LECTURE NOTES ON DESCRIPTIVE SET THEORY

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Abstract. Metric spaces, Borel and analytic sets, Baire property and measurability, dichotomies, equivalence relations. Possible topics: determinacy, group actions, rigidity, turbulence.

Contents

1. Introduction

Suppose that X is a complete separable metric space and $R_0, ..., R_n$ are relations on X. Suppose that we forget about the metric, and even the topology, and only remember the Borel sets. Which structural properties remain, i.e. which pairs of such structures can still be distinguished? If the relation is a graph, when is there a Borel measurable coloring of the graph? If the structure is an equivalence relation, when is there a Borel measurable function classifying the equivalence classes by elements of some other separable metric space?

The scope of this setting extends to many classes of mathematical structures which can be represented by elements of some Polish space, and isomorphism then corresponds to an equivalence relation on that space. An equivalence relation is *smooth* if there is a Borel measurable map which sends the equivalence classes to distinct points in a Polish space.

Example 1.1. *Graphs on the natural numbers can be represented as elements of the Cantor space* ${}^{\omega}2$. *Isomorphism of countable graphs is the most complicated isomorphism relation for classes of countable structures, in particular it is not smooth.*

Date: June 11, 2013.

Motivated by a problem in C^*-algebras, Glimm proved the following striking result in 1961.

Theorem 1.2. *(Glimm) Suppose that $G \curvearrowright X$ is a continuous action of a locally compact Polish group G on a Polish space X. If the induced orbit equivalence relation E is not smooth, then there is a continuous reduction $f\colon E_0 \to E$, i.e. a map $f\colon {}^{\omega}2 \to X$ such that $x(n) = y(n)$ for all but finitely many n if and only if $(f(x), f(y)) \in E$.*

Harrington and Kechris proved this in 1990 for arbitrary Borel equivalence relations. Subsequently Kechris and Todorcevic proved a dichotomy for analytic graphs which implies this result. We will present a proof of this by Ben Miller.

To do this, we begin with results about metric spaces and their Borel and analytic subsets.

2. Metric spaces

A standard Borel space is a set with a σ-algebra which is Borel isomorphic to $[0, 1]$. The following facts about metric spaces are useful for showing that a given space is standard Borel. Moreover, some of these facts are needed to prove results about standard Borel spaces in the following sections, i.e. given a standard Borel space we will choose an appropriate metric and work with it.

2.1. **Polish spaces.**

Definition 2.1. *A* Polish metric space *is the completion of a countable metric space. Such a space with its topology, but forgetting the metric, is called* Polish*.*

If (X, d) is metric, let $R(d)$ denotes the set of distances. A metric on a space is *compatible* if it induces its topology. Let (X, d) denote a Polish metric space (unless stated otherwise). Then $\hat{d}(x, y) = \min\{d(x, y), 1\}$ is a bounded compatible complete metric on X.

Example 2.2.
- *Countable spaces with the discrete topology.*
- *Compact metric spaces.*
- *Closed subsets of Polish spaces.*
- *Separable Banach spaces.*

Lemma 2.3. *If $(X_n)_{n\in\omega}$ is a sequence of Polish spaces, then $X = \prod_{n\in\omega} X_n$ is Polish.*

Proof. Find a compatible complete metric d_n on X_n bounded by $\frac{1}{2^{n+1}}$. Let $d\colon X \times X \to \mathbb{R}$, $d(x, y) = \sum_{n\in\omega} d_n(x_n, y_n)$, where $x = (x_n)$ and $y = (y_n)$.

To show that d induces the product topology, consider $B^d(x,\epsilon)$, where $x=(x_n)$ and $\epsilon>0$. Find $n\geq 1$ with $\frac{1}{2^{n+1}}<\frac{\epsilon}{2}$. Then $B^{d_0}(x_0,\frac{\epsilon}{2n})\times ...\times B^{d_n}(x_{n-1}),\frac{\epsilon}{2n}\times\prod_{i\geq n}X_i\subseteq B^d(x,\epsilon)$.

If $(x^{(i)})_i$ is a Cauchy sequence in X, then $(x_n^{(i)})_i$ is a Cauchy sequence in X_n for each n. Let $x_n=\lim_i^{d_n}x_n^{(i)}$. The $x=(x_n)=\lim_i^d x^{(i)}$. $\square$

Definition 2.4. *(1) A subset A of X is G_δ if it is the intersection of countably many open sets.*

(2) A subset A of X is F_σ if it is the union of countably many closed sets.

Lemma 2.5. *Every Polish space is homeomorphic to a G_δ subset of the Hilbert cube $[0,1]^\omega$.*

Proof. Let $d_n(x,y)=\frac{1}{2^{n+1}}d(x,y)$ for each n. Suppose that $\{x_n\mid n\in\omega\}\subseteq X$ is dense. Let $f\colon X\to[0,1]^\omega$, $f(x)=(f_n(x))_n$, $f_n(x)=d(x,x_n)$.

Then f is continuous, since $d(x,y)<\epsilon$ implies that $|d(x,x_n)-d(y,x_n)|<\epsilon$ and $d(f(x),f(y))=\sum_{n\in\omega}\frac{1}{2^{n+1}}|d(x,x_n)=d(y,x_n)|<\epsilon$. To see that f^{-1} (defined on the range of f) is continuous, consider $x\in X$ and any open set $U\subseteq X$ with $x\in U$. Find n and $\epsilon>0$ with $x\in B(x_n,r)\subseteq U$. If $f(x)\in\prod_{i<n}X_i\times U\times\prod_{i>n}X_i$, then $y\in U$. $\square$

Lemma 2.6. *Every open $U\subseteq X$ is Polish.*

Proof. Let $C=X\setminus U$. Consider the metric $\hat{d}(x,y)=d(x,y)+|\frac{1}{d(x,C)}-\frac{1}{d(y,C)}|$ on U.

To show that $\hat{d}$ is compatible with the topology, notice that $d\leq\hat{d}$ and hence every d-open set is $\hat{d}$-open. For the other direction suppose that $x\in U$, $d(x,C)=\delta>0$ and $\epsilon>0$. Find $\eta>0$ with $\eta+\frac{\eta}{\delta-\eta}<\epsilon$. If $d(x,y)<\eta$, then $\delta-\eta<d(y,C)<\delta+\eta$ and

$$\begin{aligned}\hat{d}(x,y)&<\eta+\max\{|\frac{1}{\delta}-\frac{1}{\delta-\eta}|,|\frac{1}{\delta}-\frac{1}{\delta+\eta}|\}\\&=\eta+\max\{|\frac{-\eta}{\delta(\delta-\eta)}|,|\frac{\eta}{\eta(\delta-\eta)}|=\eta+\frac{\eta}{\delta-\eta}<\epsilon\end{aligned}$$

and hence every $\hat{d}$-open set is d-open. $\square$

Lemma 2.7. *Every G_δ set $U\subseteq X$ is Polish.*

Proof. Let $U=\bigcap_{n\in\omega}U_n$ with U_n open. Find compatible complete metrics $d_n<\frac{1}{2^{n+1}}$ on U_n. Let $\hat{d}\colon X\times X\to\mathbb{R}$, $\hat{d}(x,y)=\sum_{n\in\omega}d_n(x,y)$.

