ORDINAL ARITHMETIC

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Abstract. We define ordinal arithmetic and show laws of Left-Monotonicity, Associativity, Distributivity, some minor related properties and the Cantor Normal Form.

1. Ordinals

Definition 1.1. A set $x$ is called transitive iff $\forall y \in x \forall z \in y : z \in x$.

Definition 1.2. A set $\alpha$ is called an ordinal iff $\alpha$ transitive and all $\beta \in \alpha$ are transitive. Write $\alpha \in \text{Ord}$.

Lemma 1.3. If $\alpha$ is an ordinal and $\beta \in \alpha$, then $\beta$ is an ordinal.

Proof. $\beta$ is transitive, since it is in $\alpha$. Let $\gamma \in \beta$. By transitivity of $\alpha$, $\gamma \in \alpha$. Hence $\gamma$ is transitive. Thus $\beta$ is an ordinal. $\square$

Definition 1.4. If $a$ is a set, define $a+1 = a \cup \{a\}$.

Remark 1.5. $\emptyset$ is an ordinal. Write $0 = \emptyset$. If $\alpha$ is an ordinal, so is $\alpha + 1$.

Definition 1.6. If $\alpha$ and $\beta$ are ordinals, say $\alpha < \beta$ iff $\alpha \in \beta$.

Lemma 1.7. For all ordinals $\alpha$, $\alpha < \alpha + 1$.

Proof. $\alpha \in \{\alpha\}$, so $\alpha \in \alpha \cup \{\alpha\} = \alpha + 1$. $\square$

Notation 1.8. From now on, $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta$ denote ordinals.

Theorem 1.9. The ordinals are linearly ordered i.e.

i. $\forall \alpha : \alpha \not< \alpha$ (strictness).

ii. $\forall \alpha \forall \beta \forall \gamma : \alpha < \beta \land \beta < \gamma \rightarrow \alpha < \gamma$ (transitivity).

iii. $\forall \alpha \forall \beta : \alpha < \beta \lor \beta < \alpha \lor \alpha = \beta$ (linearity).

Proof. "i." follows from (Found).

"ii." follows from transitivity of the ordinals.

"iii." Assume this fails. By (Found), choose a minimal $\alpha$ such that some $\beta$ is neither smaller, larger or equal to $\alpha$. Choose the minimal such $\beta$. Show towards a contradiction that $\alpha = \beta$:
Let $\gamma \in \alpha$. By minimality of $\alpha$, $\gamma < \beta \lor \beta < \gamma \lor \beta = \gamma$. If $\beta = \gamma$, $\beta < \alpha_+$. If $\beta < \gamma$ then by “ii.” $\beta < \alpha_+$. Thus $\gamma < \beta$, i.e. $\gamma \in \beta$. Hence $\alpha \subseteq \beta$.

Let $\gamma \in \beta$. By minimality of $\beta$, $\gamma < \alpha \lor \alpha < \gamma \lor \alpha = \gamma$. If $\alpha = \gamma$, $\alpha < \beta_+$. If $\alpha < \gamma$ then by “ii” $\alpha < \beta_+$. Thus $\gamma < \alpha$, i.e. $\gamma \in \alpha$. Hence $\beta \subseteq \alpha$.

**Lemma 1.10.** If $\alpha \neq 0$ is an ordinal, $0 < \alpha$, i.e. $0$ is the smallest ordinal.

**Proof.** Since $\alpha \neq 0$, by linearity $\alpha < 0$ or $0 < \alpha$, but $\alpha < 0$ would mean $\alpha \in \emptyset$. □

**Definition 1.11.** An ordinal $\alpha$ is called a successor iff there is a $\beta$ with $\alpha = \beta + 1$. Write $\alpha \in \text{Suc}$.

An ordinal $\alpha \neq \emptyset$ is called a limit if it is no successor. Write $\alpha \in \text{Lim}$.

**Remark 1.12.** By definition, every ordinal is either $\emptyset$ or a successor or a limit.

**Lemma 1.13.** For all ordinals $\alpha, \beta$: If $\beta < \alpha + 1$, then $\beta < \alpha \lor \beta = \alpha$, i.e. $\beta \leq \alpha$.

**Proof.** Let $\beta < \alpha + 1$, i.e. $\beta \in \alpha \cup \{\alpha\}$. By definition of $\cup$, $\beta \in \alpha$ or $\beta \in \{\alpha\}$, i.e. $\beta \in \alpha \lor \beta = \alpha$. □

**Lemma 1.14.** For all ordinals $\alpha, \beta$: If $\beta < \alpha$, then $\beta + 1 \leq \alpha$.

**Proof.** Suppose this fails for some $\alpha, \beta$. Then by linearity, $\beta + 1 > \alpha$, hence by the previous lemma $\alpha \leq \beta$. Hence by transitivity $\beta < \alpha \leq \beta$, contradicting strictness. □

**Lemma 1.15.** For all $\alpha$, there is no $\beta$ with $\alpha < \beta < \alpha + 1$.

**Proof.** Assume there are such $\alpha$ and $\beta$. Then, since $\beta < \alpha + 1$, $\beta \leq \alpha$, but since $\alpha < \beta$, by linearity $\alpha < \alpha$, contradicting strictness. □

**Lemma 1.16.** For all $\alpha, \beta$, if there is no $\gamma$ with $\alpha < \gamma < \beta$, then $\beta = \alpha + 1$.

**Proof.** Suppose $\beta \neq \alpha + 1$. Since $\alpha < \beta$, $\alpha + 1 \leq \beta$, so $\alpha + 1 < \beta$. Then $\alpha + 1$ is some such $\gamma$. □

**Lemma 1.17.** The operation $+1 : \text{Ord} \to \text{Ord}$ is injective.

**Proof.** Let $\alpha \neq \beta$ be ordinals. Wlog $\alpha < \beta$. Then by the previous lemmas, $\alpha + 1 \leq \beta < \beta + 1$, i.e. $\alpha + 1 \neq \beta + 1$. □

**Lemma 1.18.** $\alpha \in \text{Lim}$ iff $\forall \beta < \alpha : \beta + 1 < \alpha$ and $\alpha \neq 0$.
Proof. Let $\alpha \in \text{Lim}$, $\beta < \alpha$. By linearity, $\beta + 1 < \alpha \lor \alpha < \beta + 1 \lor \beta + 1 = \alpha$. The last case is excluded by definition of limits. So suppose $\alpha < \beta + 1$. Then $\alpha = \beta \lor \alpha < \beta$.

Since $\beta < \alpha$, $\alpha = \beta$ implies $\alpha < \alpha$, contradicting strictness.

By linearity, $\alpha < \beta$ implies $\beta < \beta$, contradicting strictness.

Thus $\beta + 1 < \alpha$.

Now suppose $\alpha \neq 0$ and $\forall \beta < \alpha : \beta + 1 < \alpha$. Assume $\alpha \in \text{Suc}$. Then there is $\beta$ such that $\alpha = \beta + 1$. Then $\beta < \beta + 1 = \alpha$, thus $\beta < \alpha$, i.e. $\beta + 1 < \alpha$. Then $\alpha = \beta + 1 < \alpha$, contradicting strictness. Hence $\alpha$ is a limit. \[\square\]

**Theorem 1.19** (Ordinal Induction). Let $\varphi$ be a property of ordinals. Suppose the following holds:

i. $\varphi(\emptyset)$ (base step).

ii. $\forall \alpha : \varphi(\alpha) \rightarrow \varphi(\alpha + 1)$ (successor step).

iii. $\forall \alpha \in \text{Lim} : (\forall \beta < \alpha : \varphi(\beta)) \rightarrow \varphi(\alpha)$ (limit step).

Then $\varphi(\alpha)$ holds for all ordinals $\alpha$.

Proof. Suppose i, ii and iii hold. Assume there is some $\alpha$ such that $\neg \varphi(\alpha)$. By (Found), take the smallest such $\alpha$.

Suppose $\alpha = \emptyset$. This contradicts i.

Suppose $\alpha \in \text{Suc}$. Then there is $\beta$ such that $\alpha = \beta + 1$, since $\beta < \beta + 1$, $\beta < \alpha$ and hence by minimality of $\alpha$, $\varphi(\beta)$. By ii, $\varphi(\alpha)$\[\square\].

