Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 8

Problem 28 (4 Points)**.** Given functions $x, y : \omega \longrightarrow 2$, we define

$$D(x,y) \ = \ \{n < \omega \mid x(n) \neq y(n)\} \quad \text{and} \quad E(x,y) \ = \ \{n < \omega \mid x(n) = y(n)\}.$$

Let M be a ground model, G be M-generic on $\mathrm{Fn}(\omega, 2, \aleph_0)$ and $x_G = \bigcup G : \omega \longrightarrow 2$. Prove the following statements.

(1) If $y : \omega \longrightarrow 2$ is an element of M, then the sets $D(x_G, y)$ and $E(x_G, y)$ are both infinite.

(2) There is a function $z : \omega \longrightarrow 2$ in $M[G]$ with the following properties.

  (a) If $y : \omega \longrightarrow 2$ is an element of M, then the sets $D(y, z)$ and $E(y, z)$ are both infinite.

  (b) The filter $\{p \in \mathrm{Fn}(\omega, 2, \aleph_0) \mid p \subseteq z\}$ is not M-generic.

Problem 29 (4 Points)**.** Let M and N be transitive with ZFC^M, ZFC^N and $\mathcal{P}(\alpha)^M \subseteq N$ for every $\alpha \in M \cap \mathrm{Ord}$. Prove that $M \subseteq N$ (Hint: *Fix* $x \in N$. *Use a bijection* $b : \mathrm{tc}(\{x\}) \longrightarrow \alpha$ *to some* $\alpha \in \mathrm{Ord}$ *and the Gödel pairing function to code* x *as a subset of* α *in* M*. Then use a Mostowski collapse to decode this subset in* N.).

Problem 30 (4 Points)**.** Let M be a ground model, P be a partial order in M, $\varphi(v_0, \ldots, v_{n-1})$ and $x_0, \ldots, x_{n-1} \in M$. Prove the following statements.

(1) Given a condition p in P and an automorphism π of P in M, we have $p \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ if and only if $\pi(p) \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$.

A partial order P is *weakly homogeneous* if for all conditions $p, q \in P$ there is an automorphism π of P such that the conditions $\pi(p)$ and q are compatible in P.

(2) If P is weakly homogeneous in M, then the following statements are equivalent.

  (a) $1_P \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$.

  (b) $p \Vdash \varphi(\check{x}_0, \ldots, \check{x}_{n-1})$ for some condition p in P.

Problem 31 (6 Points)**.** Let M be a ground model and G be an M-generic filter on $\mathrm{Fn}(\omega, 2, \aleph_0)$. Prove that $\mathrm{HOD}^{M[G]} \subseteq \mathrm{HOD}^M$ (Hint: *Show that the partial order* $\mathrm{Fn}(\omega, 2, \aleph_0)$ *is weakly homogeneous. Then use Problem 29 and 30.*).

Problem 32 (6 Points)**.** Let M be a ground model, κ be an infinite cardinal in M, X be an infinite subset of κ contained in M and G be M-generic on $P = \mathrm{Fn}(\omega \times \kappa, 2, \aleph_0)$. Set $P \restriction X = \mathrm{Fn}(\omega \times X, 2, \aleph_0)$ and $G \restriction X = G \cap P \restriction X$. Prove the following statements.

(1) $G \restriction X$ is M-generic on $P \restriction X$.

(2) If $X \subsetneq \kappa$, then $M[G \restriction X] \subsetneq M[G]$ (Hint: *Let $\alpha \in \kappa \setminus X$ and define*

$$\dot{x} \;=\; \{\langle \check{n}, p\rangle \mid n < \omega,\ p \in P,\ p(n, \alpha) = 1\} \;\in\; M.$$

Assume that $\dot{x}^G = \dot{y}^{G \restriction X}$ for some $\dot{y} \in M$ and define

$$\dot{z} \;=\; \{\langle \check{n}, p\rangle \mid n < \omega,\ p \in P \restriction X,\ p \Vdash^M_{P \restriction X} \check{n} \in \dot{y}\} \;\in\; M.$$

Show $\dot{x}^G = \dot{z}^{G \restriction X} = \dot{z}^G$ and pick $p \in G$ with $p \Vdash^M_P \dot{x} = \dot{z}$. Derive a contradiction by constructing a suitable M-generic filter $\bar{G}$ on P with $p \in \bar{G}$ and $\bar{G} \cap (P \restriction X) = G \restriction X$).

Please hand in your solutions on **Wednesday**, June 12 before the lecture.