12. Problem sheet for Set Theory, Winter 2012

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Problem 43. Use the methods of the proof of Silvers Theorem to show

(a) Let $\mathcal{F} \subseteq \prod_{\alpha<\lambda} A_\alpha$ almost disjoint, $card(A_\alpha) \leq \kappa_\alpha^{++}$. Then $card(\mathcal{F}) \leq \kappa^{++}$.

(b) Let $\omega < \lambda = cof(\kappa) < \kappa \in Card$. Let $2^\mu = \mu^{++}$ for all $\omega \leq \mu \in \kappa \cap Card$. Then $2^\kappa \leq \kappa^{++}$.

Problem 44. Suppose $S \subseteq \omega_1$ is stationary. Show

(a) There are arbitrarily large $\alpha \in S$ with $\sup(S \cap \alpha) = \alpha$.

(b) For every limit ordinal $\alpha < \omega_1$, let C_α denote the set of limit ordinals $\gamma < \omega_1$ such that for all $\beta < \gamma$ there is a closed set C with

 (i) $C \cap \beta = \emptyset$,

 (ii) $otp(C) = \alpha + 1$,

 (iii) $\max(C) = \gamma$, and

 (iv) $C \setminus \{\gamma\} \subseteq S$.

Then C_α contains a cub subset of ω_1.

Conclude that for any $\alpha < \omega_1$, there is a closed set $C \subseteq S$ with $otp(C) = \alpha + 1$.

Problem 45. Let $[\mu]^\kappa$ denote the set of all subsets of μ of cardinality κ. A subset $C \subseteq \mu^\kappa$ is called *cofinal* if for all $x \in [\mu]^\kappa$ there is some $y \in C$ with $x \subseteq y$. Show for all infinite cardinals $\kappa \leq \mu$: If C is cofinal in $[\mu]^\kappa$, then $card([\mu]^\kappa) = card(C) \cdot 2^\kappa$.

Problem 46. The *cofinality* of the partial order $([\mu]^\kappa, \subseteq)$ is the least size of a cofinal subset of $[\mu]^\kappa$.

(a) Calculate the cofinality of the partial orders $([\aleph_{n+1}]^{\aleph_n}, \subseteq)$ and $([\aleph_n]^{\aleph_0}, \subseteq)$ for all $n \in \omega$.

(b) Show that the cofinality of $([\aleph_\omega]^{\aleph_0}, \subseteq)$ is at least $\aleph_\omega^+$.

There are 6 points for each problem. Please hand in your solutions on Monday, January 14 before the lecture.