To show that $\hat{d}$ is compatible with the relative topology on U, consider $x\in X$ and $\epsilon>0$. Find n with $\frac{1}{2^n}<\frac{\epsilon}{2}$. then $\bigcap_{i<n}B^{d_i}(x,\frac{\epsilon}{2n})\cap U\subseteq B^{\hat{d}}(x,\epsilon)$. To show that $(U,\hat{d})$ is complete, suppose that (x_i) is a Cauchy sequence. Then (x_i) is a Cauchy sequence with respect to each d_n and hence its limit is in U. $\square$

Lemma 2.8. *Every Polish $U \subseteq X$ is G_δ.*

Proof. We can assume that $\overline{U} = X$. Suppose that (U_n) is a base for (X, d) and that $\hat{d}$ is a compatible complete metric on U. Let

$$A = \bigcap_{m\in\omega} \bigcap \{U_n \mid diam^{\hat{d}}(U \cap U_n) < \frac{1}{m+1}\}.$$

Then A is G_δ in X and $U \subseteq A$. To show that $A \subseteq U$, suppose that $x \in A$. Find n_i with $x \in U_{n_i}$ and $diam^{\hat{d}}(U \cap U_{n_i}) < \frac{1}{i+1}$. There is some $x_m \in U \cap \bigcap_{i\leq m} U_{n_i}$ since U is dense in X. Then (x_n) is a Cauchy sequence in $(U, \hat{d})$ and hence $x = \lim_n x_n \in U$. □

Question 2.9. *Why is $\mathbb{Q}$ not Polish?*

Exercise 2.10. *Show that every Polish space is* Baire*, i.e. the intersection of countably many dense open subsets is dense.*

Definition 2.11. *Suppose that (X, d_X) and (Y, d_Y) are Polish metric spaces. A map $f\colon X \to Y$ is* Lipschitz *if $d_Y(f(x), f(y)) \leq d_X(x, y)$ for all $x, y \in X$.*

Example 2.12. *Let $L(X, Y)$ denote the set of all Lipschitz maps from X to Y and suppose that $D_X = \{x_n \mid n \in \omega\}$ is dense in X and D_Y is dense in Y. Let*

$$d_L(f, g) = \sum_{i\in\omega} 2^{-(i+1)} d_Y(f(x_i), f(x_j))$$

for $f, g \in L(X, Y)$. Then d_L is a metric on $L(X, Y)$.

Exercise 2.13.
1. *Show that the family of $U_{x,y,n} = \{f \in L(X, Y) \mid d(f(x), y) < 2^{-n}\}$ for $x \in D_X$, $y \in D_y$, and $n \in \omega$ is a base for $(L(X, Y), d_L)$.*
2. *Show that $(L(X, Y), d_L)$ has the topology of pointwise convergence and hence it is complete.*

Example 2.14. *Let $DP(X, Y)$ denote the set of all distance-preserving maps (isometric embeddings) from X into Y with the pointwise convergence topology. Then $DP(X, Y)$ is a closed subset of $L(X, Y)$ and hence Polish.*

Example 2.15. *Let $Iso(X, Y)$ denote the set of all isometries from X onto Y with the pointwise convergence topology. Suppose that $D_X \subseteq X$ and $D_Y \subseteq Y$ are countable dense. Then $Iso(X, Y)$ is a G_δ subset of $DP(X, Y)$ and hence Polish, since $f \in Iso(X, Y)$ if and only if for all $y \in D_Y$ and all n, there is $x \in D_Y$ with $d(f(x), y) < \frac{1}{n}$.*

2.2. **Urysohn space.** A Polish metric space is *universal* if it contains an isometric copy of any other Polish metric space (equivalently, of every countable metric space). Urysohn constructed such a space with a random construction which predates the random graph.

Definition 2.16. *Suppose that (X, d) is a metric space. A function $f\colon X \to \mathbb{R}$ is* Katetov *if it measures the distances of a point in a one-point metric extension of (X, d). Equivalently*

$$|f(x) - f(y)| \leq d(x, y) \leq f(x) + f(y)$$

for all $x, y \in X$.

Definition 2.17. *Suppose that $R \subseteq \mathbb{R}$. A metric space (X, d) has the (R-)extension property (for $R \subseteq \mathbb{R}$) if for all finite $F \subseteq X$ and every Katetov $f\colon F \to \mathbb{R}$ ($f\colon F \to R$), there is $y \in X$ such that*

$$f(x) = d(x, y)$$

for all $x \in F$.

Lemma 2.18. *If (X, d_X) and (Y, d_Y) are Polish and have the extension property, then X and Y are isometric.*

Proof. Suppose that $D_X \subseteq X$ and $D_Y \subseteq Y$ are countable dense. Let $D_X = \{x_n \mid n \in \omega\}$ and $D_Y = \{y_n \mid n \in \omega\}$. We construct countable sets E_X and E_Y with $D_X \subseteq E_X \subseteq X$ and $D_Y \subseteq E_Y \subseteq Y$ and an isometry $f\colon E_X \to E_Y$ using the extension property.

We then extend f to a map $g\colon X \to Y$ as follows. If (x_n) is in E_X and $x = \lim_n x_n$, let $g(x) = \lim_n f(x_n)$. Then g is well-defined and g is an isometry. □

Definition 2.19. *(X, d) is* ultrahomogeneous *if every isometry between finite subsets extends to an isometry of X (onto X).*

Remark 2.20. *The extension property implies ultrahomogeneity by an argument as in the previous lemma.*

Suppose that K is a class of first-order structures in a countable language. An *embedding* $f\colon A \to B$ for $A, B \in K$ is an isomorphism onto its image.

Definition 2.21. *Suppose that K is a class of first-order structures in a countable language. Consider the following properties.*

- *(Hereditary property HP) If $A \in K$ and B is a finitely generated substructure of A, then B is isomorphic to some structure in K.*

- *(Joint embedding property JEP) If $A, B \in K$, then there is $C \in K$ such that both A and B are embeddable in C.*
- *(Amalgamation property AP) If $A, B, C \in K$ and $f\colon A \to B$, $g\colon A \to C$ are embeddings, then there are $D \in K$ and embeddings $h\colon B \to D$ and $i\colon C \to D$ with $hf = ig$.*

Lemma 2.22. *The class of finite rational metric spaces has the HP, JEP, and AP.*

Proof. The HP and JEP are clear. Suppose that A, B, C are finite rational metric spaces and $f\colon A \to B$, $g\colon A \to C$ are isometric embeddings. We can assume that $A = B \cap C$ and $f = g = id \restriction A$. Extend d to a metric $\hat{d}$ on $D := B \cup C$ by letting $\hat{d}(b, c) = \inf_{a \in A}(d(a, b) + d(a, c))$. Let $h = id \restriction B$ and $i = id \restriction C$. □

Lemma 2.23. *(Fraissé) For every class K of finitely generated structures with only countably many isomorphism types and with the HP, JEP, and AP, there is a unique (up to isomorphism) countable structure U such that K is the class of finitely generated substructures of U (then K is called the* age *of U) and U is ultrahomogeneous.*

We go through Fraissé's construction for the class of finite rational metric spaces. We construct finite metric spaces $\emptyset = D_0 \subseteq D_1 \subseteq ...$ such that

- if $A, B \in K$ with $A \subseteq B$, and there is an isometric embedding $f\colon A \to D_i$ for some $i \in \omega$, then there are $j > i$ and an isometric embedding $g\colon B \to D_j$ which extends f.

Take a bijection $\pi\colon \omega \times \omega \to \omega$ such that $\pi(i, j) \geq i$ for all i, j.

When D_i is defined, list as $(f_{kj}, A_{kj}, B_{kj})_{j \in \omega}$ the triples (f, A, B) (up to isomorphism) where $A \subseteq B$ and $f\colon A \to D_k$ is an isometry. Construct D_{k+1} by the amalgamation property so that if $k = \pi(i, j)$, then f_{ij} extends to an embedding of B_{ij} into D_{k+1}.

Let $U = \bigcup_i D_i$ (the rational Urysohn space). Then U has the extension property.

Definition 2.24. *(Urysohn space) Let $\mathbb{U}$ denote the completion of U.*

Lemma 2.25. *For $F \subseteq \mathbb{U}$ finite, $f\colon F \to \mathbb{R}$ Katetov, and $\epsilon > 0$, there is $y \in U$ with*

$$|d(x, y) - f(x)| < \epsilon$$

for all $x \in F$.

Proof. Suppose that $F = \{x_0, ..., x_n\}$ and $f(x_0) \geq f(x_1) \geq ... \geq f(x_n) > 0$. We can assume that $\epsilon \leq \min\{d(x_i, x_j) \mid i < j \leq n\}$.

