Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 11

Problem 41 (*Mathias forcing*, 10 Points)**.** Fix a non-principal ultrafilter $\mathcal{U}$ on ω. Let D be the set of all pairs (s, A) such that $s : n \longrightarrow \omega$ is a strictly increasing function for some $n < \omega$ and $A \in \mathcal{U}$. Given elements (s, A) and (t, B) of D, we define $(t, B) <_{P_{\mathcal{U}}} (s, A)$ to hold if $s \subseteq t$, $B \subseteq A$ and $t(k) \in A$ for all $k \in \mathrm{dom}(t) \setminus \mathrm{dom}(s)$.

(1) Show that $P_{\mathcal{U}} = (D, <_{\mathcal{U}}, (\emptyset, \omega))$ is a partial order.

(2) Show that $P_{\mathcal{U}}$ satisfies the countable chain condition.

Let M be a ground model, $\mathcal{U}$ be a non-principal ultrafilter on ω in M and G be M-generic on $P_{\mathcal{U}}^M$.

(3) Prove that $s_G = \bigcup \{s \mid \exists A \in \mathcal{U}\ (s, A) \in G\}$ is a strictly increasing function with domain ω.

(4) Prove that the set $\mathrm{ran}(s_G) \setminus A$ is finite for every $A \in \mathcal{U}$.

(5) Given an infinite subset C of ω contained in M, show that there is an $A \in \mathcal{U}$ with $C \setminus A$ infinite.

Problem 42 (10 Bonus Points)**.** Let κ be a regular uncountable cardinal. Let D denote the set of all partial functions $x : \kappa \xrightarrow{part} 2$ with the property that the set $\mathrm{dom}(x) \cap \rho$ is bounded in ρ whenever ρ is regular cardinal smaller than or equal to κ.

(1) Show that $P_\kappa = (D, \supseteq, \emptyset)$ is a partial order.

Let M be a ground model, κ be a regular uncountable cardinal in M and G be M-generic on P_κ^M.

(2) Show that $x_G = \bigcup G$ is a function with domain κ.

(3) Show that the GCH holds below κ in $M[G]$, i.e. if $\lambda < \kappa$ is a cardinal in $M[G]$, then $M[G] \models 2^\lambda = \lambda^+$ (Hint: *Generalize the argument used in the proof of Problem 37 to larger cardinalities*).

(4) Show: if λ is a cardinal in M with $M \models 2^\lambda = \lambda^+$, then $(\lambda^+)^M$ is a regular cardinal in $M[G]$ (Hint: *Show that the partial order P_κ is isomorphic to a product $Q_0 \times Q_1$ of partial orders with the property that Q_0 is λ^{++}-closed and Q_1 satisfies the λ^{++}-chain condition. Then use Lemma 142*).

Problem 43 (10 Bonus Points)**.** Let M be a ground model with $M \models \mathrm{GCH}$, F be a function in M satisfying the assumptions of *Easton's Theorem* and G be M-generic on the partial order given by *Easton's Theorem*. Calculate the value of $(2^\nu)^{M[G]}$ for every cardinal ν in $M[G]$ that is smaller than the least upper bound of $\mathrm{dom}(F)$.

Please hand in your solutions on **Wednesday**, July 10 before the lecture.