa) > \kappa$

b) $F(\kappa) \leq F(\lambda)$ *if* $\kappa \leq \lambda$

c) $\text{cof}(F(\kappa)) > \kappa$.

Then there exists a generic extension $M[G]$ of M such that M and $M[G]$ have the same cardinals and cofinalities, and for every $\kappa \in \mathrm{dom}(F)$

$$M[G] \vDash 2^{\kappa} = F(\kappa).$$

For the proof, work in M. For each $\kappa \in \mathrm{dom}(F)$, let $P_{\kappa} = \mathrm{Fn}(F(\kappa) \times \kappa, 2, \kappa)$, i.e. the set of all $p: \mathrm{dom}(p) \to 2$ such that

$$\mathrm{dom}(p) \subseteq F(\kappa) \times \kappa \quad \text{and} \quad |\mathrm{dom}(p)| < \kappa.$$

Let $(P, \leq)$ be the Easton product of P_{κ}, $\kappa \in \mathrm{dom}(F)$: A condition $p \in P$ is a function $p = (p_{\kappa} | \, \kappa \in \mathrm{dom}(F)) \in \Pi_{\kappa \in \mathrm{dom}(F)} P_{\kappa}$ such that

$$|\{\kappa \in \mathrm{dom}(F) | \, p_{\kappa} \neq \emptyset\} \cap \gamma| < \gamma$$

for every regular γ (not necessarily in $\mathrm{dom}(F)$). $s(p) := \{\kappa \in \mathrm{dom}(F) | \, p_{\kappa} \neq \emptyset\}$ is the *support* of p. Set $p \leq q$ iff

$$p_{\kappa} \leq q_{\kappa} \quad \text{for all } \kappa \in \mathrm{dom}(F).$$

Set

$$P_{\leq \lambda} = \{p \in P | \, s(p) \subseteq \lambda^{+}\}$$

and

$$P_{> \lambda} = \{p \in P | \, s(p) \subseteq \mathrm{Ord} \setminus \lambda^{+}\}.$$

Then

$$P \cong P_{> \lambda} \times P_{\leq \lambda}.$$

Hence we can for any λ view the extension by P as an extension obtained in two steps, first by $P_{> \lambda}$ and then by $P_{\leq \lambda}$.

It is easy to see that

Lemma 111. *$P_{> \lambda}$ is λ^{+}-closed.*

To prove the λ^{+}-cc for $P_{\leq \lambda}$ we use a Δ-system lemma.

Lemma 112. *Let $\kappa \geq \omega$ be a cardinal. Let $\theta \geq \kappa$ be regular such that $\forall \alpha < \theta |\alpha^{< \kappa}| < \theta$. Assume $|\mathcal{A}| \geq \theta$ and $\forall x \in \mathcal{A} \, |x| < \kappa$. Then there exists a $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathrm{card}(\mathcal{B}) = \theta$ and $\mathcal{B}$ forms a Δ-system, i.e. $\exists r \forall a \neq b \in \mathcal{B} \, a \cap b = r$.*

Proof. Exercise. □

Lemma 113. *If λ is regular then $P_{\leq\lambda}$ satisfies the λ^+-cc.*

Proof. For $p \in P_{\leq\lambda}$, let

$$d(p) = \bigcup \{\{\kappa\} \times \mathrm{dom}(p_\kappa) | \, \kappa \in \mathrm{dom}(F) \cap \lambda^+\}.$$

Then by the assumption on $s(p)$, $|d(p)| < \lambda$. Consider $A \subseteq P_{\leq\lambda}$, $|A| = \lambda^+$. Since we assume GCH, we can apply the Δ-system lemma. Hence there is $B \subseteq A$ of size λ^+ such that

$$\{d(p) | \, p \in B\}$$

forms a Δ-system with root r, $|r| < \lambda$. By GCH, $2^{|r|} \leq \lambda$. So there is $C \subseteq B$ of size λ^+ such that for all $p, q \in C$

$$p_\kappa(x) = q_\kappa(x) \quad \text{for all } \ (\kappa, x) \in r\,.$$

But then all $p \in C$ are compatible in $P_{\leqslant\lambda}$. Hence A is not an antichain. □