Let $\epsilon_0 = \frac{\epsilon}{4(n+1)}$. Find $y_0, ..., y_n \in U$ with $d(x_i, y_i) < \epsilon_0$ for $I \leq n$. Let $G = \{y_0, ..., y_n\}$. To approximate f by an admissible function $g \colon G \to \mathbb{Q}$, we increase the values of f and decrease the distances between the values.

Find $g(y_i) \in \mathbb{Q}$ with $|f(x_i) + (4i+2)\epsilon_0 - g(y_i)| \leq \epsilon_0$. Then

$$\begin{aligned} d(y_i, y_j) &\leq d(x_i, y_i) + d(x_j, y_j) + f(x_i) + f(x_j) \\ &\leq 2\epsilon_0 + f(x_i) + f(x_j) \leq g(y_i) + g(y_j) \end{aligned}$$

For $i < j \leq n$

$$\begin{aligned} g(y_i) - g(y_j) &\leq f(x_i) - f(x_j) - 4(j-i)\epsilon_0 + 2\epsilon_0 \\ &\leq d(x_i, x_j) - 2\epsilon_0 \leq d(y_i, y_j) \end{aligned}$$

and

$$\begin{aligned} g(y_j) - g(y_i) &\leq f(x_j) - f(x_i) + 4(j-i)\epsilon_0 + 2\epsilon_0 \\ &\leq (4n+1)\epsilon_0 = \epsilon - 2\epsilon + 0 \\ &\leq d(x_i, x_j) - d(x_i, y_i) - d(x_j, y_j) \leq d(y < i, y_j). \end{aligned}$$

Find $y \in U$ realizing g by the extension property. Then

$$\begin{aligned} |d(x_i, y) - f(x_i)| &= |d(y_i, y) - f(x_i)| + d(x, y_i) \\ &= |g(y_i) - f(x_i)| + d(x, y_i) \leq (4n+3)\epsilon_0 + \epsilon_0 = \epsilon \end{aligned}$$

□

Lemma 2.26. $\mathbb{U}$ *has the extension property.*

Proof. Suppose that $F = \{x_0, ..., x_m\} \subseteq \mathbb{U}$. Suppose that $f \colon F \to \mathbb{R}$ is admissible and $f(x_0) \geq f(x_1) \geq ... \geq f(x_n) = \epsilon > 0$. We define a sequence (y_n) in U with

$$|d(x_i, y_n) - f(x_i)| \leq 2^{-n}\epsilon$$

for $i \leq m$. Find y_0 using that f is admissible. If y_n is defined, extend f to g by letting $g(y_n) = 2^{-n}\epsilon$. Now g is admissible by the inductive assumption on y_n. Find y_{n+1} realizing g up to $2^{-(n+1)}\epsilon$. Then $|d(y_n, y_{n+1}) - 2^{-n}\epsilon| \leq 2^{-(n+1)}\epsilon$, so $d(y_n, y_{n+1}) < 2^{-n+1}\epsilon$. Let $y = \lim_n y_n$. Then $d(x_i, y) = f(x_i)$ for $i \leq n$. □

A Polish metric space is *universal* if it contains an isometric copy of any other Polish metric space.

Exercise 2.27. *Show that any ultrahomogeneous universal Polish metric space has the extension property and hence is isometric to* $\mathbb{U}$.

For studying isometry groups we need a variant of the Fraissé construction, the *Katetov construction*, which begins from any Polish metric space (X, d) instead of the empty set.

Definition 2.28. *We say that a Katetov function $f\colon X \to \mathbb{R}$ has* finite support *if for some finite $S \subseteq X$, $f(x) = \inf\{f(y) + d(x, y) : y \in S\}$ for all $x \in X$. Let $E(X)$ denote the set of finitely supported Katetov functions on X and d_E the function defined by*

$$d_E(f, g) = \sup\{|f(x) - g(x)| : x \in X\}.$$

Exercise 2.29. *d_E is a metric on $E(X)$.*

Lemma 2.30. *$(E(X), d_E)$ is separable.*

Proof. Suppose that $D \subseteq X$ is dense and countable. It follows from the proof of Lemma 2.25 that $E(D)$ is dense in $E(X)$. □

We go through the Katetov contruction. Let $X_0 = X$ and $X_{n+1} = E(X_n)$ for $n \geq 1$. As in the Fraissé construction, $\bigcup_{n\in\omega} X_n$ has the extension property. Then its completion has the extension property by Lemma 2.26.

Example 2.31. *When we consider $\mathbb{R}$ as a subset of $\mathbb{R}^2$, the Katetov construction over $\mathbb{R}$ adds no element of $\mathbb{R}^2 \setminus \mathbb{R}$ to $\mathbb{R}$.*

2.3. **Ultrametric spaces.**

Definition 2.32. *An* ultrametric space *is a metric space (X, d) which satisfies the ultrametric inequality*

$$d(x, z) \leq \max\{d(x, y), d(y, z)\}$$

for all $x, y \in X$.

Example 2.33. *The Baire space ${}^{\omega}\omega$ with the ultrametric $d\colon {}^{\omega}\omega \times {}^{\omega}\omega \to \mathbb{R}$, $d(x, y) = 2^{-\min\{i\in\omega | x(i)\neq y(i)\}}$ for $x \neq y$ in ${}^{\omega}\omega$.*

Example 2.34. *The Cantor space ${}^{\omega}2$ with the ultrametric inherited from the Baire space.*

Example 2.35. *$Sym(\omega)$ with the ultrametric $d\colon Sym(\omega) \times Sym(\omega) \to \mathbb{R}$,*

$$d(x, y) = 2^{-\max\{\Delta(x,y), \Delta(x^{-1}, y^{-1})\}}$$

and $\Delta(x, y) = \min\{i \in \omega \mid x(i) \neq y(i)\}$ for $x \neq y$ in $Sym(\omega)$.

Example 2.36. *The* space of models *or* logic space *for a countable relational language. Suppose* $\mathcal{L} = \{R_i \mid i \in \omega\}$ *and* $R_i \subseteq \omega^{n_i}$*. Let*

$$Mod(\mathcal{L}) = \prod_{i \in \omega} 2^{(\omega^{n_i})}$$

where each $M \in Mod(\mathcal{L})$ *codes a sequence of characteristic functions of relations on* ω *and hence a model of* $\mathcal{L}$*.*

Note that we can define the logic space for any countable language by representing functions and constants by relations.

Lemma 2.37. *The set of distences* $R(d)$ *for a complete ultrametric* d *on a separable space is countable.*

Proof. Suppose (X, d) is ultrametric Polish. Find $D \subseteq X$ countable dense. Let $R = \{d(x, y) \mid x, y \in D\}$. We show that $R = R(d)$. Suppose that $x \neq y$ in X. Suppose that $x', y' \in X$ with $d(x, x'), d(y, y') < \frac{d(x,y)}{3}$. Then $d(x, y) = d(x', y') \in R$. □

Suppose that $R \subseteq \mathbb{R}$ is countable. We can construct an ultrametric Urysohn space $\mathbb{U}_R^{ult}$ similar as the Urysohn space $\mathbb{U}$ by a Fraisse construction or a Katetov construction.[1]

Exercise 2.38. *Suppose* d *is the standard ultrametric on the Baire space.Then* $({}^{\omega}\omega, d)$ *has the* $R(d)$*-Urysohn property.*

We now prove that every Polish space is a bijective continuous image of a closed subset of the Baire space.