Suppose $\alpha \in \text{Lim}$. By minimality of $\alpha$, all $\beta < \alpha$ satisfy $\varphi(\beta)$. Thus by iii, $\varphi(\alpha)$\[\square\].

Hence there can’t be any such $\alpha$. \[\square\]

**Definition 1.20.** Let $\omega$ be the (inclusion-)smallest set that contains 0 and is closed under $+1$, i.e. $\forall x \in \omega : x + 1 \in \omega$.

More formally, $\omega = \bigcap \{w \mid 0 \in w \land \forall v \in w : v + 1 \in w\}$.

**Remark 1.21.** $\omega$ is a set by the Axiom of Infinity.

**Theorem 1.22.** $\omega$ is an ordinal.

Proof. Consider $\omega \cap \text{Ord}$. This set contains 0 and is closed under $+1$, as ordinals are closed under $+1$. So $\omega$ must by definition be a subset of $\omega \cap \text{Ord}$, i.e. $\omega$ contains only ordinals.

Hence it suffices to show that $\omega$ is transitive. Consider $\omega' = \{x \mid x \in \omega \land \forall y \in x : y \in \omega\}$. Clearly, $0 \in \omega'$. Let $x \in \omega'$ and show that $x + 1 \in \omega'$.

By definition, $x + 1 \in \omega$. Let $y \in x + 1$, i.e. $y = x \lor y \in x$. If $y = x$, $y \in \omega$. If $y \in x$ then $y \in \omega$ by definition of $\omega'$. Hence $x + 1 \in \omega'$.

Thus $\omega'$ contains 0 and is closed under $+1$, i.e. $\omega \subseteq \omega'$. But $\omega' \subseteq \omega$ by definition, hence $\omega = \omega'$, i.e. $\omega$ is transitive. \[\square\]
Theorem 1.23. \( \omega \) is a limit, in particular, it is the smallest limit ordinal.

Proof. \( \omega \neq 0 \), since \( 0 \in \omega \). Let \( \alpha < \omega \). Then \( \alpha + 1 < \omega \) by definition.

Assume \( \gamma < \omega \) is a limit ordinal. Since \( \gamma \neq \emptyset \), \( 0 \notin \gamma \). Also, as a limit, \( \gamma \) is closed under \(+1\). Hence \( \gamma \) contradicts the minimality of \( \omega \). \( \square \)

2. Ordinal Arithmetic

Definition 2.1. Define an ordinal \( 1 := 0 + 1 = \{0\} \).

Lemma 2.2. \( 1 \in \omega \).

Proof. \( 0 \in \omega \) and \( \omega \) is closed under \(+1\). \( \square \)

Definition 2.3. Let \( \alpha, \beta \) be ordinals. Define ordinal addition recursively:

i. \( \alpha + 0 = \alpha \).

ii. If \( \beta \in \text{Suc} \), \( \beta = \gamma + 1 \), define \( \alpha + \beta = (\alpha + \gamma) + 1 \).

iii. If \( \beta \in \text{Lim} \), define \( \alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma) \).

Remark 2.4. By this definition, the sum \( \alpha + 1 \) of an ordinal \( \alpha \) and the ordinal \( 1 = \{0\} \) is the same as \( \alpha + 1 = \alpha \cup \{\alpha\} \).

Definition 2.5. Let \( \alpha, \beta \) be ordinals. Define ordinal multiplication recursively:

i. \( \alpha \cdot 0 = 0 \).

ii. If \( \beta \in \text{Suc} \), \( \beta = \gamma + 1 \), define \( \alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha \).

iii. If \( \beta \in \text{Lim} \), define \( \alpha \cdot \beta = \bigcup_{\gamma < \beta} (\alpha \cdot \gamma) \).

Definition 2.6. Let \( \alpha, \beta \) be ordinals. Define ordinal exponentiation recursively:

i. \( \alpha^0 = 1 \).

ii. If \( \beta \in \text{Suc} \), \( \beta = \gamma + 1 \), define \( \alpha^\beta = (\alpha^\gamma) \cdot \alpha \).

iii. If \( \beta \in \text{Lim} \) and \( \alpha > 0 \), define \( \alpha^\beta = \bigcup_{\gamma < \beta} (\alpha^\gamma) \). If \( \alpha = 0 \), define \( \alpha^0 = 0 \).

Lemma 2.7. If \( A \) is a set of ordinals, \( \bigcup A \) is an ordinal.

Proof. Let \( A \) be a set of ordinals, define \( a = \bigcup A \).

Let \( x \in y \in a \), then there is an \( \alpha \in A \) such that \( x \in y \in \alpha \), so \( x \in \alpha \) hence \( x \in a \). Thus, \( a \) is transitive. Let \( z \in a \). There is \( \alpha \in A \) such that \( z \in \alpha \), hence \( z \) is transitive.

Thus \( a \) is transitive and every element of \( a \) is transitive, i.e. \( a \) is an ordinal. \( \square \)
Remark 2.8. By induction and this lemma, the definitions of +, · and exponentiation above are well-defined, i.e. if \( \alpha, \beta \) are ordinals, \( \alpha + \beta, \alpha \cdot \beta \) and \( \alpha^\beta \) are again ordinals.

Definition 2.9. Let \( A \) be a set of ordinals. The supremum of \( A \) is defined as: \( \sup A = \min \{ \alpha \mid \forall \beta \in A : \beta \leq \alpha \} \).

Lemma 2.10. Let \( A \) be a set of ordinals, then \( \sup A = \bigcup A \).

Proof. Since sup \( A \) is again an ordinal, it is just the set of all ordinals smaller than it. Hence by linearity, \( \sup A = \{ \alpha \mid \exists \beta \in A : \alpha < \beta \} \).
Which equals \( \bigcup A \) by definition. \( \square \)

Lemma 2.11. Let \( A \) be a set of ordinals. If \( \sup A \) is a successor, then \( \sup A \in A \).

Proof. Assume \( \sup A = \alpha + 1 \notin A \), then for all \( \beta \in A, \beta < \alpha + 1 \), i.e. \( \beta \leq \alpha \). Then \( \sup A = \alpha < \alpha + 1 = \sup A \).

Lemma 2.12. Let \( A \) be a set of ordinals, \( B \subseteq A \) such that \( \forall \alpha \in A \exists \beta \in B : \alpha \leq \beta \). Then \( \sup A = \sup B \).

Proof. Show \( \{ \gamma \mid \forall \alpha \in A : \gamma \geq \alpha \} = \{ \gamma \mid \forall \beta \in B : \gamma \geq \beta \} \). Then the minima of these sets, and hence the suprema of \( A \) and \( B \), are equal. Suppose \( \gamma \geq \alpha \) for all \( \alpha \in A \). Then, since \( B \subseteq A \), \( \gamma \geq \alpha \) for all \( \beta \in B \).
Suppose \( \gamma \geq \beta \) for all \( \beta \in B \). Let \( \alpha \in A \), then there is some \( \beta \in B \) with \( \beta \geq \alpha \), hence \( \gamma \geq \beta \geq \alpha \).
Thus \( \gamma \geq \alpha \) for all \( \alpha \in A \).

Lemma 2.13. If \( \gamma \) is a limit, \( \bigcup \gamma = \sup \gamma = \bigcup_{\alpha < \gamma} \alpha = \sup \alpha_{< \gamma} \alpha = \gamma \).

Proof. We’ve shown a more general form of the first equality, the second and third are just a different ways of writing the same set. Assume \( \gamma \neq \sup \alpha_{< \gamma} \alpha \), i.e. \( \gamma < \sup \alpha_{< \gamma} \alpha \) or \( \sup \alpha_{< \gamma} \alpha < \gamma \) by linearity.
In the first case, there is \( \alpha < \gamma \) such that \( \gamma < \alpha \), i.e. \( \gamma < \gamma \)
contradicting strictness.
In the second case, \( (\sup \alpha_{< \gamma} \alpha) + 1 < \gamma \), since \( \gamma \) is a limit. But then, by definition of sup, \( (\sup \alpha_{< \gamma} \alpha) + 1 \leq \sup \alpha_{< \gamma} \alpha \) while \( \sup \alpha_{< \gamma} \alpha < (\sup \alpha_{< \gamma} \alpha) + 1 \), again contradicting strictness. \( \square \)

Lemma 2.14. For all \( \alpha \), \( 0 + \alpha = \alpha \).