Lemma 114. *Let $G \times H$ be an M-generic filter on $P \times Q$ where P is λ^+-closed and Q satisfies the λ^+-cc. Then every function $f: \lambda \to M$ in $M[G \times H]$ is in $M[H]$. In particular,*

$$\mathcal{P}^{M[G\times H]}(\lambda) = \mathcal{P}^{M[H]}(\lambda)$$

Proof. Let $\dot{f} \in M$ be a $P \times Q$-name such that $\dot{f}^{G\times H} = f$. Assume w.l.o.g. that for some $A \in M$

$$1 \Vdash \dot{f} : \check{\lambda} \to \check{A}.$$

For $\alpha < \lambda$, let

$$D_\alpha = \{p \in P | \, \exists W \text{ max. antichain in } Q \wedge \exists (a^\alpha_{(p,q)} | \, q \in W) \forall q \in W (p, q) \Vdash \dot{f}(\check{\alpha}) = \check{a}^\alpha_{(p,q)}\}.$$

Then $p \leq q \in D_\alpha \to p \in D_\alpha$. Every D_α is dense in P: Let $p_0' \in P$. Since $1 \Vdash \dot{f} : \check{\lambda} \to \check{A}$, there exists $(p_0, q_0) \leq (p_0', 1)$ and $a_0 \in A$ such that $(p_0, q_0) \Vdash \dot{f}(\check{\alpha}) = \check{a}_0$. We construct by induction sequences $\langle p_i | \, i < \delta \rangle$ and $\langle q_i | \, i < \delta \rangle$ such that

$$p_i \leq p_j \quad \text{for all } i \leq j < \delta,$$

$$(p_i, q_i) \Vdash \dot{f}(\check{\alpha}) = \check{a}_i \quad \text{for some } a_i \in A,$$

and

$$\{q_i|\ i<\delta\} \text{ is a maximal antichain in } Q.$$

Hence $\delta<\lambda^+$ by the λ^+-cc. Assume that $\langle p_i|\ i<\eta\rangle$ and $\langle q_i|\ i<\eta\rangle$ have already been constructed. Then let $p'_\eta\in P$ be by λ^+-closedness such that $p'_\eta\leq p_i$ for all $i<\eta$. If $\{q_i|\ i<\eta\}$ is a maximal antichain, $p'_\eta\in D_\alpha$ and $p'_\eta\leq p'_0$. So we are done. If $\{q_i|\ i<\eta\}$ is not maximal, we pick some q'_η such that q'_η is incompatible with all $i<\eta$. Consider (p'_η, q'_η). Since $1\Vdash\dot{f}\colon\check{\lambda}\to\check{A}$, there exists $(p_\eta,q_\eta)\leq(p'_\eta,q'_\eta)$ and $a_\eta\in A$ such that

$$(p_\eta,q_\eta)\Vdash\dot{f}(\check{\alpha})=\check{a}_\eta.$$

This proves the density of D_α.
Since P is λ^+-closed, it follows that $\bigcap\ \{D_\alpha|\ \alpha<\lambda\}$ is dense. So there exists some $p\in G$ such that $p\in D_\alpha$ for all $\alpha<\lambda$. We pick (in M) for each $\alpha<\lambda$ a maximal antichain $W_\alpha\subseteq Q$ and a family $\{a^\alpha_{(p,q)}|\ q\in W_\alpha\}$ such that

$$(p,q)\Vdash\dot{f}(\check{\alpha})=\check{a}^\alpha_{(p,q)}$$

for all $q\in W_\alpha$. By the genericity of H, for every α there is a unique $q\in W_\alpha$ such that $q\in H$, and we have for every $\alpha<\lambda$

$$f(\alpha)=a^\alpha_{(p,q)}\text{where } q \text{ is the unique } q\in W_\alpha\cap H.$$

However, this defines f in $M[H]$. □

Now, we can finish the proof of Easton's theorem.