Lemma 2.39. *Suppose that* (X, d) *is Polish,* $U \subseteq X$ *is open, and* $\epsilon > 0$*. Then there is a sequence* $(U_n)_{n \in \omega}$ *of open sets such that* $U = \bigcup_{n \in \omega} U_n = \bigcup_{n \in \omega} \overline{U_n}$ *and* $diam(U_n) < \epsilon$ *for all* n*.*

Proof. See Marker: Lecture notes on descriptive set theory, Lemma 1.6. □

Lemma 2.40. *Suppose that* (X, d) *is Polish,* $F \subseteq X$ *is* F_σ*, and* $\epsilon > 0$*. Then there is a sequence* $(F_n)_{n \in \omega}$ *of* F_σ *sets such that* $F = \bigcup_{n \in \omega} F_n = \bigcup_{n \in \omega} \overline{F_n}$ *and* $diam(F_n) < \epsilon$ *for all* n*.*

Proof. See Marker: Lecture notes on descriptive set theory, Lemma 1.18. □

Lemma 2.41. *For every Polish space* (X, d)*, there is a closed set* $C \subseteq {}^{\omega}\omega$ *and a continuous bijection* $f \colon C \to X$*.*

[1] See Gao-Shao: Polish ultrametric Urysohn spaces and their isometry groups

Proof. See Marker: Lecture notes on descriptive set theory, Lemma 1.19. □

It is easy to see that every closed subset of the Baire space is a continuous image of the Baire space (from a retraction). Hence every Polish space is a continuous image of the Baire space. This also follows from a modification of the construction above.

2.4. **Hyperspaces.** To represent closed subsets of a Polish space X as elements of another Polish space, we need to define a metric measuring the distance between closed sets. Let us first consider compact sets.

Example 2.42. *(Hausdorff metric) Suppose that (X,d) is Polish. Let $K(X) = \{K \subseteq X \mid K$ is compact $\}$. Let*

$$d_H(K,L) := \begin{cases} \max\{\max\{d(x,L) \mid x \in K\}, \max\{d(K,y) \mid y \in L\}\} & \text{if } K, L \neq \emptyset \\ 0 & \text{if } K = L = \emptyset \\ 1 & \text{if exactly one of} K, L \text{is } \emptyset. \end{cases}$$

Lemma 2.43. *$(K(X), d_H)$ is Polish.*

Proof. It is easy to see that d_H is a metric. To show that $(K(X), d_H)$ is separable, suppose that $D \subseteq X$ is countable dense. Then $\{K \subseteq D \mid K \text{ is finite}\}$ is dense in $K(X)$ (use that every compact subset of a metric space is totally bounded).

To show that $(K(X), d_H)$ is complete, suppose that (K_n) is a Cauchy sequence. Let $K = \{\lim_n x_n \mid x_n \in K_n,\ (x_n) \text{ is Cauchy}\}$. It is straightforward to check that $K = \lim_n K_n$. □

Borel subsets of a topological space are the elements of the σ-algebra generated by the open sets.

Remark 2.44. *Suppose that (X,d) is Polish. Let $F(X) = \{C \subseteq X \mid C$ is closed$\}$. Let*

$$E_U = \{C \in F(X) \mid C \cap U \neq \emptyset\}$$

$$E^U = \{C \in F(X) \mid C \subseteq U\}$$

where $U \subseteq X$ is open.

(1) *The family of all sets E_U and E^U form a subbase for $(K(X), d_H)$. For example, to show that $U_\epsilon(L) := \{K \mid \max\{d(x,L) \mid x \in K\} < \epsilon\}$ is a union of sets of the form E_V, cover L with a finite union of open balls of radius δ for any given $\delta < \epsilon$ using compactness. This topology on $F(X)$ is called the* Vietoris topology.

(2) *The* Effros Borel structure *on $F(X)$ is defined as the σ-algebra generated by the sets E_U (equivalently by the sets E^U).*

Definition 2.45. *(Gromov-Hausdorff metric) Suppose that K, L are compact metric spaces. Let*

$$d_{GH}(K,L) = \inf\{d_H^X(f(K), g(L) \mid f\colon K \to X,\ g\colon L \to X \text{ are isometric embeddings }\}.$$

Fact 2.46. *(Gromov)*[2] *Let $\mathcal{C}$ denote the set of isometry classes of compact metric spaces. Then $(\mathcal{C}, d_{GH})$ is a compact metric space.*

To define a Polish topology on $F(X)$, suppose that (U_n) is a base for X. Suppose that Y is a metric compactification of X (i.e. Y is a compact metric space and X is dense in Y) (see 2.5). We consider the topology τ on $F(X)$ induced by $(K(Y), d_H)$ via the pullback of $C \mapsto \overline{C}$. Note that this might depend on the choice of the compactification.

Lemma 2.47. *$(F(X), \tau)$ is Polish.*

Proof. Let $A = \{\overline{C} \mid C \in F(X)\} \subseteq K(Y)$. Then for all $C \in K(Y)$, $C \in A$ if and only if for all n

$$C \cap U_n \neq \emptyset \Rightarrow C \cap U_n \cap X \neq \emptyset.$$

Then A is G_δ in $K(Y)$ and hence Polish. Moreover $(F(X), \tau)$ is homeomorphic to (A, τ_H), where τ_H is the topology induced by d_H. □

Remark 2.48. *It is easy to see that τ generates the Effros Borel structure on $F(X)$. Every set $(E^U)^X$ is of the form $(E^U)^Y$, every set $(E^U)^Y$ is a countable intersection of sets of the form $(E^U)^X$, and $(E^U)^X = (E^U)^Y$.*

The Wijsman topology on $F(X)$ is the least topology which makes all maps $C \mapsto d(x, C)$ for $x \in X$ continuous. Like τ it is also Polish and generates the Effros Borel structure, and moreover it only depends on the metric on X.

3. Borel and analytic sets

Suppose that $X = (X, d)$ is a Polish space.

Definition 3.1. *(perfect) A set $A \subseteq X$ is* perfect *if it is uncountable, closed, and has no isolated points.*

Lemma 3.2. *Suppose that (X, d) is a perfect Polish space. Then there is a continuous injection $f\colon {}^\omega 2 \to X$.*

Proof. We construct a family $(U_s)_{s \in {}^{<\omega}2}$ of open nonempty subsets of X such that for all $s \in {}^{<\omega}2$ and $i < 2$

[2] See Heinonen: Geometric embeddings of metric spaces, section 2: Gromov-Hausdorff convergence

(1) $\overline{U_{s^\frown i}} \subseteq U_s$ and
(2) $U_{s^\frown i} \cap U_{s^\frown (1-i)} = \emptyset$.

For each $x \in {}^\omega 2$, let $f(x)$ denote the unique element of $\bigcap_{n\in\omega} U_{x\upharpoonright n}$. Then $f\colon {}^\omega 2 \to X$ is continuous and injective. □

Lemma 3.3. *Suppose that $U \subseteq X$ is open and that $\epsilon > 0$. There is a sequence $(U_n)_{n\in\omega}$ of open sets with $U = \bigcup_{n\in\omega} U_n = \bigcup_{n\in\omega} \overline{U_n}$ and $diam(U_n) < \epsilon$ for all n.*

Proof. Suppose that $D \subseteq X$ is countable and dense. Let $(U_i)_{i\in\omega}$ list all $B(x, \frac{1}{n})$ with $x \in D$, $\frac{1}{n} < \frac{\epsilon}{2}$, and $\overline{U_n} \subseteq U$. We claim that $U = \bigcup_{i\in\omega} U_i$. Suppose that $x \in U$. Find n with $B(x, \frac{2}{n}) \subseteq U$. Find $y \in D \cap B(x, \frac{1}{n})$. Then $x \in B(y, \frac{1}{n})$ and $\overline{B(y, \frac{1}{n})} \subseteq U$, so $B(y, \frac{1}{n}) = U_i$ for some i. □

Lemma 3.4. *Suppose that $F \subseteq X$ is F_σ and $\epsilon > 0$. There is a sequence $(F_n)_{n\in\omega}$ of pairwise disjoint F_σ sets with $F = \bigcup_{n\in\omega} F_n = \bigcup_{n\in\omega} \overline{F_n}$ and $diam(F_n) < \epsilon$ for all n.*

Proof. Let $F = \bigcup_{h\in\omega} C_n$ with $C_0 \subseteq C_1 \subseteq ...$ closed. Then $F = C_0 \sqcup (C_1 \setminus C_0) \sqcup ...$ and $\overline{C_{n+1} \setminus C_n} \subseteq C_{n+1} \subseteq F$. Hence it is sufficient to show that for $A \subseteq X$ closed and $B \subseteq X$ open, there is a sequence $F_n)_{n\in\omega}$ of pairwise disjoint F_σ sets with $A \cap B = \bigcup_{n\in\omega} F_n$ and $diam(F_n) < \epsilon$ for all n.

