Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke — Problem sheet 5

Problem 18 (4 Points)**.** Let M be a transitive set with ZFC^M, P be a partial order in M and G be a filter on P. Prove the following statements.

(1) If $x \in M[G]$, then there is a function $f \in M[G]$ with $\mathrm{dom}(f) \in \mathrm{Ord}$ and $x \subseteq \mathrm{ran}(f)$.

(2) If $\mathrm{ZF}^{M[G]}$, then $\mathrm{ZFC}^{M[G]}$.

Problem 19 (4 Points)**.** Let P be a partial order. A condition p in P is an *atom in P* if all stronger conditions are compatible, i.e. if $q, r \in P$ with $q, r \leq p$, then there is an $s \in P$ with $s \leq q, r$. We say that P is *atomless* if there are no atoms in P.
Show that the following statements are equivalent for every transitive set M with ZFC^M and every partial order P in M.

(1) P is atomless.

(2) M does not contain a filter on P that is M-generic for P.

(Hint: *Given a filter G on P, consider the subset $P \setminus G$*).

Problem 20 (6 Points)**.** Let P be a partial order. An *antichain in P* is a subset of P whose elements are pairwise incompatible in P. We call an antichain *maximal* if it is not a proper subset of another antichain in P.

(1) Prove that every antichain in a partial order is contained in a maximal antichain.

(2) Explicitly construct an infinite antichain in the partial order $\mathrm{Fn}(\omega, 2, \aleph_0)$.

(3) Prove that every atomless partial order contains an infinite antichain.

Problem 21 (6 Points)**.** Given a partial order P, we call a subset U of P *open in P* if U is downwards-closed in P, i.e. if $p \in U$, $q \in P$ and $q \leq p$, then $q \in U$.
Show that the following statements are equivalent for every transitive set M with ZFC^M, every partial order P in M and every filter G on P.

(1) G is M-generic for P.

(2) If $D \in M$ is dense and open in P, then $D \cap G \neq \emptyset$.

(3) If $A \in M$ is a maximal antichain in P, then $A \cap G \neq \emptyset$.

(Hint: *To prove the implication* $(3) \rightarrow (1)$*, start with a dense subset $D \in M$, find an antichain $A \in M$ that is a maximal antichain in D and show that A is a maximal antichain in P*).

Please hand in your solutions on Wednesday, May 15 before the lecture.