Proof. By induction on \( \alpha \). Since \( 0 + 0 = 0 \), the base step is trivial.
Suppose \( \alpha = \beta + 1 \) and \( 0 + \beta = \beta \). Then \( 0 + \alpha = 0 + (\beta + 1) = (0 + \beta) + 1 = \beta + 1 = \alpha \).
Suppose \( \alpha \in \text{Lim} \) and for all \( \beta < \alpha \), \( 0 + \beta = \beta \). Then \( 0 + \alpha = \bigcup_{\beta<\alpha} (0 + \beta) = \bigcup_{\beta<\alpha} \beta = \alpha \). \( \square \)

Lemma 2.15. For all \( \alpha \), \( 1 \cdot \alpha = \alpha \cdot 1 = \alpha \).
Proof. \(\alpha \cdot 1 = \alpha \cdot (0 + 1) = (\alpha \cdot 0) + \alpha = \alpha\). Prove \(1 \cdot \alpha = \alpha\) by induction on \(\alpha\). Since \(1 \cdot 0 = 0\), the base step holds.

Suppose \(\alpha = \beta + 1\) and \(1 \cdot \beta = \beta\). Then \(1 \cdot \alpha = (1 \cdot \beta) + 1 = \beta + 1 = \alpha\).

Suppose \(\alpha\) is a limit and for all \(\beta < \alpha\), \(1 \cdot \beta = \beta\). Then \(1 \cdot \alpha = \bigcup_{\beta < \alpha} (1 \cdot \beta) = \bigcup_{\beta < \alpha} \beta = \alpha\). \(\square\)

Lemma 2.16. For all \(\alpha\), \(\alpha^1 = \alpha\).

Proof. \(\alpha^1 = \alpha^0 \cdot 1 = 1 \cdot \alpha = \alpha\). \(\square\)

Lemma 2.17. For all \(\alpha\), \(1^\alpha = 1\).

Proof. If \(\alpha = 0\), \(1^\alpha = 1\) by definition. If \(\alpha = \beta + 1\), \(1^{\beta+1} = 1^\beta \cdot 1 = 1\).

If \(\alpha\) is a limit, \(1^\alpha = \sup_{\beta < \alpha} 1^\beta = \sup_{\beta < \alpha} 1 = 1\). \(\square\)

Lemma 2.18. Let \(\alpha\) be an ordinal. If \(\alpha > 0\), \(0^0 = 0\). Otherwise \(0^\alpha = 1\).

Proof. \(0^0 = 1\) by definition, so let \(\alpha > 0\). If \(\alpha = \beta + 1\), \(0^\alpha = 0^\beta \cdot 0 = 0\).

If \(\alpha\) is a limit, \(0^\alpha = 0\) by definition. \(\square\)

Theorem 2.19 (Subtraction). For all \(\beta\leq\alpha\) there is some \(\gamma\leq\alpha\) with \(\beta + \gamma = \alpha\).

Proof. By induction on \(\alpha\). \(\alpha = 0\) is trivial. Suppose \(\alpha = \delta + 1\) and \(\beta \leq \alpha\). If \(\beta = \alpha\), set \(\gamma = 0\). So suppose \(\beta < \alpha\), i.e. \(\beta \leq \delta\). Find \(\gamma' \leq \beta\) with \(\beta + \gamma' = \delta\). Set \(\gamma = \gamma' + 1\), then \(\beta + \gamma = \beta + (\gamma' + 1) = (\beta + \gamma') + 1 = \delta + 1 = \alpha\).

If \(\alpha\) is a limit and \(\beta < \alpha\) then for all \(\delta < \alpha\), \(\beta \leq \delta\), find \(\gamma_\delta\) such that \(\beta + \gamma_\delta = \delta\). If \(\delta < \beta\), set \(\gamma_\delta = 0\). Set \(\gamma = \sup_{\beta < \delta \leq \alpha} \gamma_\delta\). If \(\gamma\) is a successor, then there is some \(\delta\) with \(\gamma = \gamma_\delta\). But \(\delta + 1 < \alpha\) and as in the successor case, \(\gamma_{\delta+1} = \gamma_\delta + 1 > \gamma_\delta = \gamma\), so this can’t be the supremum.

Also, \(\gamma \not= 0\), since if it were, for all \(\beta < \delta < \gamma\), \(\beta = \delta\), i.e. there are no such \(\delta\). This implies \(\beta + 1 = \alpha\), but \(\alpha\) is no successor.

So, \(\gamma\) is a limit. In particular for all \(\delta < \alpha\), \(\gamma_\delta < \gamma\): If there were any \(\delta < \gamma\) with \(\gamma_\delta = \gamma\), then since \(\gamma \not= 0\), \(\beta < \delta\). Then again \(\gamma_{\delta+1} = \gamma_\delta + 1 > \gamma_\delta = \gamma\), contradicting that \(\gamma\) is the supremum. Hence, \(\beta + \gamma = \sup_{\epsilon < \gamma} (\beta + \epsilon) = \sup_{\gamma_\delta < \gamma} (\beta + \gamma_\delta) = \sup_{\gamma_\delta < \gamma} \delta = \sup_{\delta < \alpha} \delta = \alpha\). \(\square\)

Theorem 2.20. \(\omega\) is closed under \(+\), \(\cdot\) and exponentiation, i.e. \(\forall n, m \in \omega : n + m \in \omega \land n \cdot m \in \omega \land n^m \in \omega\).

Proof. By induction on \(m\). Since \(\omega\) does not contain any limits, we may omit the limit step.
First consider addition. If \( m = 0 \) then \( n + m = m \in \omega \). Suppose \( m = k + 1 \). \( n + m = (n + k) + 1 \). By induction \( n + k \in \omega \) and since \( \omega \) is closed under \(+\), \((n + k) + 1 \in \omega\).

Now consider multiplication. If \( m = 0 \), \( n \cdot 0 = 0 \in \omega \). Suppose \( m = k + 1 \). \( n \cdot m = (n \cdot k) + n \). By induction \( n \cdot k \in \omega \) and since \( \omega \) is closed under \(*\), \( n^k \cdot n \in \omega \).

Finally consider exponentiation. If \( m = 0 \), \( n^0 = 1 \in \omega \). Suppose \( m = k + 1 \). \( n^m = n^k \cdot n \). By induction, \( n^k \in \omega \) and since \( \omega \) is closed under \( \cdot \), \( n^k \cdot n \in \omega \).

\[ \square \]

3. Monotonicity Laws

3.1. Comparisons of Addition.

**Lemma 3.1.** If \( \alpha \) and \( \beta \) are ordinals, and \( \alpha \leq \beta \), then \( \alpha + 1 \leq \beta + 1 \).

**Proof.** Assume \( \alpha \leq \beta \) and \( \alpha + 1 > \beta + 1 \). By transitivity it suffices to now derive a contradiction. Since \( \beta + 1 < \alpha + 1 \), \( \beta + 1 = \alpha \lor \beta + 1 < \alpha \).

If \( \beta + 1 = \alpha \), \( \beta + 1 \leq \beta \), but \( \beta < \beta + 1 \).

If \( \beta + 1 < \alpha \), by transitivity \( \beta + 1 \leq \beta \), but \( \beta < \beta + 1 \).

\[ \square \]

**Lemma 3.2.** If \( \alpha \) and \( \beta \) are ordinals, then \( \alpha \leq \alpha + \beta \).

**Proof.** By induction on \( \beta \): If \( \beta = 0 \), \( \alpha = \alpha + \beta \).

If \( \beta = \gamma + 1 \) and \( \alpha \leq \alpha + \gamma \), then \( \alpha + \beta = (\alpha + \gamma) + 1 \geq \alpha + 1 \geq \alpha \).

If \( \beta \in \text{Lim} \) and for all \( \gamma < \beta \), \( \alpha \leq \alpha + \gamma \), then: \( \alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma) = \sup_{\gamma < \beta} (\alpha + \gamma) \geq \sup_{\gamma < \beta} \alpha = \alpha \).

\[ \square \]

**Lemma 3.3.** If \( \alpha \) and \( \beta \) are ordinals, then \( \beta \leq \alpha + \beta \).

**Proof.** By induction on \( \beta \): \( \beta = 0 \) is trivial, since 0 is the smallest ordinal.

If \( \beta = \gamma + 1 \) and \( \gamma \leq \alpha + \gamma \), then \( \alpha + \beta = (\alpha + \gamma) + 1 \geq \gamma + 1 = \beta \).