Write $B = \bigcup_{n\in\omega} B_n = \bigcup_{n\in\omega} \overline{B_n}$ as in the previous lemma. Then $F_n = A \cap (B_n \setminus \bigcup_{i<n} B_i)$ works. □

Lemma 3.5. *There is a continuous bijection $f\colon C \to X$ for some closed $C \subseteq {}^{]\omega}\omega$.*

Proof. We construct level by level a tree $T \subseteq {}^{<\omega}\omega$ and a family $(X_t)_{t\in T}$ of nonempty F_σ sets such that $X_0 = X$ and for all $s \in T \cap {}^n\omega$

- $X_s = \bigsqcup_{s^\frown i \in T} X_{s^\frown i}$,
- $\overline{X_{s^\frown i}} \subseteq X_s$, and
- $diam(X_s) < \frac{1}{2^n}$.

We write $[T] = \{x \in {}^\omega\omega \mid \forall n\ x \upharpoonright n \in T\}$. For each $x \in [T]$, let $f(x)$ denote the unique element of $\bigcap_{n\in\omega} X_{x\upharpoonright n}$. Then f is continuous by the last condition. For each n, there is a unique $s \in {}^n 2$ with $x \in X_s$ by the first condition, and hence f is surjective. □

Corollary 3.6. *There is a continuous surjection $f\colon {}^\omega\omega \to X$.*

Proof. It is easy to see that there is a continuous retraction $r\colon {}^\omega\omega \to [T]$, i.e. such that $r \upharpoonright [T] = id \upharpoonright [T]$. □

Definition 3.7. *Suppose that $A \subseteq X$ and $x \in X$. The x is a* condensation point *of A if $A \cap U$ is uncountable for every open $U \subseteq X$ with $x \in X$.*

Lemma 3.8. *Suppose that $A \subseteq X$ is uncountable and $C \subseteq X$ is the set of condensation points of A. Then $A \cap C$ is uncountable.*

Proof. It is easy to see that C is closed. For each $x \in X \setminus C$, there is a basic open set $U \subseteq X \setminus C$ with $x \in U$ such that $A \cap U$ is countable. Hence $A \setminus C$ is countable. □

Lemma 3.9. *Suppose that $A \subseteq X$ is uncountable. Then there are disjoint open sets $U, V \neq \emptyset$ such that $A \cap U$ and $A \cap V$ are uncountable.*

Proof. Find condensation points $x \neq y$ of A and disjoint open neighborhoods of x and y. □

Lemma 3.10. *Suppose that $f\colon X \to Y$ is continuous and $A = f[X]$ si uncountable. Then there are open set $U, V \subseteq X$ such that $f[U]$ and $f[V]$ are disjoint and uncountable.*

Proof. Find disjoint open sets $U, V \subseteq Y$ such that $A \cap U$ and $A \cap V$ are uncountable. Consider $f^{-1}[U]$ and $f^{-1}[V]$. □

Proposition 3.11. *Every uncountable analytic set has a perfect subset.*

Proof. Suppose that $f\colon {}^\omega\omega \to X$ is continuous and $A = f[{}^\omega\omega]$. We construct by the previous lemma $(t_s)_{s \in {}^{<\omega}2}$ in ${}^{<\omega}\omega$ such that for all $s \in {}^n2$ and $i < 2$

- $t_s \subsetneq t_{s^\frown i}$,
- $f[U_{t_s}]$ is uncountable, and
- $f[U_{t_{s^\frown 0}} \cap U_{t_{s^\frown 1}}] = \emptyset$ (so in particular $t_{s^\frown 0} \perp t_{s^\frown 1}$).

Let $g\colon {}^\omega 2 \to {}^\omega\omega$, where $g(x)$ is the unique element of $\bigcap_{n\in\omega} U_{t_{x \upharpoonright n}}$. Then g is a continuous injection. Moreover fg is injective by the last requirement. Since ${}^\omega 2$ is compact, $fg[{}^\omega 2]$ is closed in X. Since fg is injective, $fg[{}^\omega 2]$ is perfect. □

Lemma 3.12. *Suppose that $Y \subseteq X$ is Borel and $f\colon X \to Y$ is an injection such that the preimages and images of Borel sets are Borel. Then there is a Borel isomorphism between X and Y.*

Proof. Let $A_0 = X$, $A_{n+1} = f[A_n]$, $B_0 = Y$, $B_{n+1} = f[B_n]$. Then $f[A_n \setminus B_n] = A_{n+1} \setminus B_{n+1}$. The sets $A_n \setminus B_n$ are disjoint since $f[A_n] \subseteq B_n$. Let $A = \bigcup_{n\in\omega}(A_n \setminus B_n)$. Let $g\colon X \to Y$, where

$$g(x) = \begin{cases} f(x) & \text{if } x \in A \\ x & \text{otherwise.} \end{cases}$$

Then $g \upharpoonright A = f \upharpoonright A\colon A \to A$ is a Borel isomorphism and $g \upharpoonright (X \setminus A) = id \upharpoonright (X \setminus A)$ is a Borel isomorphism. □

Proposition 3.13. *All uncountable Polish spaces are Borel isomorphic.*

Proof. Suppose that X is an uncountabl Polish space. We construct a continuous injection $f\colon {}^{\omega}1 \to X$. Then f maps closed sets to closed sets. and hence Borel sets to Borel sets.

We have constructed a continuous bijection $g\colon C \to X$, where $C \subseteq {}^{\omega}\omega$ is closed. Note that the specific function which we contructed earlier maps basic open sets to F_σ sets and hence Borel sets to Borel sets.

To apply the previous lemma, it is sufficient that there is an injection ${}^{\omega}\omega \to {}^{\omega}2$ which maps Borel sets to Borel sets. Let $h\colon {}^{\omega}\omega \to {}^{\omega}2$, where $h(x) = 0^{x(0)}10^{x(1)}10^{x(2)}1\ldots$. □

Stefan proved the following lemmas.

Lemma 3.14. *Suppose that $A_n \subset X$ is Borel for each $n \in \omega$. Then there is a finer Polish topology with the same Borel sets in which each A_n is both closed and open.*

Lemma 3.15. *Suppose that A_0 and A_1 are disjoint analytic subsets of X. Then there are disjoint Borel subsets of X with $A_i \subseteq B_i$ for $i < 2$.*

Lemma 3.16. *Suppose that $f\colon X \to Y$ is a function and $A = \{(x, y) \in X \times Y \mid f(x) = y\}$ its graph. Then the following are equivalent:*

(1) f is Borel measurable.
(2) A is Borel.
(3) A is analytic.

Proof. To prove (2) from (1), find a base $(U_n)_{n\in\omega}$ for Y. Then $(x, y) \in A$ if for all n, if $x \in f^{-1}[U_n]$, then $y \in U_n$.

To prove (1) from (3), suppose that $B \subseteq Y$ is Borel. Then $x \in f^{-1}[B]$ if and only if $\exists y \in B\ (x, y) \in A$ if and only if $\forall y \in Y\ ((x, y) \in A \Rightarrow y \in B)$. □

Lemma 3.17. *suppose that $(A_n)_{n\in\omega}$ is a sequence of analytic subsets of X. Then there is a sequence $(B_n)_{n\in\omega}$ of pairwise disjoint Borel subsets of X with $A_n \subseteq B_n$ for all n. So $f^{-1}[B]$ is analytic and coanalytic, so it is Borel.*

Proof. Find B_n Borel with $A_n \subseteq B_n$ and $B_n \cap A_i = \emptyset$ for all $i \neq n$ by separation of analytic sets. Let $C_n = B_n \setminus \bigcup_{i\neq n} B_i$. □

Suppose that $X = (X, d_X)$ and $Y = (Y, d_Y)$ are Polish spaces.