Let μ be a regular cardinal in M. We show that μ is regular in $M[G]$. If not take a function f that maps some $\lambda<\mu$, regular in M, cofinally into μ. We consider P as the product $P=P_{>\lambda}\times P_{\leq\lambda}$. Then $G=G_{>\lambda}\times G_{\leq\lambda}$ and $M[G]=M[G_{>\lambda}][G_{\leq\lambda}]$. By the previous lemma, f is in $M[G_{\leq\lambda}]$ and so μ is not regular in $M[G_{\leq\lambda}]$. However, this is a contradiction since $P_{\leq\lambda}$ satisfies the λ^+-cc.

It remains to prove that $(2^\kappa)^{M[G]}=F(\kappa)$, for each $\kappa\in\mathrm{dom}(F)$. The projection

$$G_\kappa=\{p_\kappa\mid p\in G\}$$

is M-generic for $P_\kappa=\mathrm{Fn}(F(\kappa)\times\kappa,2,\kappa)$. As in previous arguments this induces an injection of $F(\kappa)$ into $\mathcal{P}(\kappa)$. Hence $(2^\kappa)^{M[G]}\geqslant F(\kappa)$.

By Lemma 114, $(2^\kappa)^{M[G]}=(2^\kappa)^{M[G_{\leq\lambda}]}$. And

$$(2^\kappa)^{M[G_{\leq\lambda}]}$$

However, like before $(2^\lambda)^{M[G_{\leq\lambda}]} = F(\lambda)$.

19 Models of ¬AC

So far we have only considered constructions of models of AC. To obtain models of ¬AC we use a relativised version of the HOD construction, that we already introduced in an exercise.

Definition 115. *Define*

$$\mathrm{OD}(s) = \{y | \exists\alpha \in \mathrm{Ord}\, \exists\varphi \in \mathrm{Fml}\, \exists a \in \mathrm{Asn}((\alpha \cup s) \cap V_\alpha)\ \ y = \{z \in V_\alpha | (V_\alpha, \in) \vDash \varphi[a\frac{z}{0}]\}\},$$

and

$$\mathrm{HOD}(s) = \{x | \mathrm{TC}(\{x\}) \subseteq \mathrm{OD}(s)\}.$$

So $\mathrm{HOD}(s)$ *is built from parameters in* $\mathrm{Ord} \cup s$ *much like* HOD *was built from parameters in* Ord.

We shall examine which axioms of set theory hold in $\mathrm{HOD}(s)$. Just as before we get that $\mathrm{Ord} \subseteq \mathrm{HOD}$ and HOD is transitive.

Lemma 116. *Let z be definable from $x_1, \ldots, x_{n-1}$ by the $\in$-formula $\varphi(v_1, \ldots, v_n)$:*

$$\forall v_n (v_n = z \leftrightarrow \varphi(x_1, \ldots, x_{n-1}, v_n)). \tag{6}$$

Let $x_1, \ldots, x_n \in \mathrm{OD}(s)$ and $z \subseteq \mathrm{HOD}(s)$. Then $z \in \mathrm{HOD}(s)$.

Proof. $\mathrm{TC}(\{z\}) = \{z\} \cup \mathrm{TC}(z) \subseteq \{z\} \cup \mathrm{HOD}(s)$. So it suffices to prove that $z \in \mathrm{OD}(s)$. By the definition of $\mathrm{OD}(s)$ choose

$$\alpha_1, \ldots, \alpha_{n-1} \in \mathrm{Ord},\ \varphi_1, \ldots, \varphi_{n-1} \in \mathrm{Fml},\ \text{and}\ a_1, \ldots, a_{n-1},$$

such that for $i = 1, \ldots, n-1$

$$a_i \in \mathrm{Asn}((\alpha_i \cup s) \cap V_{\alpha_i})\ \text{and}\ x_i = \{w \in V_{\alpha_i} | (V_{\alpha_i}, \in) \vDash \varphi_i[a_i\frac{w}{0}]\}.$$