If \( \beta \in \text{Lim} \) and for all \( \gamma < \beta \), \( \gamma \leq \alpha + \gamma \), then: \( \alpha + \beta = \bigcup_{\gamma < \beta} (\alpha + \gamma) = \sup_{\gamma < \beta} (\alpha + \gamma) = \beta \).

\[ \square \]

**Lemma 3.4.** If \( \gamma \) is a limit, then for all \( \alpha \): \( \alpha + \gamma \) is a limit.

**Proof.** \( \gamma \neq 0 \), so \( \alpha + \gamma \geq \gamma > 0 \), i.e. \( \alpha + \gamma \neq 0 \). So let \( x \in \alpha + \gamma \). Show that \( x + 1 \leq \alpha + \gamma \).

\( x \in \alpha + \gamma = \bigcup_{\beta \leq \gamma} (\alpha + \beta) \), i.e. there is \( \beta < \gamma \) such that \( x \in \alpha + \beta \).

By a previous lemma, \( x + 1 \leq \alpha + \beta \). If \( x + 1 \in \alpha + \beta \), \( x + 1 < \alpha + \gamma \).

So suppose \( \alpha + \beta = x + 1 \). Since \( \gamma \) is a limit, \( \beta + 1 < \gamma \) and by definition \( \alpha + (\beta + 1) = (\alpha + \beta) + 1 \), and \( x + 1 \in (\alpha + \beta) + 1 \), hence \( x + 1 \in \alpha + \gamma \).

\[ \square \]

**Lemma 3.5.** Suppose \( \gamma \) is a limit, \( \alpha, \beta \) are ordinals and \( \beta < \gamma \). then \( \alpha + \beta < \alpha + \gamma \).
Proof. By definition, $\alpha + \gamma = \bigcup_{\delta \leq \gamma} (\alpha + \delta)$. Since $\gamma$ is a limit, $\beta + 1 < \gamma$. Also by definition: $\alpha + \beta < (\alpha + \beta) + 1 = \alpha + (\beta + 1) \in \{\alpha + \delta \mid \delta < \gamma\}$. Hence $\alpha + \beta \in \bigcup_{\delta < \gamma} (\alpha + \delta)$. \hfill \Box

**Lemma 3.6.** Suppose $\alpha, \beta, \gamma$ are ordinals and $\beta < \gamma$. then $\alpha + \beta < \alpha + \gamma$.

Proof. By induction over $\gamma$. $\gamma = 0$ is clear, since there is no $\beta < 0$. And the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta + 1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, notice: $\alpha + \beta = \alpha + \delta < (\alpha + \delta) + 1 = \alpha + (\delta + 1) = \alpha + \gamma$.

If $\beta < \delta$, apply induction: $\alpha + \beta < \alpha + \delta$. Hence $\alpha + \beta < (\alpha + \delta) + 1 = \alpha + (\delta + 1) = \alpha + \gamma$. \hfill \Box

**Theorem 3.7 (Left-Monotonicity of Ordinal Addition).** Let $\alpha, \beta, \gamma$ be ordinals. The following are equivalent:

i. $\beta < \gamma$.

ii. $\alpha + \beta < \alpha + \gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha + \beta < \alpha + \gamma$ and not $\beta < \gamma$. By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha + \gamma = \alpha + \beta < \alpha + \gamma \cdot \frac{1}{\beta}$. If $\gamma < \beta$, by the forward direction, $\alpha + \gamma < \alpha + \beta < \alpha + \gamma \cdot \frac{1}{\beta}$. \hfill \Box

**Lemma 3.8.** $1 + \omega = \omega$.

Proof. $1 + \omega$ is a limit by a lemma above, so $\omega \leq 1 + \omega$ (since $\omega$ is the smallest limit). $\omega$ is a limit, so $1 + \omega = \sup_{\alpha < \omega} 1 + \alpha$. Since $\omega$ is closed under $+$, $1 + \alpha < \omega$ for all $\alpha < \omega$, hence $\sup_{\alpha < \omega} \leq \omega$. It follows that $1 + \omega = \omega$. \hfill \Box

**Remark 3.9.** Right-Monotonicity does not hold: Clearly, $0 < 1$ and we’ve seen that $0 + \omega = \omega$ and $1 + \omega = \omega$. So $0 + \omega \neq 1 + \omega$.

3.2. Comparisons of Multiplications.

**Lemma 3.10.** For all $\alpha, \beta$: $\alpha + \beta = 0$ iff $\alpha = \beta = 0$.

Proof. Reverse direction is trivial. So suppose $\alpha + \beta = 0$ and not $\alpha = \beta = 0$. If $\beta = 0$, then $0 = \alpha + \beta = \alpha$ and if $\beta > 0$, by Left-Monotonicity $0 \leq \alpha + 0 < \alpha + \beta = 0 \cdot \frac{1}{\beta}$. \hfill \Box

**Lemma 3.11.** If $\alpha$ and $\beta \neq 0$ are ordinals, then $\alpha \leq \alpha \cdot \beta$.

Proof. By induction on $\beta$. Suppose $\beta = \gamma + 1$, then $\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha \geq \alpha$, by induction and the corresponding lemma on addition.

Suppose $\beta$ is a limit. $\alpha \cdot \beta = \bigcup_{\gamma < \beta} (\alpha \cdot \gamma) = \sup_{\gamma < \beta} (\alpha \cdot \gamma) \geq \sup_{\gamma < \beta} \alpha = \alpha$. \hfill \Box
Lemma 3.12. If $\alpha \neq 0$ and $\beta$ are ordinals, then $\beta \leq \alpha \cdot \beta$.

Proof. By induction on $\beta$. $\beta = 0$ is trivial. Suppose $\beta = \gamma + 1$, then:

$$\alpha \cdot \beta = (\alpha \cdot \gamma) + \alpha > \alpha \cdot \gamma$$

(since $\alpha > 0$ and by Left-Monotonicity)

$$\geq \gamma$$

(by induction).

And $\alpha \cdot \beta > \gamma$ implies $\alpha \cdot \beta \geq \gamma + 1 = \beta$.

Suppose $\beta$ is a limit. $\alpha \cdot \beta = \bigcup_{\gamma < \beta} (\alpha \cdot \gamma) = \sup_{\gamma < \beta} (\alpha \cdot \gamma) \geq \sup_{\gamma < \beta} \gamma = \beta$.

□

Lemma 3.13. If $\gamma$ is a limit, then for all $\alpha \neq 0$: $\alpha \cdot \gamma$ is a limit.

Proof. $\gamma \neq 0$, so $\alpha \cdot \gamma \geq \gamma > 0$, i.e. $\alpha \cdot \gamma \neq 0$. So let $x \in \alpha \cdot \gamma$. Show that $x + 1 < \alpha \cdot \gamma$.

$$x \in \alpha \cdot \gamma = \bigcup_{\beta < \gamma} (\alpha \cdot \beta), \text{ i.e. there is } \beta < \gamma \text{ such that } x \in \alpha \cdot \beta.$$ By a previous lemma, $x + 1 \leq \alpha \cdot \beta$. If $x + 1 \in \alpha \cdot \beta$, $x + 1 < \alpha \cdot \gamma$.

So suppose $\alpha \cdot \beta = x + 1$. Since $\gamma$ is a limit, $\beta + 1 < \gamma$ and by definition $\alpha \cdot (\beta + 1) = (\alpha \cdot \beta) + \alpha$, and $x + 1 \in (\alpha \cdot \beta) + 1 \leq (\alpha \cdot \beta) + \alpha$ by Left-Monotonicity (since $\alpha \geq 1$). Hence $x + 1 \in \alpha \cdot \gamma$.

□

Lemma 3.14. Suppose $\gamma$ is a limit, $\alpha \neq 0$ and $\beta$ are ordinals and $\beta < \gamma$. Then $\alpha \cdot \beta < \alpha \cdot \gamma$.

Proof. By definition, $\alpha \cdot \gamma = \bigcup_{\delta < \gamma} (\alpha \cdot \delta)$. Since $\gamma$ is a limit, $\beta + 1 < \gamma$. By Left-Monotonicity: $\alpha \cdot \beta < (\alpha \cdot \beta) + \alpha = \alpha \cdot (\beta + 1) \in \{\alpha \cdot \delta \mid \delta < \gamma\}$.

