Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 6

Problem 22 (12 Points). Let $P = \langle P, <_P, 1_P \rangle$ and $Q = \langle Q, <_Q, 1_Q \rangle$ be partial orders. We say that a function $\pi : Q \to P$ is a *complete embedding* if the following statements hold.

(a) $\pi(1_Q) = 1_P$.

(b) If $q_0, q_1 \in Q$ with $q_1 \leq_Q q_0$, then $\pi(q_1) \leq_P \pi(q_0)$.

(c) Given $q_0, q_1 \in Q$, the conditions q_0 and q_1 are incompatible in Q if and only if the conditions $\pi(q_0)$ and $\pi(q_1)$ are incompatible in P.

(d) The image of every maximal antichain in Q under π is a maximal antichain in P.

We say that a function $\pi : Q \to P$ is a *dense embedding* if the above statements (a)-(c) hold and the image of Q under π is a dense subset of P.

(1) Show that every dense embedding is a complete embedding.

Let $\mathrm{Add}(\omega) = \langle {}^{<\omega}\omega, \supseteq, \emptyset \rangle$ denote the partial order consisting of functions $s : m \to \omega$ with $m < \omega$ ordered by reverse inclusion.

(2) Explicitly construct a dense embedding of $\mathrm{Add}(\omega)$ into $\mathrm{Fn}(\omega, 2, \aleph_0)$.

(3) Let P be a countable atomless partial order. Prove that there is a dense embedding of $\mathrm{Add}(\omega)$ into P (Hint: *Use a previous exercise to show that there is an infinite antichain below every condition in P. Fix an enumeration $\langle p_n \mid n < \omega \rangle$ of P and define a function π by recursion. Set $\pi(\emptyset) = 1_P$. If $\pi(s)$ is defined for some $s : n \to \omega$, then extend π in a way such that $\{\pi(s^\frown\langle m \rangle) \mid m < \omega\}$ is a maximal antichain below $\pi(s)$ in P. Moreover, if the conditions p_n and $\pi(s)$ are compatible in P, then ensure that there is an $m < \omega$ with $\pi(s^\frown\langle m \rangle) \leq_P p_n$. Show that the resulting function is a dense embedding.*).

(4) Use the above result and the Löwenheim-Skolem Theorem to show that for every atomless partial order P there is a function $\pi : \mathrm{Add}(\omega) \to P$ satisfying the above statements (a)-(c).

Problem 23 (4 Points). Let M be a transitive set with ZFC^M, $P, Q \in M$ be partial orders and $\pi : Q \to P$ be a function contained in M. Prove the following statements.

(1) If π is a complete embedding in M and G is M-generic for P, then the preimage of G under π is a filter that is M-generic for Q.

(2) If π is a dense embedding in M and H is M-generic for Q, then the set $\pi[H] = \{p \in P \mid \exists q \in H \; \pi(q) \leq_P p\}$ is a filter that is M-generic for P.

Problem 24 (4 Points)**.** Let $P = \langle P, <_P, 1_P\rangle$ and $Q = \langle Q, <_Q, 1_Q\rangle$ be partial orders. We define the product of P and Q to be the partial order

$$P \times Q \ = \ \langle P \times Q, \leq_{P\times Q}, (1_P, 1_Q)\rangle$$

with

$$(p_1, q_1) \leq_{P\times Q} (p_0, q_0) \iff p_1 \leq_P p_0 \ \wedge \ q_1 \leq_Q q_0$$

for all $p_0, p_1 \in P$ and $q_0, q_1 \in Q$.

(1) Show that the map

$$\pi_P : P \longrightarrow P \times Q : \ p \longmapsto (p, 1_Q)$$

is a complete embedding of P into $P \times Q$.

(2) Explicitly construct a dense embedding of $\mathrm{Fn}(\omega, 2, \aleph_0)$ into the product $\mathrm{Fn}(\omega, 2, \aleph_0) \times \mathrm{Fn}(\omega, 2, \aleph_0)$.

Please hand in your solutions on **Monday**, May 27 before the lecture.