Choose sufficiently high, pairwise distinct $j_1, \ldots, j_{n-1} < \omega$ which are intended to be indices for "new" variables $v_{j_1}, \ldots, v_{j_{n-1}}$ in the GÖDELised language. Let $\varphi_i^{v_{j_i}} \in \mathrm{Fml}$ be obtained from $\varphi_i \in \mathrm{Fml}$ by relativising all quantifiers to the term for the VON NEUMANN level $V_{v_{j_i}}$. Let α be a limit ordinal $> \alpha_1, \ldots, \alpha_{n-1}$ such that $z \in V_\alpha$ and such that the formula φ is V_α-absolute. Let a_i^* be the assignment obtained by adding the assigment $j_i \mapsto \alpha_i$. Then a_i^* is an assignment in $\alpha \cup s$ and

$$x_i = \{w \in V_{\alpha_i} | (V_{\alpha_i}, \in) \vDash \varphi_i[a_i \frac{w}{0}]\} = \{w \in V_\alpha | (V_\alpha, \in) \vDash \varphi_i^{v_{j_i}}[a_i^* \frac{w}{0}]\}.$$

By renaming variables we may assume that the formulas $\varphi_i^{v_{j_i}}$ do not share variables except the variable v_0 and that they do not contain the variables $v_1, \ldots, v_n$. One may also assume that the assignments a_i^* are all merged into a single assignment a. Then

$$\begin{aligned} z &= \{u \mid V_\alpha \vDash \exists v_n \exists v_1 \ldots \exists v_{n-1}(u \in v_n \wedge \varphi(v_1, \ldots, v_{n-1}, v_n) \wedge \\ &\qquad \forall w(w \in v_1 \leftrightarrow \varphi_1^{v_{j_1}}[a^* \frac{w}{0}]) \wedge \ldots \wedge \forall w(w \in v_{n-1} \leftrightarrow \varphi_{n-1}^{v_{j_{n-1}}}[a^* \frac{w}{0}]))\} \\ &= \{u \mid (V_\alpha, \in) \vDash \exists v_n \exists v_1 \ldots \exists v_{n-1}(u \in v_n \wedge \varphi(v_1, \ldots, v_{n-1}, v_n) \wedge \\ &\qquad \forall w(w \in v_1 \leftrightarrow \varphi_1^{v_{j_1}} \frac{w}{v_0}) \wedge \ldots \wedge \forall w(w \in v_{n-1} \leftrightarrow \varphi_{n-1}^{v_{j_{n-1}}} \frac{w}{v_0}))\, [a^*]\} \end{aligned}$$

Here we assume that the formula behind $(V_\alpha, \in) \vDash$ is GÖDELised and an element of FML. Hence $z \in \mathrm{OD}(s)$. □

Theorem 117. *Let $s = \mathrm{TC}(\{r\})$ for some set $r \in V$. Then* $\mathrm{ZF}^{\mathrm{HOD}(s)}$.

Proof. The transitivity of s implies

(1) $s \subseteq \mathrm{HOD}(s)$.

Also

(2) The class $\mathrm{HOD}(s) = \mathrm{HOD}(\mathrm{TC}(\{r\}))$ is definable from the parameter $r \in s$.

Using the criteria of Theorem 3 we check certain closure properties of $\mathrm{HOD}(s)$.

a) Extensionality holds in $\mathrm{HOD}(s)$, since $\mathrm{HOD}(s)$ is transitive.

b) Let $x, y \in \mathrm{HOD}(s)$. Then $\{x, y\}$ is definable from x, y, and $\{x, y\} \subseteq \mathrm{HOD}(s)$. By Lemma 116, $\{x, y\} \in \mathrm{HOD}(s)$, i.e. $\mathrm{HOD}(s)$ is closed under unordered pairs. This implies Pairing in $\mathrm{HOD}(s)$.

c) Let $x \in \mathrm{HOD}(s)$. Then $\bigcup x$ is definable from x, and $\bigcup x \subseteq \mathrm{TC}(\{x\}) \subseteq \mathrm{HOD}(s)$. So $\bigcup x \in \mathrm{HOD}(s)$, and so Union holds in $\mathrm{HOD}(s)$.

d) Let $x \in \mathrm{HOD}(s)$. Then $\mathcal{P}(x) \cap \mathrm{HOD}(s)$ is definable from x and $r \in \mathrm{HOD}(s)$, and $\mathcal{P}(x) \cap \mathrm{HOD}(s) \subseteq \mathrm{HOD}(s)$. So $\mathcal{P}(x) \cap \mathrm{HOD}(s) \in \mathrm{HOD}(s)$ and Powerset holds in $\mathrm{HOD}(s)$.