Theorem 3.18 (Lusin-Suslin)**.** *If $f\colon X \to Y$ is Borel, $A \subseteq X$ is Borel, and $f \restriction A$ is injective, then $f[A]$ is Borel.*

Proof. Let us first argue that it is sufficient to prove this for $X = {}^\omega\omega$, closed sets $A \subseteq X$, and continuous f. We refine the topology on X to a Polish topology with the same Borel sets such that f is continuous and A is closed. Then there is a closed $C \subseteq {}^\omega\omega$ and a continuous bijection $g\colon C \to A$. There is a continuous $h\colon {}^\omega\omega$ with $h \upharpoonright C = id$. Now consider $fgh\colon {}^\omega\omega \to Y$ and $C \subseteq {}^\omega\omega$ instead of A.

Now suppose that $f\colon {}^\omega\omega \to Y$ is continuous, $A \subseteq X$ is closed, and $f \upharpoonright A$ is injective. Let $A_s = f[A \cap N_s]$ for $s \in {}^{<\omega}\omega$. Then each A_s is analytic. Let $B_\emptyset = Y$. For $n > 0$ find disjoint Borel sets $(B_s)_{s \in {}^N\omega}$ with $A_s \subseteq B_s \subseteq \overline{A_s}$ for $s \in {}^n\omega$ by the version of the separation theorem in the previous lemma. We can assume that $B_s \subseteq \overline{A_s}$ by intersection B_s with $\overline{A_s}$ if necessary. Let $C_t = \bigcap_{s \subseteq t} B_s$ for all $t \in {}^{<\omega}\omega$. Then $A_t \subseteq C_t$.

We claim that $f[A] = \bigcap_{n\in\omega} \bigcup_{s \in {}^n\omega} C_s$. If $y \in f[A]$, find $x \in A$ with $f(x) = y$. Then $y \in f[A \cap N_{x\upharpoonright n} \subseteq C_{x \upharpoonright n}$ for all $n \in \omega$.

If $y \in \bigcap_{n\in\omega} \bigcup_{s\in {}^n\omega} C_s$, then there is $x \in {}^\omega\omega$ with $y \in \bigcap_{n\in\omega} C_{x\upharpoonright n}$ by the choice of C_s. Then $y \in \bigcap_{n\in\omega} \overline{A_{x\upharpoonright n}}$. So $A \cap N_{x\upharpoonright n} \neq \emptyset$ for all n. Since A is closed, this implies that $x \in A$. We claim that $f(x) = y$. If $f(x) \neq y$, find an open neighborhood U of $f(x)$ with $y \neq \overline{U}$. Find n with $f[N_{x\upharpoonright n}] \subseteq U$ by the continuity of f. This contradicts the assumption that $y \in C_{x\upharpoonright n} \subseteq \overline{A_{x\upharpoonright n}}$. □

Exercise 3.19. *Read about Lusin schemes in section 7C of Kechris' book.*

We would now like to show that analytic and coanalytic sets have the Baire property and are universally measurable, i.e. measurable with respect to every σ-finite Borel measure on X. A measure μ is σ-finite if there are Borel sets $X_n \subseteq X$ for $n \in \omega$ with $X = \bigcup_{n\in\omega} X_n$ and $\mu(X_n) < \infty$ for all n.

Definition 3.20. *Suppose that $\mathcal{S}$ is a σ-algebra on a set X and $A, \hat{A} \subseteq X$ with $\hat{A} \in \mathcal{S}$. Then $\hat{A}$ is an* $\mathcal{S}$-cover *of A if*

(1) $A \subseteq \hat{A}$ and
(2) if $A \subseteq B \in \mathcal{S}$, then every subset of $\hat{A} \setminus B$ is in $\mathcal{S}$.

We say that $\mathcal{S}$ admits covers *if every $A \subseteq X$ has an $\mathcal{S}$-cover.*

Lemma 3.21. *Suppose that X is a Polish space and $\mathcal{S}$ is the set of all $A \subseteq X$ with the Baire property. Then $\mathcal{S}$ admits covers.*

Proof. Suppose that $A \subseteq X$ and let $U = \bigcup\{V \subseteq X \mid V$ is basic open, $(X \setminus A) \cap V$ is comeager in $V\}$. Let $C = X \setminus U$. Then $A \setminus C = A \cap U$ is meager by the choice of U. Find a meager F_σ set $D \subseteq X$ with $A \setminus C \subseteq D$ and let $\hat{A} = C \cup D$.

Suppose that $B \supseteq A$ has the Baire property and that $\hat{A} \setminus B$ is nonmeager. Find a basic open set V such that $(\hat{A} \setminus B) \cap V$ is comeager in V. Then $V \subseteq U$. So $D \cap V$ is comeager in V, contradicting the choice of D. □

Lemma 3.22. *Suppose that X is a Polish space and μ is a σ-finite Borel measure on X. Suppose that $\mathcal{S}$ is the set of all μ-measurable $A \subseteq X$ (i.e. such that the inner and outer measures $\mu_*(A)$ and $\mu^*(A)$ coincide). Then $\mathcal{S}$ admits covers.*

Proof. We can assume that $\mu(X) < \infty$. Let $\mu^*(A) = \inf\{\mu(B) \mid B \subseteq X \text{ Bore, } A \subseteq B\}$ denote the outer measure of $A \subseteq X$. Then there is a Borel set $\hat{A} \subseteq X$ with $A \subseteq \hat{A}$ and $\mu(\hat{A}) = \mu^*(A)$. Suppose that $B \supseteq A$ is μ-measurable and $\mu(\hat{A} \setminus B) > 0$. Then $A \subseteq \hat{A} \cap B$ and $\mu(\hat{A} \cap B) < \mu(\hat{A})$, contradicting the choice of $\hat{A}$. □

Definition 3.23. *(Suslin operation) Suppose that $(C_s)_{s \in {}^{<\omega}\omega}$ is a family of subsets of a set X with $C_s \supseteq C_t$ for all $s \subseteq t$. Let*

$$\mathcal{A}(C_s) = \bigcup_{x \in {}^{\omega}\omega} \bigcap_{n \in \omega} C_{x \upharpoonright n}.$$

Lemma 3.24. *Suppose that $A \subseteq X$ is analyitic. Then there is a family $(C_s)_{s \in {}^{<\omega}\omega}$ of closed subsets of X such that $C_s \supseteq C_t$ for all $s \subseteq t$ and $A = \mathcal{A}(C_s)$.*

Proof. Suppose that $A \neq \emptyset$ and that $f \colon {}^{\omega}\omega \to X$ is continuous with $f[{}^{\omega}\omega] = A$. Let $C_s = \overline{f[N_s]}$ for $s \in {}^{<\omega}\omega$. Then $A \subseteq \mathcal{A}(C_s)$.

To show that $\mathcal{A}(C_s) \subseteq A$, suppose that $y \in \bigcap_{n \in \omega} C_{x \upharpoonright n} = \bigcap_{n \in \omega} \overline{f[N_{x \upharpoonright n}]}$. Find $x_n \in N_{x \upharpoonright n}$ with $d(f(x_n), y) < \frac{1}{2^n}$. Then $\lim_n x_n = x$. So $f(x) = y$ and hence $y \in f[U]$. □

Lemma 3.25. *Suppose that $\mathcal{S}$ is a σ-algebra which admits covers. Then $\mathcal{S}$ is closed under the Suslin operation.*

Proof. Suppose that $A = \mathcal{A}(C_s)$ with $C_s \in \mathcal{S}$ for all $s \in {}^{<\omega}\omega$ and $C_s \supseteq C_t$ if $s \subseteq t$. Let

$$C^s = \bigcup_{s \subseteq x} \bigcap_{n \in \omega} C_{x \upharpoonright n} \subseteq C_s$$

for each $s \in {}^{<\omega}\omega$. Then $A = C^{\emptyset}$. Find an $\mathcal{S}$-cover $\hat{C}^s$ for C^s with $\hat{C}^s \subseteq C_s$. Let

$$D_s = \hat{C}^s \setminus \bigcup_{n \in \omega} \hat{C}^{s^\frown n}.$$

Since $C^s = \bigcup_{n \in \omega} C^{s^\frown n} \subseteq \bigcup_{n \in \omega} \hat{C}^{s^\frown n}$, every subset of D_s is in $\mathcal{S}$ and hence every subset of $D := \bigcup_{s \in {}^{<\omega}\omega} D_s$ is in $\mathcal{S}$.