Hence $\alpha \cdot \beta \in \bigcup_{\delta < \gamma} \alpha \cdot \delta$.

□

Lemma 3.15. Suppose $\alpha \neq 0$ and $\beta, \gamma$ are ordinals and $\beta < \gamma$. Then $\alpha \cdot \beta < \alpha \cdot \gamma$.

Proof. By induction over $\gamma$. $\gamma = 0$ is clear and the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta + 1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, apply Left-Monotonicity:

$$\alpha \cdot \beta = \alpha \cdot \delta < (\alpha \cdot \delta) + \alpha = \alpha \cdot (\delta + 1) = \alpha \cdot \gamma.$$ If $\beta < \delta$, apply induction: $\alpha \cdot \delta < \alpha \cdot \delta$. Hence (by Left-Monotonicity)

$$\alpha \cdot \beta < \alpha \cdot \delta < (\alpha \cdot \delta) + \alpha = \alpha \cdot (\delta + 1) = \alpha \cdot \gamma.$$

□

Theorem 3.16 (Left-Monotonicity of Ordinal Multiplication). Let $\alpha, \beta, \gamma$ be ordinals. The following are equivalent:

i. $\beta < \gamma \wedge \alpha > 0$.

ii. $\alpha \cdot \beta < \alpha \cdot \gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha \cdot \beta < \alpha \cdot \gamma$ and not $\beta < \gamma$. If $\alpha = 0$, then $\alpha \cdot \beta = 0 = \alpha \cdot \gamma$ and so $\alpha > 0$. By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha \cdot \gamma = \alpha \cdot \beta < \alpha \cdot \gamma$. If $\gamma < \beta$, by the forward direction, $\alpha \cdot \gamma < \alpha \cdot \beta < \alpha \cdot \gamma$.
Lemma 3.17. Let $2 = 1 + 1$. $2 \cdot \omega = \omega$.

Proof. Since $\omega$ is the smallest limit and by a lemma above $2 \cdot \omega$ is a limit, $\omega \leq 2 \cdot \omega$. Since $\omega$ is closed under $\cdot$, for all $\alpha < \omega$, $2 \cdot \alpha \in \omega$. Hence $2 \cdot \omega = \sup_{\alpha < \omega} 2 \cdot \alpha \leq \omega$. □

Remark 3.18. Right-Monotonicity does not hold: Clearly, $1 < 2$ and since $1 \cdot \omega = \omega$ and $2 \cdot \omega = \omega$, $1 \cdot \omega \neq 2 \cdot \omega$.

3.3. Comparisons of Exponentation.

Lemma 3.19. If $\alpha$ and $\beta \neq 0$ are ordinals, then $\alpha \leq \alpha^\beta$.

Proof. If $\alpha = 0$ the lemma is trivial. So suppose $\alpha > 0$.

By induction on $\beta$. Suppose $\beta = \gamma + 1$, then $\alpha^\beta = (\alpha^\gamma) \cdot \alpha \geq \alpha$, by induction and the corresponding lemma on multiplication.

Suppose $\beta$ is a limit. $\alpha^\beta = \bigcup_{\gamma<\beta} (\alpha^\gamma) = \sup_{\gamma<\beta} (\alpha^\gamma) \geq \sup_{\gamma<\beta} \alpha = \alpha$. □

Lemma 3.20. If $\alpha > 1$ and $\beta$ are ordinals, then $\beta \leq \alpha^\beta$.

Proof. If $\beta = 0$ the lemma is trivial. So suppose $\beta > 0$.

By induction on $\beta$. Suppose $\beta = \gamma + 1$, then:

$$\alpha^\beta = (\alpha^\gamma) \cdot \alpha \geq \alpha^\gamma \cdot 1 \quad \text{(by Left-Monotonicity)}$$

$$= \alpha^\gamma \geq \gamma \quad \text{(by induction)}.$$

And $\alpha^\beta > \gamma$ implies $\alpha^\beta \geq \gamma + 1 = \beta$.

Suppose $\beta$ is a limit. $\alpha^\beta = \bigcup_{\gamma<\beta} (\alpha^\gamma) = \sup_{\gamma<\beta} (\alpha^\gamma) \geq \sup_{\gamma<\beta} \gamma = \beta$. □

Lemma 3.21. If $\gamma$ is a limit, then for all $\alpha > 1$: $\alpha^\gamma$ is a limit.

Proof. $\gamma \neq 0$, so $\alpha^\gamma \geq \gamma > 0$, i.e. $\alpha^\gamma \neq 0$. So let $x \in \alpha^\gamma$. Show that $x + 1 < \alpha^\gamma$.

$x \in \alpha^\gamma = \bigcup_{\gamma<\beta} (\alpha^\beta)$, i.e. there is $\beta < \gamma$ such that $x \in \alpha^\beta$. By a previous lemma, $x + 1 \leq \alpha^\beta$. If $x + 1 \in \alpha^\beta$, $x + 1 < \alpha^\gamma$.

So suppose $\alpha^\beta = x + 1$. Since $\gamma$ is a limit, $\beta + 1 < \gamma$ and by definition $\alpha^{\beta+1} = (\alpha^\beta) \cdot \alpha$, and $x + 1 \in (\alpha^\beta) + 1 \leq \alpha^\beta + \alpha^\beta \leq \alpha^\beta \cdot 2 \leq \alpha^\beta \cdot \alpha$ by Left-Monotonicity (since $\alpha \geq 2$). Hence $x + 1 \in \alpha \cdot \gamma$. □

Lemma 3.22. Suppose $\gamma$ is a limit, $\alpha > 1$ and $\beta$ are ordinals and $\beta < \gamma$. Then $\alpha^\delta < \alpha^\gamma$.

Proof. By definition, $\alpha^\gamma = \bigcup_{\delta<\gamma} (\alpha^\delta)$. Since $\gamma$ is a limit, $\beta + 1 < \gamma$. By Left-Monotonicity: $\alpha^\beta < (\alpha^\beta) \cdot \alpha = \alpha^{\beta+1} \in \{\alpha^\delta \mid \delta < \gamma\}$. Hence $\alpha^\delta \in \bigcup_{\delta<\gamma} \alpha^\delta$. □
Lemma 3.23. Suppose $\alpha > 1$ and $\beta, \gamma$ are ordinals and $\beta < \gamma$. Then $\alpha^\beta < \alpha^\gamma$.

Proof. By induction over $\gamma$. $\gamma = 0$ is clear and the previous lemma is the limit step. So we need to cover the successor step. Suppose $\gamma = \delta + 1$. Then $\beta < \gamma$ means $\beta \leq \delta$. If $\beta = \delta$, apply Left-Monotonicity: $\alpha^\beta = \alpha^\delta < (\alpha^\delta) \cdot \alpha = \alpha^{\delta+1} = \alpha^\gamma$.

If $\beta < \delta$, apply induction: $\alpha^\beta < \alpha^\delta$. Hence (by Left-Monotonicity) $\alpha^\beta < \alpha^\gamma$.

Theorem 3.24 (Left-Monotonicity of Ordinal Exponentiation). Let $\alpha, \beta, \gamma$ be ordinals and $\alpha > 0$. The following are equivalent:

i. $\beta < \gamma \land \alpha > 1$.

ii. $\alpha^\beta < \alpha^\gamma$.

Proof. The previous lemma shows the forward direction. So assume $\alpha^\beta < \alpha^\gamma$ and not $\beta < \gamma$. If $\alpha = 1$, $\alpha^\beta = 1 = \alpha^{\beta}$.

By linearity, $\gamma \leq \beta$. If $\gamma = \beta$, $\alpha^\gamma = \alpha^\beta < \alpha^\gamma$. If $\gamma < \beta$, by the forward direction, $\alpha^\gamma < \alpha^\beta < \alpha^\gamma$.

Lemma 3.25. Let $0 < n \in \omega$. $n^\omega = \omega$.

Proof. $\omega$ is the smallest limit and $n^\omega$ is a limit by a lemma above. So $\omega \leq n^\omega$. Since $\omega$ is closed under exponentiation, for all $\alpha < \omega$, $n^\alpha \in \omega$. Then $n^\omega = \sup_{\alpha < \omega} n^\alpha \leq \omega$.