e) $\omega \in \mathrm{HOD}(s)$ implies that Infinity holds in $\mathrm{HOD}(s)$.

f) Let $\varphi(x, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}(s)$. Then $\{x \in a | \varphi^{\mathrm{HOD}(s)}(x, \vec{w})\}$ is a set by Separation in V, and it is definable from $\vec{w}$, a and $r \in \mathrm{HOD}(s)$. Moreover $\{x \in a | \varphi^{\mathrm{HOD}(s)}(x, \vec{w})\} \subseteq \mathrm{HOD}(s)$. So $\{x \in a | \varphi^{\mathrm{HOD}(s)}(x, \vec{w})\} \in \mathrm{HOD}(s)$, and Separation for the formula φ holds in $\mathrm{HOD}(s)$.

g) Let $\varphi(x, y, \vec{w})$ be an $\in$-formula and $\vec{w}, a \in \mathrm{HOD}(s)$. Assume that

$$\forall x, y, y' \in \mathrm{HOD}(s)(\varphi^{\mathrm{HOD}(s)}(x, y, \vec{w}) \wedge \varphi^{\mathrm{HOD}(s)}(x, y', \vec{w}) \rightarrow y = y').$$

Then $\{y | \exists x \in a \varphi^{\mathrm{HOD}(s)}(x, y, \vec{w})\} \cap \mathrm{HOD}(s)$ is a set by Replacement and Separation in V. It is definable from $\vec{w}$, a and $r \in \mathrm{HOD}(s)$. Moreover $\{y | \exists x \in a \varphi^{\mathrm{HOD}(s)}(x, y, \vec{w})\} \cap \mathrm{HOD}(s) \subseteq \mathrm{HOD}(s)$. So $\{y | \exists x \in a \varphi^{\mathrm{HOD}(s)}(x, y, \vec{w})\} \cap \mathrm{HOD}(s) \in \mathrm{HOD}(s)$, and Replacement for φ holds in $\mathrm{HOD}(s)$.

h) Foundation holds in $\mathrm{HOD}(s)$ since $\mathrm{HOD}(s)$ is an $\in$-model. □

Hence $\mathrm{HOD}(s)$ is an inner model of ZF set theory. We shall see that in general $\mathrm{HOD}(s)$ is *not* a model of AC.

Fix a ground model M and the forcing

$$P = (P, <, 1) = \mathrm{Fn}(\omega \times \omega, 2, \omega)^M = \mathrm{Fn}(\omega \times \omega, 2, \omega),$$

partially ordered by reverse inclusion. P is the partial order for adjoining ω many Cohen reals $a_i \subseteq \omega$. Let G be M-generic on P, $f = \bigcup G: \omega \times \omega \rightarrow 2$, and for $i < \omega$ define

$$a_i = \{n \in \omega | f(i, n) = 1\}.$$

Let $A = \{a_i | i < \omega\}$, $s = \mathrm{TC}(\{A\}) = \{A\} \cup A \cup \omega$. Set $N = (\mathrm{HOD}(s))^{M[G]}$. By the previous lemma:

Lemma 118. *N is transitive, $A \in N$ and ZF^N.*

We shall see that A does not have a wellorder in N. Note that A is the "unordered" *set* $\{a_i | i < \omega\}$ but not the *sequence* $(a_i | i < \omega)$. The basic idea is that the Cohen reals a_i behave in very similar ways so that one may *permute* them without changing the overall properties of the model N. This permutability of the a_i is reflected in a symmetry property of the forcing P.

Lemma 119. *Let $\pi: \omega \leftrightarrow \omega$ be a permutation. Let $\pi': P \leftrightarrow P$ be the induced map*

$$\pi'(p) = \{((\pi(i), n), p(i, n)) | (i, n) \in \mathrm{dom}(p)\}.$$

Then

a) $\pi' \colon (P, <, 1) \leftrightarrow (P, <, 1)$ *is an order isomorphism and* $(\pi^{-1})' \circ \pi' = \mathrm{id} \upharpoonright P$.

b) *If* D *is dense in* P *then* $\pi'[D]$ *is dense in* P.

c) *If* G *is* M*-generic on* P *then* $\pi'[G]$ *is* M*-generic on* P*; moreover* $M[G] = M[\pi'[G]]$.