We claim that $\hat{A} \setminus A = \hat{C}^{\emptyset} \setminus C^{\emptyset} \subseteq D$ and hence $A \in \mathcal{S}$. Suppose that $y \in (\hat{A} \setminus A) \setminus D$. For all $s \in {}^{<\omega}\omega$, if $y \in \hat{C}^s \setminus D$, then $y \in \hat{C}^{s^\frown n}$ for some n. Find $x \in {}^{\omega}\omega$ with $y \in \hat{C}^{x \upharpoonright n} \subseteq C_{x \upharpoonright n}$ for all $n \in \omega$. Then $y \in \bigcap_{n \in \omega} C_{x \upharpoonright n} \subseteq A$, contradicting the choice of y. □

The previous lemmas imply

Proposition 3.26. *All analytic and coanalytic subsets of Polish spaces have the Baire property and are universally measurable.*

4. The G_0 dichotomy

We present Ben Miller's proof of the G_0 dichotomy and some consequences.

As usual, X and Y denote Polish spaces, unless stated otherwise. Suppose that G is a graph on X. We say that $A \subseteq X$ is G-discrete if there are no $x \neq y$ in A with $(x,y) \in G$. A κ-coloring of G is a map $c\colon X \to \kappa$ such that $c^{-1}[\{\alpha\}]$. A homomorphism from a graph G on X to a graph H on Y is a map $f\colon X \to Y$ such that $(x,y) \in G$ implies $(f(x), f(y)) \in H$.

Supose that $I \subseteq 2^{<\omega}$ is dense, i.e. for every $s \in 2^{<\omega}$ there is some $t \in I$ with $s \subseteq t$. Let G_I denote the graph consisting of all pairs $(s^\frown i^\frown x, s^\frown(1-i)^\frown x)$, where $s \in I$, $i < 2$, and $x \in {}^\omega 2$. We fix a sequence $(s_n)_{n\in\omega}$ with $s_n \in 2^n$ such that $I = \{s_n \mid n \in \omega\}$ is dense and let $G_0 = G_I$.

Lemma 4.1. *(Ben Miller) Suppose that $A \subseteq {}^\omega 2$ is nonmeager and has the Baire property. Then A is not G_0-discrete.*

Theorem 4.2. *(Kechris-Solecki-Todorcevic) Suppose that X is a Hausdorff space and G is an analytic graph on X. Then either*

(1) there is a Borel ω-coloring of G or

(2) there is a continuous homomorphism from G_0 to G.

The space ${}^\omega\kappa$ is the product of κ with the discrete topology.

Definition 4.3. *Suppose that X is a topological space. A set $A \subseteq X$ is* κ-Suslin *if $A = f[{}^\omega\omega]$ for some continuous map $f\colon {}^\omega\kappa \to X$ or $A = \emptyset$. A set $A \subseteq X$ is* co-κ-Suslin *if $X \setminus A$ is κ-Suslin.*

The ω-Suslin subsets of Polish spaces are the analytic sets. The G_0 dichotomy and many if its consequences can be generalized to κ-Suslin graphs on Hausdorff spaces, which we don't pursue here.

Lemma 4.4 (Mycielski)**.** *Suppose that E is a meager binary relation on X. Then there is a perfect $C \subseteq X$ with $(x,y) \notin E$ for all $x \neq y$ in X.*

Proof. Suppose that $U_n \subseteq X$ is open dense for each$n \in \omega$ and $E \cap \bigcap_{n\in\omega} U_n = \emptyset$. We can assume that $U_0 \supseteq U_1 \supseteq ....$ We construct $(U_s)_{s\in 2^{<\omega}}$ with $U_s \subseteq X$ nonempty open such that for all $s^\frown i \neq t^\frown j$ in 2^{n+1}

(1) $U_{s^\frown i} \cap U_{t^\frown j} = \emptyset$,

(2) $diam(U_{s^\frown i}) < 2^{-n}$, and

(3) $U_{s^\frown i} \times U_{t^\frown j} \subseteq U_n$.

Let $U_\emptyset = X$. If U_n is defined for all $s \in {}^n2$, first find $(U_{s^\frown i})_{s \in 2^n,\ i<2}$ satisfying (1) and (2). For (3) we successively shrink each pair $U_{s^\frown i}, U_{t^\frown j}$ with $s^\frown i \neq t^\frown j$ in 2^{n+1}.

Let $f\colon {}^\omega 2 \to X$, $f(x) = \bigcap_{n\in\omega} U_{x\upharpoonright n}$. Then f is continuous and injective. Let $C = f[2^\omega]$. Then C is compact and perfect.

We claim that $(f(x), f(y)) \notin E$ for all $x \neq y$ in 2^ω. Suppose that $x(n) \neq y(n)$. Then $(f(x), f(y)) \in U_{x\upharpoonright m+1} \times U_{y\upharpoonright m+1} \subseteq U_m$ for all $m \geq n$. So $(f(x), f(y)) \notin E$. □

Lemma 4.5. *If $U \subseteq X \times Y$ is open dense, then $S := \{x \in X \mid U_x$ is dense $\}$ is comeager.*

Proof. Suppose that $W \subseteq Y$ is basic open and nonempty. We claim that $S_W := \{x \in X \mid U_X \cap W \neq \emptyset\}$ is comeager. Then $S = \bigcap_W S_W$ is comeager.

Otherwise let $T_W := \{x \in X \mid U_X \cap W = \emptyset\}$. There is a nonempty basic open set $U \subseteq X$ such that $T_W \cap U$ is comeager in U. Then $T_W \times W$ is nonmeager and $(T_W \times W) \cap U = \emptyset$, contradicting the assumption that U is comeager. □

Lemma 4.6 (Kuratowski-Ulam)**.** *A set $A \subseteq X \times Y$ with the Baire property is comeager (nonmeager) if and only if $\{x \in X \mid A_x$ is comeager (nonmeager)$\}$ is comeager (nonmeager).*

Proof. Suppose that $A \subseteq X \times Y$ is comeager, $A \subseteq \bigcap_{n\in\omega} U_n$, U_n open dense in $X \times Y$. Then $\{x \in X \mid A_x$ is comeager$\} \subseteq \bigcap_{n\in\omega}\{x \in X \mid (U_n)_x$ is comeager in $Y\}$ is comeager in X by the previous lemma.

Suppose that $A \subseteq X \times Y$ is nonmeager. Since A has the Baire property, there is a basic open set $U \times V \subseteq X \times Y$ such that $A \cap (U \times V)$ is comeager in $U \times V$. Then $\{x \in X \mid A_x \cap V$ is comeager in $V\}$ is comeager in U by the previous argument, so nonmeager.

If A is not comeager, then $(X \times Y) \setminus A$ is nonmeager, so $\{x \in X \mid A_x$ is not comeager$\} = \{x \in X \mid ((X \times Y) \setminus A)_x$ is nonmeager$\}$ is nonmeager and hence $\{x \in X \mid A_x$ is comeager$\}$ is not comeager.

If A is meager, then $(X \times Y) \setminus A$ is comeager, so $\{x \in X \mid A_x$ is meager$\} = \{x \in X \mid ((X \times Y) \setminus A)_x$ is comeager$\}$ is comeager and hence $\{x \in X \mid A_x$ is nonmeager$\}$ is meager. □

Theorem 4.7 (Silver)**.** *Suppose that X is a Hausdorff space and E is a coanalytic equivalence relation on X. Then either*

(1) E has countably many equivalence classes or

(2) there is a perfect set $C \subseteq X$ of pairwise E-inequivalent points.

