Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke — Problem sheet 7

Problem 25 (12 Points)**.** Let (X,τ) be a non-empty topological space. We let $\mathrm{ro}(X,\tau)$ denote the set of all regular open subsets of X (i.e. $int(cl(A)) = A$). Define $U \vee V = int(cl(U \cup V))$ and $U' = int(X \setminus U)$ for all $U, V \in \mathrm{ro}(X,\tau)$.

(1) Show that

$$\mathbb{B}(X,\tau) \ = \ \langle \mathrm{ro}(X,\tau), \subseteq, \cap, \vee, \emptyset, X, ' \ \rangle$$

is a complete boolean algebra.

Given a partial order $P = \langle P, \leq_P, 1_P \rangle$, we define τ_P to be the set of all subsets of P that are open in P (see Problem 21).

(2) Show that (P, τ_P) is a topological space.

Given a boolean algebra $\mathbb{B} = \langle B, \leq, \wedge, \vee, 0, 1, ' \ \rangle$, we define $\mathbb{B}^*$ to be the partial order $\langle B \setminus \{0\}, \leq, 1 \rangle$.

(3) Show that the map

$$\pi_P : P \longrightarrow \mathrm{ro}(P, \tau_P) \setminus \{\emptyset\}; \ p \longmapsto int(cl(\{q \in P \mid q \leq_P p\}))$$

is a dense embedding of P into the partial order $\mathbb{B}(P, \tau_P)^*$.

Problem 26 (8 Points)**.** A partial order P is *separative* if for all conditions p and q in P with $p \not\leq q$ there is a condition r in P that is stronger than p and incompatible with q.

(1) Show: if $\mathbb{B}$ is a boolean algebra, then $\mathbb{B}^*$ is separative.

(2) Show that a partial order P is separative if and only if the following statements hold.

  (a) The embedding π_P constructed in part (3) of Problem 25 is injective.

  (b) $\forall p, q \in P \ [p \leq q \ \longleftrightarrow \ \pi_P(p) \subseteq \pi_P(q)$.

(3) If P is a partial order, then there is a surjective complete embedding of P into a separative partial order (Hint: *Show that*

$$p \approx_{sep} q \iff \forall r \ [\ p \textit{ and } r \textit{ are compatible in } P \\ \longleftrightarrow \ \ q \textit{ and } r \textit{ are compatible in } P]$$

defines an equivalence relation on P*. Then define a suitable ordering of the quotient* $P/\approx_{sep}$).

Problem 27 (4 Points)**.** Let M be a ground model and P be a partial order contained in M. We fix a formula $\varphi(v_0, v_1)$, a condition p in P and names $\dot{y}, \dot{z} \in M$. Prove the following statements.

(1) $p \Vdash \forall x \in \dot{y}\ \varphi(x, \dot{z})$ if and only if $p \Vdash [\dot{x} \in \dot{y} \longrightarrow \varphi(\dot{x}, \dot{z})]$ for every $\dot{x} \in \mathrm{dom}(\dot{y})$.
(2) $p \Vdash \exists x\ \varphi(x, \dot{z})$ if and only if the set $\{q \in P \mid \exists \dot{x} \in M\ q \Vdash \varphi(\dot{x}, \dot{z})\}$ is dense below p in P.
(3) $p \Vdash \exists x \in \dot{y}\ \varphi(x, \dot{z})$ if and only if the set $\{q \in P \mid \exists \dot{x} \in \mathrm{dom}(\dot{y})\ q \Vdash \varphi(\dot{x}, \dot{z})\}$ is dense below p in P.
(4) $p \Vdash \dot{y} \in \dot{z}$ if and only if $p \Vdash \dot{x} \subseteq \dot{z}$ with $\dot{x} = \{(\dot{y}, 1)\}$.

Please hand in your solutions on **Wednesday**, June 05 before the lecture.