Remark 3.26. Right-Monotonicity does not hold: Define $3 = 2 + 1 \in \omega$. Clearly, $2 < 3$ and since $2^\omega = \omega$ and $3^\omega = \omega$, $2^\omega \neq 3^\omega$.

4. Associativity, Distributivity and Commutativity

Theorem 4.1. $+ \cdot$ and exponentiation are not commutative, i.e. there are $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ such that $\alpha + \beta \neq \beta + \alpha$, $\gamma \cdot \delta \neq \delta \cdot \gamma$ and $\varepsilon^\zeta \neq \zeta^\varepsilon$.

Proof. Let $\alpha = 1$, $\beta = \omega$, $\gamma = 2$, $\delta = \omega$, $\varepsilon = 0$, $\zeta = 1$.

$1 + \omega = \omega$ as shown above. $\omega \in \omega \cup \{\omega\} = \omega + 1$, so $\alpha + \beta < \beta + \alpha$.

$2 \cdot \omega = \omega$ as shown above. By Left-Monotonicity, $\omega < \omega + \omega = \omega \cdot 2$.

So $\gamma \cdot \delta < \delta \cdot \gamma$.

$0^\delta = 0^\delta \cdot 0 = 0$, but $1^0 = 1$ by definition. Hence $\varepsilon^\zeta \neq \zeta^\varepsilon$.

Theorem 4.2 (Associativity of Ordinal Addition). Let $\alpha, \beta, \gamma$ be ordinals. Then $(\alpha + \beta) + \gamma = \alpha + (\beta + \gamma)$.
Proof. By induction on \( \gamma \). \( \gamma = 0 \) is trivial. Suppose \( \gamma = \delta + 1 \).

\[
(\alpha + \beta) + (\delta + 1) = ((\alpha + \beta) + \delta) + 1 \quad \text{(by definition)}
\]
\[
= (\alpha + (\beta + \delta)) + 1 \quad \text{(by induction)}
\]
\[
= \alpha + ((\beta + \delta) + 1) \quad \text{(by definition)}
\]
\[
= \alpha + (\beta + (\delta + 1)) \quad \text{(by definition)}
\]
\[
= \alpha + (\beta + \gamma).
\]

Now suppose \( \gamma \) is a limit, in particular \( \gamma > 1 \). Then \( \beta + \gamma \) is a limit, so \( \alpha + (\beta + \gamma) \) and \( (\alpha + \beta) + \gamma \) are limits.

\[
(\alpha + \beta) + \gamma = \sup_{\varepsilon < \gamma} ((\alpha + \beta) + \varepsilon) \quad \text{(by definition)}
\]
\[
= \sup_{\beta + \varepsilon < \beta + \gamma} ((\alpha + \beta) + \varepsilon) \quad \text{(by Left-Monotonicity)}
\]
\[
= \sup_{\beta + \varepsilon < \beta + \gamma} (\alpha + (\beta + \varepsilon)) \quad \text{(by induction)}
\]
\[
= \sup_{\delta < \beta + \gamma} (\alpha + \delta) \quad \text{(see below)}
\]
\[
= \alpha + (\beta + \gamma) \quad \text{(by definition)}.
\]

Recall Lemma 2.12. Write \( B = \{\alpha + (\beta + \varepsilon) \mid \beta + \varepsilon < \beta + \gamma\} \) and \( A = \{\alpha + \delta \mid \delta < \beta + \gamma\} \). Clearly \( B \subseteq A \).

Let \( \alpha + \delta \in A \). Let \( \varepsilon = \min\{\zeta \mid \beta + \zeta \geq \delta\} \). Obviously \( \varepsilon \leq \gamma \). Assume \( \varepsilon = \gamma \), then for each \( \zeta < \gamma \), \( \beta + \zeta < \delta \). Then \( \delta < \beta + \gamma = \sup_{\zeta < \gamma} \beta + \zeta \leq \delta \). Hence, \( \varepsilon < \gamma \), i.e. \( \beta + \varepsilon < \beta + \gamma \). By construction, \( \delta < \beta + \varepsilon \). Thus, by Left-Monotonicity, \( \alpha + \delta \leq \alpha + (\beta + \varepsilon) \in B \). Thus, the conditions of Lemma 2.12 are satisfied. \( \square \)

**Theorem 4.3** (Distributivity). Let \( \alpha, \beta, \gamma \) be ordinals. Then \( \alpha \cdot (\beta + \gamma) = \alpha \cdot \beta + \alpha \cdot \gamma \).

Proof. Note that the theorem is trivial if \( \alpha = 0 \), so suppose \( \alpha > 0 \).

Proof by induction on \( \gamma \). \( \gamma = 0 \) is trivial. Suppose \( \gamma = \delta + 1 \).

\[
\alpha \cdot (\beta + (\delta + 1)) = \alpha \cdot ((\beta + \delta) + 1) \quad \text{(by definition)}
\]
\[
= \alpha \cdot (\beta + \delta) + \alpha \quad \text{(by definition)}
\]
\[
= \alpha \cdot \beta + \alpha \cdot \delta + \alpha \quad \text{(by induction)}
\]
\[
= \alpha \cdot \beta + \alpha \cdot (\delta + 1) \quad \text{(by definition)}.
\]

Suppose \( \gamma \) is a limit. Hence \( \alpha \cdot \gamma \) and \( \beta + \gamma \) are limits.
\[ \alpha \cdot (\beta + \gamma) = \sup_{\delta < \beta + \gamma} \alpha \cdot \delta \quad \text{(by definition)} \]
\[ = \sup_{\beta + \varepsilon < \beta + \gamma} (\alpha \cdot (\beta + \varepsilon)) \quad \text{(see below)} \]
\[ = \sup_{\varepsilon < \gamma} (\alpha \cdot (\beta + \varepsilon)) \quad \text{(by Left-Monotonicity)} \]
\[ = \sup_{\alpha \cdot \varepsilon < \alpha \cdot \gamma} (\alpha \cdot \beta + \alpha \cdot \varepsilon) \quad \text{(by induction)} \]
\[ = \sup_{\varepsilon < \gamma} (\alpha \cdot \beta + \alpha \cdot \varepsilon) \quad \text{(by Left-Monotonicity)} \]
\[ = \alpha \cdot \beta + \alpha \cdot \gamma \quad \text{(by definition)} \]

Recall Lemma 2.12. Write \( B = \{ \alpha \cdot (\beta + \varepsilon) \mid \beta + \varepsilon < \beta + \gamma \} \) and \( A = \{ \alpha \cdot \delta \mid \delta < \beta + \gamma \} \). Clearly \( B \subseteq A \). Let \( \alpha \cdot \delta \in A \). Let \( \varepsilon = \min \{ \eta \mid \beta + \eta \geq \delta \} \). Obviously \( \varepsilon \leq \gamma \). Assume \( \varepsilon = \gamma \), then for each \( \eta < \gamma \), \( \beta + \eta < \delta \). Then \( \delta < \beta + \gamma = \sup_{\eta < \gamma} \beta + \eta \leq \delta \frac{\gamma}{\varepsilon} \). Hence, \( \varepsilon < \gamma \), i.e. \( \beta + \varepsilon < \beta + \gamma \). By construction, \( \delta \leq \beta + \varepsilon \). Thus, by Left-Monotonicity, \( \alpha \cdot \delta \leq \alpha \cdot (\beta + \varepsilon) \in B \). Thus, the conditions of Lemma 2.12 are satisfied.

Write \( B = \{ \alpha \cdot \beta + \alpha \cdot \varepsilon \mid \alpha \cdot \varepsilon < \alpha \cdot \gamma \} \) and \( A = \{ \alpha \cdot \beta + \zeta \mid \zeta < \alpha \cdot \gamma \} \). Clearly \( B \subseteq A \). Let \( \alpha \cdot \beta + \zeta \in A \). Let \( \varepsilon = \min \{ \eta \mid \alpha \cdot \eta \geq \zeta \} \). Obviously \( \varepsilon \leq \gamma \). Assume \( \varepsilon = \gamma \), then for each \( \eta < \gamma \), \( \alpha \cdot \eta < \zeta \). Then \( \zeta < \alpha \cdot \gamma = \sup_{\eta < \gamma} \alpha \cdot \eta \leq \zeta \frac{\gamma}{\varepsilon} \). Hence, \( \varepsilon < \gamma \), i.e. \( \alpha \cdot \varepsilon < \alpha \cdot \gamma \). By construction, \( \zeta \leq \alpha \cdot \varepsilon \). Thus, by Left-Monotonicity, \( \alpha \cdot \beta + \zeta \leq \alpha \cdot \beta + \alpha \cdot \varepsilon \in B \). Thus, the conditions of Lemma 2.12 are satisfied.