Proof. The condition are mutually exclusive. Let $G = X^2 \setminus E$. Suppose that $c\colon X \to \omega$ is a coloring of G. Then $c^{-1}[\{n\}]$ is contained in some E-class for every $n \in \omega$. So E has countably many equivalence classes.

Otherwise there is a continuous homomorphism $f\colon {}^{\omega}2 \to X$ from G_0 to G by the G_0 dichotomy. Then $F = (f \times f)^{-1}[E]$ is an equivalence relation on ${}^{\omega}2$ with $G_0 \cap F = \emptyset$.

We claim that F is meager. By Kuratowski-Ulam, it is sufficient to show that every equivalence with the Baire property class is meager. Suppose that $x \in {}^{\omega}2$ and $[x]_F$ is nonmeager and has the Baire property. Then $[x]_F$ is not G_0-discrete by a previous lemma. Find $y, z \in [x]_F$ with $(y, z) \in G_0 \cap F$. There is a continuous $g\colon {}^{\omega}2 \to {}^{\omega}2$ with $(g(x), g(y)) \in F$ for all $x \neq y$ in ${}^{\omega}2$ by Mycielski's theorem. Let $h = fg$. Then $(h(x), h(y)) \notin E$ for all $x \neq y$ in ${}^{\omega}2$. So $C = h[{}^{\omega}2]$ satisfies (2). □

Theorem 4.8 (Lusin-Novikov)**.** *Suppose that X, Y are Polish spaces, $R \subseteq X \times Y$ is analytic and R_x is countable for all $x \in X$. then there are partial functions $f_n\colon X \to Y$ with relatively Borel graphs $R_n \subseteq R$ (i.e. $R_n = R \cap B_n$ for some Borel set B_n) such that $R = \bigcup_{n\in\omega} R_n$.*

Proof. Let $G \subseteq {}^{\omega}2 \times {}^{\omega}2$ where $((x_0, y_0), (x_1, y_1)) \in G$ if $x_0 = x_1$, $y_0 \neq y_1$, and $(y_0, y_1) \in R_{x_0}$.

Suppose that $f\colon {}^{\omega}2 \to X \times Y$ is a homomorphism from G_0 to G. If $x, y \in {}^{\omega}2$ and $(x, y) \in G$, then $f(x)_0 = f(y)_0$. Since the connected G_0-component of 0^∞ is dense in ${}^{\omega}2$, $f(x)_0 = f(y)_0$ for all $x, y \in {}^{\omega}2$. Let $z = f(x)_0$.

Let $f_1\colon {}^{\omega}2 \to Y$, $f_1(x) = f(x)_1$. Let $H = \{(x, y) \in Y \times Y \mid x \neq y \text{ and } (x \in R_z \text{ or } y \in R_z)\}$. Then f_1 is a homomorphism from G_0 to H. Let $I = (f_1 \times f_1)^{-1}[H]$. Then $G_0 \subseteq I$. Now each I_x is analytic and hence it has the Baire property. It is easy to check that the complement of H is an equivalence relation, so the complement of I is an equivalence relation as well. Using this and the fact the nonmeager sets with the Baire property are not G_0-discrete, if some I_x was not comeager, then there have to be $y, z \in I_x$ with $(y, z) \in G_0$, contradicting the assumption that f_1 is a homomorphism from G_0 to H. So every I_x is is comeager, and hence I is comeager by the Kuratowski-Ulam theorem.

Then there is a continuous map $g\colon {}^{\omega}2 \to {}^{\omega}2$ with $(g(x), g(y)) \in I$ for all $x \neq y$ in ${}^{\omega}2$ by Mycielski's theorem. Let $h = f_1 g$. Then $h(x), h(y) \in R_z$ and $h(x) \neq h(y)$ for all $x \neq y$ in ${}^{\omega}2$, by the definition of I as $I = (f_1 \times f_1)^{-1}[H]$ and by the definition of G and H. This contradicts the assumption that R_z is countable.

By the G_0 dichotomy, there is a Borel ω-coloring $c\colon {}^\omega 2 \times {}^\omega 2 \to \omega$ of G. Let $R_n = c^{-1}[\{n\}] \cap R$. Then $y = z$ for all $(x,y),(x,z) \in R_n$. □

If R is Borel, then R_n is Borel and hence also $dom(R_n)$ is Borel for all n, since injective images of Borel sets under continuous (and even Borel measurable) maps are Borel by a previous lemma.

The Lusin-Novikov theorem can also be shown for Hausdorff spaces X, Y with a very similar proof.

Definition 4.9 (Topological group)**.** *A topological group $G = (G, \cdot, 1)$ is a group with a topology such that $\cdot$ and $^{-1}$ are continuous. We will equip every countable group with the discrete topology, making it into a topological group.*

Definition 4.10 (Group actions)**.** *Suppose that $G = (G, \cdot, 1)$ is a group.*

(1) A Borel (continuous) action *of G on X is a Borel (continuous) map $G \times X \to X$, $(g, x) \mapsto g \cdot x$ with $(gh) \cdot x = g \cdot (h \cdot x)$ for all $g, h \in G$ and $x \in X$.*
(2) If G acts on X, let $E_G = \{(x, y) \mid \exists g \in G \ g \cdot x = y\}$.

Definition 4.11. *Let $Aut(X)$ denote the group of Borel automorphisms of X.*

Definition 4.12. *An equivalence relation E is* countable *if all its equivalence classes are countable.*

Lemma 4.13. *There is a partition $X^2 \setminus id_X = \bigsqcup_{n\in\omega} A_n \times B_n$ with A_n, B_n Borel and $A_n \cap B_n = \emptyset$.*

Proof. It is sufficient to prove this for $X = {}^\omega 2$, since all uncountable Polish spaces are Borel isomorphic. Let $(s_n)_{n\in\omega}$ enumerate ${}^{<\omega}2$ without repetitions. Let $A_n = N_{s_n^\frown 0}$ and $B_n = N_{s_n^\frown 1}$. □

Theorem 4.14 (Feldman-Moore)**.** *Suppose that E is a countable Borel equivalence relation on X. Then there is a countable group G and a Borel action of G on X with $E = E_G$.*

Proof. Let $E = \bigcup_{n\in\omega} R_n$ where each R_n is Borel and $(R_n)_x$ is countable for all $x \in X$ and $n \in \omega$ by Lusin-Novikov. We can assume that the sets R_n are pairwise disjoint.

Let $X^2 \setminus id_X = \bigsqcup_{k\in\omega} A_k \times B_k$ with $A_k \cap B_k = \emptyset$ and A_k, B_k Borel by the previous lemma. Let

$$E_{m,n,k} = \{(x,y) \in E \mid (x,y) \in R_m,\ (y,x) \in R_n,\ x \neq y,\ (x,y) \in A_k \times B_k\}.$$

Then $E \setminus id_X = \bigsqcup_{m,n,k} E_{m,n,k}$. Each $E_{m,n,k}$ is the graph of some Borel isomorphism $f_{m,n,k} \colon A_{m,n,k} \to B_{m,n,k}$. In fact $f_{m,n,k} = f_n \restriction A_{m,n,k}$ and $f^{-1}_{m,n,k} = f_m \restriction B_{m,n,k}$.

We extend each $f_{m,n,k}$ to a Borel automorphism of X as follows. Let

$$h_{m,n,k}(x) = \begin{cases} f_{m,n,k}(x) & \text{if } x \in A_{m,n,k}, \\ f^{-1}_{m,n,k}(x) & \text{if } x \in B_{m,n,k}, \\ x & \text{otherwise.} \end{cases}$$

Let $G \subseteq Aut(X)$ denote the (countable) group generated by all $h_{m,n,k}$. Then $E \subseteq G$ by the definition of G, and $E_G \subseteq E$ since E is an equivalence relation. $\square$