Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 10

Problem 37 (4 Points)**.** Construct a dense embedding (see Problem 22) of the partial order $\mathrm{Fn}(\omega_1, 2^\omega, {<}\aleph_1)$ into the partial order $\mathrm{Fn}(\omega_1, 2, {<}\aleph_1)$.

Problem 38 (8 Points)**.** A sequence $\langle S_\alpha \mid \alpha < \omega_1\rangle$ is a $\diamondsuit$-sequence if the following statements hold.

- S_α is a subset of α for every $\alpha < \omega_1$.
- If X is a subset of ω_1, then the set $\{\alpha < \omega_1 \mid S_\alpha = X \cap \alpha\}$ is a stationary subset of ω_1.

Let M be a ground model with $(\mathrm{CH})^M$ and let G be M-generic on $\mathrm{Fn}(\omega_1, \omega_1, {<}\aleph_1)^M$. Show that there is a $\diamondsuit$-sequence in $M[G]$ (Hint: *Start in M and fix an enumeration $\langle B_\alpha \mid \alpha < \omega_1\rangle$ of all bounded subsets of ω_1. In $M[G]$, let $c : \omega_1 \longrightarrow \omega_1$ be the function corresponding to G and define $S_\alpha = B_{c(\alpha)} \cap \alpha$. Given $\dot{C}, \dot{X} \in M$ and $q \in \mathrm{Fn}(\omega_1, \omega_1, {<}\aleph_1)$ with $q \Vdash$ "$\dot{C} \subseteq \omega_1$ is club $\wedge$ $\dot{X} \subseteq \omega_1$", use the the fact that $\mathrm{Fn}(\omega_1, \omega_1, {<}\aleph_1)$ is ω_1-closed to find $p \in \mathrm{Fn}(\omega_1, \omega_1, {<}\aleph_1)$, $\alpha < \omega_1$ and $Y \subseteq \alpha$ with $\mathrm{dom}(p) = \alpha$, $p \leq q$ and $p \Vdash$ "$\alpha \in \dot{C}$ $\wedge$ $\check{Y} = \dot{X} \cap \check{\alpha}$ "*).

Problem 39 (8 Points)**.** Given an ordinal α, we let ${}^{<\alpha}2$ denote the set of all functions $s : \beta \longrightarrow 2$ with $\beta < \alpha$. We call $T \subseteq {}^{<\alpha}2$ a *subtree of* ${}^{<\alpha}2$ if T is closed under initial segments and we define $ht(T) = \mathrm{lub}\{\mathrm{dom}(s) \mid s \in T\}$. A subtree T of ${}^{<\alpha}2$ is *normal* if the following statements hold for all $s \in T$.

- If $\mathrm{dom}(s) + 1 < ht(T)$ and $i < 2$, then $s \cup \{(\mathrm{dom}(s), i)\} \in T$.
- If $\mathrm{dom}(s) \leq \alpha < ht(T)$, then there is an $\bar{s} \in T$ with $s \subseteq \bar{s}$ and $\mathrm{dom}(\bar{s}) = \alpha$.

A subtree T of ${}^{<\omega_1}2$ is a *Souslin tree* if $ht(T) = \omega_1$ and the partial order $\langle T, \supseteq, \emptyset\rangle$ satisfies the countable chain condition.

We let $\mathbb{S}$ denote the partial order consisting of countable normal subtrees of ${}^{<\omega_1}2$ ordered by end-extension, i.e. we have $t_0 \leq t_1$ if $t_1 = t_0 \cap {}^{<ht(t_1)}2$. Prove the following statements.

(1) $\mathbb{S}$ is ω_1-closed.

(2) If (CH) holds, then $\mathbb{S}$ satisfies the $\aleph_2$-chain condition.

(3) Let M be a ground model with $(\mathrm{CH})^M$ and let G be M-generic on $\mathbb{S}^M$. Then $T^G = \bigcup G$ is a Souslin tree in $M[G]$ (Hint: *First show that $ht(T^G) = \omega_1$ holds in $M[G]$. Let $\dot{T} \in M$ denote the canonical name for T^G and assume, towards a contradiction, that there is $\dot{A} \in M$ and $t_0 \in \mathbb{S}^M$ with $t_0 \Vdash$ "$\dot{A}$ is an uncountable maximal antichain in $\dot{T}$". Find $\alpha < \omega_1$, $t_1 \leq t_0$ and $B \subseteq {}^{<\alpha}2$ such that $\alpha = ht(t_1)$, $t_1 \Vdash$ "$\check{B} = \dot{A} \cap {}^{<\check{\alpha}}2$" and B is a maximal antichain in $\langle t_1, \supseteq, \emptyset\rangle$. Then construct a condition t_2 in $\mathbb{S}^M$ such that $ht(t_2) = \alpha + 1$ and for every $s \in t_2$ with* $\mathrm{dom}(s) = \alpha$ *there is an $\bar{s} \in B$ with $\bar{s} \subseteq s$*).

Problem 40 (4 Points)**.** Let M be a ground model and P be a partial order in M. Prove: if P is ω_1-closed in M, then P *preserves stationary subsets of* ω_1, i.e. we have

$$1_P \Vdash \text{"}\check{S} \textit{ is a stationary subset of } \omega_1\text{"}$$

for every stationary subset S of ω_1 in M (Hint: *Assume, towards a contradiction, that there is* $p \in P$ *and* $\dot{C} \in M$ *with* $p \Vdash$ "$\dot{C}$ *is a club subset of* ω_1 *with* $\dot{C} \cap \check{S} = \emptyset$". *Working in* M*, use this assumption to construct a descending sequence* $\langle p_\alpha \mid \alpha < \omega_1 \rangle$ *of conditions in* P *and a strictly increasing continuous sequence* $\langle \gamma_\alpha \mid \alpha < \omega_1 \rangle$ *of countable ordinals such that* $p_\alpha \Vdash$ "$\check{\gamma}_\alpha \in \dot{C}$" *holds for every* $\alpha < \omega_1$).

Please hand in your solutions on **Wednesday**, June 26 before the lecture.