Proof. Easy. □

Let uns now introduce some canonical names for the generic objects so far.

Definition 120. *For* $i < \omega$ *let*

$$\dot{a}_i = \{(\check{n}, p) | n < \omega, p \in P, p(i, n) = 1\}.$$

Let $\dot{A} = \{(\dot{a}_i, 1) | i < \omega\}$*. Obviously* $\dot{a}_i^G = a_i$ *and* $\dot{A}^G = A$ *.*

Permutions $\pi \colon \omega \leftrightarrow \omega$ act on the forcing construction as follows.

Lemma 121. $(\dot{a}_{\pi(i)})^{\pi'[G]} = \dot{a}_i^G$ *and* $\dot{A}^{\pi'[G]} = \dot{A}^G$.

Proof.

$$\begin{aligned}(\dot{a}_{\pi(i)})^{\pi'[G]} &= \{n < \omega | \exists p \in \pi'[G]\ p(\pi(i), n) = 1\} \\ &= \{n < \omega | \exists p \in G\, p(i, n) = 1\} \\ &= \dot{a}_i^G.\end{aligned}$$

$$\dot{A}^{\pi'[G]} = \{(\dot{a}_i)^{\pi'[G]} | i < \omega\} = \{\dot{a}_{\pi^{-1}(i)}^G | i < \omega\} = \{\dot{a}_i^G | i < \omega\} = \dot{A}^G.$$

□

Lemma 122. *Let* $\varphi(u_0, \ldots, u_{k-1}, v_0, \ldots, v_{l-1}, w)$ *be an* $\in$*-formula,* $\alpha_0, \ldots, \alpha_{k-1} \in \mathrm{Ord}$*, and* $i_0, \ldots, i_{l-1} < \omega$*. For* $p \in P$ *holds*

$$p \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{i_0}, \ldots, \dot{a}_{i_{l-1}}, \dot{A}) \text{ iff } \pi'(p) \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{\pi(i_0)}, \ldots, \dot{a}_{\pi(i_{l-1})}, \dot{A}).$$

Proof. Assume $p \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{i_0}, \ldots, \dot{a}_{i_{l-1}}, \dot{A})$. Let H be M-generic on P with $\pi'(p) \in H$. Then $(\pi')^{-1}[H]$ is M-generic on P with $p \in (\pi')^{-1}[H]$, and $M[H] = M[(\pi')^{-1}[H]]$. By assumption

$$M[(\pi')^{-1}[H]] \vDash \varphi(\alpha_0, \ldots, \alpha_{k-1}, \dot{a}_{i_0}^{(\pi')^{-1}[H]}, \ldots, \dot{a}_{i_{l-1}}^{(\pi')^{-1}[H]}, \dot{A}^{(\pi')^{-1}[H]}).$$

Then

$$M[H] \vDash \varphi(\alpha_0, \ldots, \alpha_{k-1}, \dot{a}^{(\pi')^{-1}[H]}_{(\pi')^{-1} \circ \pi'(i_0)}, \ldots, \dot{a}^{(\pi')^{-1}[H]}_{(\pi')^{-1} \circ \pi'(i_{l-1})}, \dot{A}^{(\pi')^{-1}[H]})$$

and

$$M[H] \vDash \varphi(\alpha_0, \ldots, \alpha_{k-1}, \dot{a}^H_{\pi(i_0)}, \ldots, \dot{a}^H_{\pi(i_{l-1})}, \dot{A}^H).$$

Hence

$$\pi'(p) \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{\pi(i_0)}, \ldots, \dot{a}_{\pi(i_{l-1})}, \dot{A}).$$

