Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke — Problem sheet 9

Problem 33 (*Hechler Forcing*, 8 Points)**.** We define a partial order P_H by the following clauses.

- A condition in P_H is a pair $p = (s_p, E_p)$ such that $s_p : n \longrightarrow \omega$ for some $n < \omega$ and E_p is a finite set of functions from ω to ω.
- We define $p \leq_{P_H} q$ to hold if $s_q \subseteq s_p$, $E_q \subseteq E_p$ and $f(k) < s_p(k)$ for all $k \in \mathrm{dom}(s_p) \setminus \mathrm{dom}(s_q)$ and $f \in E_q$.

Prove the following statements.

(1) P_H satisfies the countable chain condition.

(2) If M is a ground model and G is M-generic on P_H^M, then there is a function $x : \omega \to \omega$ in $M[G]$ with the property that the set $\{n < \omega \mid x(n) \leq y(n)\}$ is finite for every function $y : \omega \longrightarrow \omega$ in M.

(3) If M is a ground model and G is M-generic on P_H^M, then there is a $H \in M[G]$ that is M-generic on $\mathrm{Fn}(\omega, 2, \aleph_0)$.

Problem 34 (4 Points)**.** Let M be a ground model and P be a partial order in M. We use the Recursion Theorem in M to construct a class M^P in M that satisfies the following properties.

- If $\dot{x} \in M^P$, then $\dot{x} \subseteq M^P \times P$.
- If $\dot{x} \in M^P$ and $\dot{y} \in \mathrm{dom}(\dot{x})$, then the set $\{p \in P \mid \langle \dot{y}, p\rangle \in \dot{x}\}$ is an antichain in P that is an element of M.

Show that for every $\dot{x} \in M$ there is a $\dot{y} \in M^P$ with $1_P \Vdash \dot{x} = \dot{y}$.

Problem 35 (6 Points)**.** Let M be a ground model, κ be a cardinal of uncountable cofinality in M and P be a partial order in M. Prove: if P satisfies the countable chain condition in M, then P *preserves stationary subsets of* κ, i.e. we have

$$1_P \ \Vdash \ \check{S} \textit{ is a stationary subset of } \check{\kappa}$$

for every stationary subset S of κ in M (Hint: *Given a condition p in P and $\dot{C} \in M$ with $p \Vdash$ "$\dot{C}$ club in $\check{\kappa}$", prove that there is a club subset C of κ in M with $p \Vdash \check{C} \subseteq \dot{C}$ by using the countable chain condition to show that for every $\beta < \kappa$ there is a $\beta < \gamma < \kappa$ with $p \Vdash \check{\gamma} \in \dot{C}$.*).

Problem 36 (*The Maximality Principle*, 6 Points)**.** Let M be a ground model, P be a partial order in M and p be a condition in P. Prove that for very $\in$-formula $\varphi(v_0, \ldots, v_n)$ and all $\dot{x}_0, \ldots, \dot{x}_{n-1} \in M$ with $p \Vdash \exists x \ \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}, x)$ there is an $\dot{x}_n \in M$ with $p \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_n)$ (Hint: *Consider maximal antichains in the set*

$$\{q \leq_P p \mid \exists \dot{x} \in M \ q \Vdash \varphi(\dot{x}_0, \ldots, \dot{x}_{n-1}, \dot{x})\}$$

contained in M and show that they are maximal in P. Use these antichains to construct the name $\dot{x}_n$).

Please hand in your solutions on **Wednesday**, June 19 before the lecture.