**Theorem 4.4** (Associativity of Ordinal Multiplication). Let \( \alpha, \beta, \gamma \) be ordinals. Then \( (\alpha \cdot \beta) \cdot \gamma = \alpha \cdot (\beta \cdot \gamma) \).

**Proof.** Note that the theorem is trivial if \( \beta = 0 \). So suppose \( \beta > 0 \). Proof by induction on \( \gamma \). \( \gamma = 0 \) is trivial. Suppose \( \gamma = \delta + 1 \).

\[ (\alpha \cdot \beta) \cdot (\delta + 1) = ((\alpha \cdot \beta) \cdot \delta) + (\alpha \cdot \beta) \quad \text{(by definition)} \]
\[ = (\alpha \cdot (\beta \cdot \delta)) + (\alpha \cdot \beta) \quad \text{(by induction)} \]
\[ = \alpha \cdot ((\beta \cdot \delta) + \beta) \quad \text{(by Distributivity)} \]
\[ = \alpha \cdot (\beta \cdot (\delta + 1)) \quad \text{(by definition)} \]
\[ = \alpha \cdot (\beta \cdot \gamma) \].
Now suppose \( \gamma \) is a limit, in particular \( \gamma > 1 \). Then \( \beta \cdot \gamma \) is a limit, so \( \alpha \cdot (\beta \cdot \gamma) \) and \( (\alpha \cdot \beta) \cdot \gamma \) are limits.

\[
(\alpha \cdot \beta) \cdot \gamma = \sup_{\varepsilon < \gamma} ((\alpha \cdot \beta) \cdot \varepsilon) \quad \text{(by definition)}
\]

\[
= \sup_{\beta \varepsilon < \beta \cdot \gamma} ((\alpha \cdot \beta) \cdot \varepsilon) \quad \text{(by Left-Monotonicity)}
\]

\[
= \sup_{\delta < \beta \gamma} (\alpha \cdot (\beta \cdot \delta)) \quad \text{(by induction)}
\]

\[
= \alpha \cdot (\beta \cdot \gamma) \quad \text{(by definition)}.
\]

Recall Lemma 2.12. Write \( B = \{\alpha \cdot (\beta \cdot \varepsilon) \mid \beta \cdot \varepsilon < \beta \cdot \gamma\} \) and \( A = \{\alpha \cdot \delta \mid \delta < \beta \cdot \gamma\} \). Clearly \( B \subseteq A \). If \( A = \emptyset \), \( B = A \).

Let \( \alpha \cdot \delta \in A \). Let \( \varepsilon = \min\{\zeta \mid \beta \cdot \zeta \geq \delta\} \). Obviously \( \varepsilon \leq \gamma \). Assume \( \varepsilon = \gamma \), then for each \( \zeta < \gamma, \beta \cdot \zeta < \delta \). Then \( \delta < \beta \cdot \gamma = \sup_{\zeta < \gamma} \beta \cdot \zeta \leq \delta \). Hence, \( \varepsilon < \gamma \), i.e. \( \beta \cdot \varepsilon < \beta \cdot \gamma \). By construction, \( \delta \leq \beta \cdot \varepsilon \). Thus, by Left-Monotonicity, \( \alpha \cdot \delta \leq \alpha \cdot (\beta \cdot \varepsilon) \in B \). Thus, the conditions of Lemma 2.12 are satisfied.

\[ \square \]

**Notation 4.5.** As of now, we may omit bracketing ordinal addition and multiplication.

**Remark 4.6.** Ordinal exponentiation is not associative, i.e. there are \( \alpha, \beta, \gamma \) with \( \alpha^{(\beta \cdot \gamma)} \neq (\alpha^{\beta})^{\gamma} \).

**Proof.** Let \( \alpha = \omega, \beta = 1, \gamma = \omega \). Then \( \beta^{\gamma} = 1 \), i.e. \( \alpha^{(\beta^{\gamma})} = \alpha^1 = \omega \). But \( \alpha^{\beta} = \omega \), hence \( (\alpha^{\beta})^{\gamma} = \omega^{\omega} \). And \( \omega < \omega^{\omega} \) by Left-Monotonicity. \( \square \)

**Theorem 4.7.** Let \( \alpha, \beta, \gamma \) be ordinals. Then \( \alpha^{\beta + \gamma} = \alpha^\beta \cdot \alpha^\gamma \).

**Proof.** Recall that \( \beta + \gamma = 0 \) iff \( \beta = \gamma = 0 \), so the theorem holds for \( \alpha = 0 \). Also note that the theorem is trivial for \( \alpha = 1 \), so suppose \( \alpha > 1 \). Proof by induction on \( \gamma \). \( \gamma = 0 \) is trivial. Suppose \( \gamma = \delta + 1 \).

\[
\alpha^{\beta + \delta + 1} = \alpha^{\beta + \delta} \cdot \alpha \quad \text{(by definition)}
\]

\[
= \alpha^\beta \cdot \alpha^\delta \cdot \alpha \quad \text{(by induction)}
\]

\[
= \alpha^\beta \cdot \alpha^{\delta + 1} \quad \text{(by definition)}.
\]
ORDINAL ARITHMETIC

Suppose $\gamma$ is a limit. Then $\alpha^\gamma$ and $\alpha^{\beta+\gamma}$ are limits.

$$\alpha^{\beta+\gamma} = \sup_{\delta<\beta+\gamma} \alpha^\delta \quad \text{(by definition)}$$

$$= \sup_{\beta+\varepsilon<\beta+\gamma} \alpha^{\beta+\varepsilon} \quad \text{(see below)}$$

$$= \sup_{\varepsilon<\gamma} \alpha^{\beta+\varepsilon} \quad \text{(by Left-Monotonicity)}$$

$$= \sup_{\varepsilon<\gamma} (\alpha^{\beta} \cdot \alpha^{\varepsilon}) \quad \text{(by induction)}$$

$$= \sup_{\zeta<\alpha^{\gamma}} (\alpha^{\beta} \cdot \zeta) \quad \text{(see below)}$$

$$= \alpha^{\beta} + \alpha^{\gamma} \quad \text{(by definition).}$$

Recall Lemma 2.12. Write $B = \{\alpha^{\beta+\varepsilon} \mid \beta+\varepsilon < \beta+\gamma\}$ and $A = \{\alpha^{\delta} \mid \delta < \beta+\gamma\}$. Clearly $B \subseteq A$. Let $\alpha^{\delta} \in A$. Let $\varepsilon = \min \{\eta \mid \beta+\eta \geq \delta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\beta+\eta < \beta+\gamma$. Thus, $\varepsilon < \gamma$, i.e. $\beta+\varepsilon < \beta+\gamma$. By construction, $\delta \leq \beta+\varepsilon$. Hence, $\delta < \beta+\gamma = \sup_{\eta<\gamma} \beta+\eta \leq \delta^{\frac{1}{\gamma}}$. Hence, $\varepsilon < \gamma$, i.e. $\beta+\varepsilon < \beta+\gamma$. By Left-Monotonicity, $\alpha^{\delta} \leq \alpha^{\beta+\varepsilon} \in B$. Thus, the conditions of Lemma 2.12 are satisfied.