The converse direction follows by considering π^{-1} instead of π. $\square$

Lemma 123. *In N there is* no *wellorder of the set A.*

Proof. Assume that N possesses a wellorder of A. Then take $f \in N$ and $\eta \in \mathrm{Ord}$ such that $f\colon \eta \leftrightarrow A$. Then $f \in N = (\mathrm{HOD}(s))^{M[G]}$ where $s = \mathrm{TC}(\{A\}) = \{A\} \cup A \cup \omega$. So take $\alpha_0, \ldots, \alpha_{k-1} \in \mathrm{Ord}$ and $i_0, \ldots, i_{l-1} < \omega$ such that f is definable in $M[G]$ by the $\in$-formula φ and the parameters $\alpha_0, \ldots, \alpha_{k-1}, a_{i_0}, \ldots, a_{i_{l-1}}, A$:

$$f(\xi) = b \text{ iff } M[G] \vDash \varphi(\alpha_0, \ldots, \alpha_{k-1}, a_{i_0}, \ldots, a_{i_{l-1}}, A, \xi, b).$$

Consider some $i_* \in \omega \setminus \{i_0, \ldots, i_{l-1}\}$ and $\xi < \eta$ such that $f(\xi) = a_{i_*}$. Then

$$M[G] \vDash \varphi(\alpha_0, \ldots, \alpha_{k-1}, a_{i_0}, \ldots, a_{i_{l-1}}, A, \xi, a_{i_*}).$$

Take $p \in G$ such that

$$p \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{i_0}, \ldots, \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, \dot{a}_{i_*})$$

and

$p \Vdash \forall y \forall y' (\varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{i_0}, \ldots, \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, y) \wedge \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{i_0}, \ldots, \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, y') \rightarrow y = y')$.

Take $i_{**} \in \omega \setminus \{i_0, \ldots, i_{l-1}, i_*\}$ such that $\mathrm{dom}(p) \cap (\{i_{**}\} \times \omega) = \emptyset$. Define a permutation π: $\omega \leftrightarrow \omega$,

$$\pi(i) = \begin{cases} i_{**} \text{ iff } i = i_* \\ i_* \text{ iff } i = i_{**} \\ i \text{ else} \end{cases}$$

By the previous lemma

$$\pi'(p) \Vdash \varphi(\check{\alpha}_0, \ldots, \check{\alpha}_{k-1}, \dot{a}_{\pi(i_0)}, \ldots, \dot{a}_{\pi(i_{l-1})}, \dot{A}, \check{\xi}, \dot{a}_{\pi(i_*)}),$$

i.e.

$$\pi'(p) \Vdash \varphi(\check{\alpha}_0, ..., \check{\alpha}_{k-1}, \dot{a}_{i_0}, ..., \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, \dot{a}_{i_{**}}).$$

Consider $(i, n) \in \mathrm{dom}(p) \cap \mathrm{dom}(\pi'(p))$. The choice of i_{**} and π implies that $i \neq i_*, i_{**}$. Then $\pi(i) = i$ and $\pi'(p)(i, n) = \pi'(p)(\pi(i), n) = p(i, n)$. Hence p and $\pi'(p)$ are compatible in P. Take $q \leqslant p, \pi'(p)$. The previous forcing statements imply

$$q \Vdash \varphi(\check{\alpha}_0, ..., \check{\alpha}_{k-1}, \dot{a}_{i_0}, ..., \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, \dot{a}_{i_*}),$$

$$q \Vdash \varphi(\check{\alpha}_0, ..., \check{\alpha}_{k-1}, \dot{a}_{i_0}, ..., \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, \dot{a}_{i_{**}}),$$

and

$q \Vdash \forall y \forall y' (\varphi(\check{\alpha}_0, ..., \check{\alpha}_{k-1}, \dot{a}_{i_0}, ..., \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, y) \wedge \varphi(\check{\alpha}_0, ..., \check{\alpha}_{k-1}, \dot{a}_{i_0}, ..., \dot{a}_{i_{l-1}}, \dot{A}, \check{\xi}, y') \rightarrow y = y')$.

Then

$$q \Vdash \dot{a}_{i_*} = \dot{a}_{i_{**}}.$$

This is a contradiction since the weakest condition 1 forces that the Cohen reals $\dot{a}_i$ are pairwise distinct. □

Theorem 124. (Paul Cohen) *If* ZF *is consistent then* ZF + ¬AC *is consistent. Hence the Axiom of Choice is independent from* ZF*.*