Models of Set Theory I - Summer 2013

Prof. Dr. Peter Koepke, Dr. Philipp Lücke Problem sheet 1

Problem 1 (4 Points)**.** Let W be a term and φ be an $\in$-formula which do not have common variables. Show: if φ is a tautology derivable from the sequent calculus (see *Mathematical Logic. An Introduction (Summer 2012)*, page 20), then φ holds in W.

Problem 2 (4 Points)**.** Assume ZF. Let W be a transitive, non-empty class. Show:

(1) $(\mathit{Union})^W \;\longleftrightarrow\; \forall x \in W \bigcup x \in W$.

(2) Let ψ be the instance of the Replacement schema for the $\in$-formula $\varphi(x, y, \vec{w})$. Then ψ^W is equivalent to

$$\forall \vec{w} \in W \; \big(\forall x, y, y' \in W \; \big[\big(\varphi^W(x, y, \vec{w}) \wedge \varphi^W(x, y', \vec{w})\big) \to y = y'\big] \longrightarrow \forall a \in W \; \{y \mid \exists x \in a \; \varphi^W(x, y, \vec{w})\} \cap W \in W\big).$$

Problem 3 (4 Points)**.** Assume ZF. Show:

(1) For every ordinal α, we have $V_\alpha \models$ *Extensionality, Union, Separation, and Foundation.*

(2) If α is a limit ordinal, then $V_\alpha \models$ *Pairing and Powerset.*

Problem 4 (8 points)**.** Assume ZFC. Given a cardinal κ, we define

$$\mathrm{H}_\kappa \;=\; \{x \mid \mathrm{card}(\mathrm{TC}(\{x\})) < \kappa\}.$$

Examine which ZFC-axioms hold in H_κ for various infinite cardinals κ.

Please hand in your solutions on Wednesday, April 17 before the lecture.