Write $B = \{\alpha^{\beta} \cdot \alpha^{\varepsilon} \mid \alpha^{\varepsilon} < \alpha^{\gamma}\}$ and $A = \{\alpha^{\beta} \cdot \zeta \mid \zeta < \alpha^{\gamma}\}$. Clearly $B \subseteq A$. Let $\alpha^{\beta} \cdot \zeta \in A$. Let $\varepsilon = \min \{\eta \mid \alpha^{\beta} \cdot \eta \geq \zeta\}$. Obviously $\varepsilon \leq \gamma$. Assume $\varepsilon = \gamma$, then for each $\eta < \gamma$, $\alpha^{\beta} \cdot \eta < \zeta$. Then $\zeta < \alpha^{\gamma} = \sup_{\eta<\gamma} \alpha^{\eta} \leq \eta^{\frac{1}{\gamma}}$. Hence, $\varepsilon < \gamma$, i.e. $\alpha^{\varepsilon} \leq \alpha^{\gamma}$. By construction, $\zeta \leq \alpha^{\beta} \cdot \varepsilon$. Thus, by Left-Monotonicity, $\alpha^{\beta} \cdot \zeta \leq \alpha^{\beta} + \alpha^{\varepsilon} \in B$. Thus, the conditions of Lemma 2.12 are satisfied. \qed

5. THE CANTOR NORMAL FORM

Lemma 5.1. If $\alpha < \beta$ and $n, m \in \omega \setminus \{0\}$, $\omega^{\alpha} \cdot n < \omega^{\beta} \cdot m$.

Proof. $\alpha + 1 \leq \beta$, so $\omega^{\alpha+1} \leq \omega^{\beta}$ by Left-Monotonicity (of exponentiation). Hence (by Left-Monotonicity of multiplication), $\omega^{\alpha} \cdot n < \omega^{\alpha} \cdot \omega = \omega^{\alpha+1} \leq \omega^{\beta} \leq \omega^{\beta} \cdot m$. \qed

Lemma 5.2. If $\alpha_0 > \alpha_1 > \ldots > \alpha_n$, and $m_1, \ldots, m_n \in \omega$, then $\omega^{\alpha_0} > \sum_{1 \leq i \leq n} \omega^{\alpha_i} \cdot m_i$.

Proof. If any $m_i = 0$ it may just be omitted from the sum. So suppose all $m_i > 0$. $n = 0$ and $n = 1$ are the trivial cases. Consider $n = 2$:

$\omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_2} \cdot m_2 \leq \omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_1} \cdot m_1$ by the lemma above and Left-Monotonicity of addition. Then again by the previous lemma $\omega^{\alpha_1} \cdot m_1 \cdot 2 < \omega^{\alpha_0}$.
Continue via induction: Suppose the lemma holds for \( n \). Then consider the sequence \( \alpha_1, \ldots, \alpha_n \). It follows that \( \sum_{2 \leq i \leq n+1} \omega^{\alpha_i} \cdot m_i < \omega^{\alpha_1} \).

By the \( n = 2 \) case, \( \omega^{\alpha_0} > \omega^{\alpha_1} \cdot m_1 + \omega^{\alpha_1} \) and by Left-Monotonicity of addition, \( \omega^{\alpha_0} \cdot m_1 + \omega^{\alpha_1} > \sum_{1 \leq i \leq n} \omega^{\alpha_i} \cdot m_i \).

\( \square \)

**Theorem 5.3 (Cantor Normal Form (CNF)).** For every ordinal \( \alpha \), there is a unique \( k \in \omega \) and unique tuples \( (m_0, \ldots, m_k) \in (\omega \setminus \{0\})^k \), \( (\alpha_0, \ldots, \alpha_k) \) of ordinals with \( \alpha_0 > \ldots > \alpha_k \) such that:

\[
\alpha = \omega^{\alpha_0} \cdot m_0 + \ldots + \omega^{\alpha_k} \cdot m_k
\]

**Proof.** Existence by induction on \( \alpha \): If \( \alpha = 0 \), then \( k = 0 \). Suppose that every \( \beta < \alpha \) has a CNF. Let \( \hat{\alpha} = \sup \{ \gamma \mid \omega^\gamma \leq \alpha \} \) and let \( \hat{m} = \sup \{ m \in \omega \mid \omega^\alpha \cdot m \leq \alpha \} \). Note that \( \omega^{\hat{\alpha}} \leq \alpha \): If not, then \( \alpha \in \omega^{\hat{\alpha}} \).

Then there is \( \gamma, \omega^\gamma \leq \alpha \) with \( \alpha \in \gamma \). But since \( \omega^{\alpha+1} > \omega^\alpha \geq \alpha \), \( \gamma < \alpha + 1 \), i.e. \( \gamma \leq \alpha \).

Also note that \( \hat{m} \in \omega \): If not, then \( \hat{m} = \omega \), hence: \( \alpha < \omega^{\hat{\alpha}+1} = \omega^{\hat{\alpha}} \cdot \omega = \sup_{n \in \omega} \omega^{\hat{\alpha}} \cdot n \leq \alpha \).

By construction, \( \omega^{\hat{\alpha}} \cdot \hat{m} \leq \alpha \), so there is \( \varepsilon \leq \alpha \) with \( \alpha = \omega^{\hat{\alpha}} \cdot \hat{m} + \varepsilon \).

Show that \( \varepsilon < \alpha \): Suppose not, then \( \varepsilon \geq \alpha \), hence \( \varepsilon \geq \omega^{\hat{\alpha}} \), so there is \( \zeta < \varepsilon \) with \( \varepsilon = \omega^{\hat{\alpha}} + \zeta \), i.e. \( \alpha = \omega^{\hat{\alpha}} \cdot \hat{m} + \omega^{\hat{\alpha}} + \zeta \).

By left-distributivity, \( \alpha = \omega^{\hat{\alpha}} \cdot (\hat{m} + 1) + \zeta \geq \omega^{\hat{\alpha}} \cdot (\hat{m} + 1) \), contradicting the choice of \( \hat{m} \).

Thus, by induction, \( \varepsilon \) has a CNF \( \sum_{i \leq l} \omega^{\beta_i} \cdot n_i \). Note that \( \beta_0 \leq \alpha \): If not, \( \beta_0 > \alpha \), i.e. by the choice of \( \hat{\alpha} \), \( \omega^{\beta_0} > \alpha \), so \( \varepsilon \geq \omega^{\beta_0} > \alpha \).

Now state the CNF of \( \alpha \): If \( \beta_0 < \alpha \) set \( k = l + 1 \), \( \alpha_0 = \hat{\alpha}, m_0 = \hat{m} \) and \( \alpha_i = \beta_{i-1}, m_i = n_{i-1} \) for \( 1 \leq i \leq k \). And if \( \beta_0 = \hat{\alpha} \) set \( k = l \), \( m_0 = n_0 + \hat{m} \), \( \alpha_0 = \hat{\alpha} \) and \( \alpha_i = \beta_i, m_i = n_i \) for \( 1 \leq i \leq k \).

**Uniqueness:** Suppose not and let \( \alpha \) be the minimal counterexample.

Let \( \alpha = \omega^{\alpha_0} \cdot m_0 + \ldots + \omega^{\alpha_m} \cdot m_m = \omega^{\beta_0} \cdot n_0 + \ldots + \omega^{\beta_n} \cdot n_n \). Obviously \( \alpha > 0 \), i.e. the sums are not empty.

Show \( \alpha_0 = \beta_0 \): Suppose not, wlog assume \( \alpha_0 > \beta_0 \). Consider the previous lemma. Then \( \alpha \geq \omega^{\alpha_0} \cdot m_0 > \omega^{\beta_0} \cdot n_0 + \ldots + \omega^{\beta_n} \cdot n_n = \alpha_k \).

Then show \( m_0 = n_0 \): Suppose not, wlog assume \( m_0 < n_0 \). Then, again by the previous lemma, \( \omega^{\alpha_0} > \sum_{1 \leq i \leq m} \omega^{\alpha_i} \cdot m_i \). So, by Left-Monotonicity of addition, \( \alpha < \omega^{\alpha_0} \cdot m_0 + \omega^{\alpha_0} \), i.e. \( \alpha < \omega^{\alpha_0} \cdot (m_0 + 1) \leq \omega^{\alpha_0} \cdot n_0 \leq \alpha_k \).

So \( \omega^{\alpha_0} \cdot m_0 = \omega^{\beta_0} \cdot n_0 \), so by Left-Monotonicity, \( \omega^{\alpha_1} \cdot m_1 + \ldots + \omega^{\alpha_m} \cdot m_m = \omega^{\beta_1} \cdot n_1 + \ldots + \omega^{\beta_n} \cdot n_n \). These terms are strictly smaller than \( \alpha \) by the previous lemma. By minimality of \( \alpha, m = n \), and the \( \alpha \)'s, \( \beta \)'s, \( m \)'s and \( n \)'s are equal. Thus \( \alpha \) has a unique CNF \( \hat{\alpha} \). \( \square \)