# Set Theory

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*Die Mengenlehre ist das Fundament*
*der gesamten Mathematik*
(Felix Hausdorff,
*Grundzüge der Mengenlehre, 1914*)

## 1 Introduction

Georg Cantor characterized sets as follows:

> Unter einer *Menge* verstehen wir jede Zusammenfassung $M$ von bestimmten, wohlunterschiedenen Objekten $m$ unsrer Anschauung oder unseres Denkens (welche die "Elemente" von $M$ genannt werden) zu einem Ganzen.

Felix Hausdorff in *Grundzüge* formulated shorter:

> Eine Menge ist eine Zusammenfassung von Dingen zu einem Ganzen, d.h. zu einem neuen Ding.

Sets are ubiquitous in mathematics. According to Hausdorff

> Differential- und Integralrechnung, Analysis und Geometrie arbeiten in Wirklichkeit, wenn auch vielleicht in verschleiernder Ausdrucksweise, beständig mit unendlichen Mengen.

In current mathematics, *many* notions are explicitly defined using sets. The following example indicates that notions which are not set-theoretical *prima facie* can be construed set-theoretically:

> $f$ is a real funktion $\equiv$ $f$ is a **set** of ordered pairs $(x, f(x))$ of real numbers, such that ... ;
>
> $(x, y)$ is an ordered pair $\equiv$ $(x, y)$ is a **set** ...$\{x, y\}$... ;
>
> $x$ is a real number $\equiv$ $x$ is a left half of a Dedekind cut in $\mathbb{Q}$ $\equiv$ $x$ is a **subset** of $\mathbb{Q}$, such that ... ;
>
> $r$ is a rational number $\equiv$ $r$ is an **ordered pair** of integers, such that ... ;
>
> $z$ is an integer $\equiv$ $z$ is an **ordered pair** of natural numbers (= non-negative integers);
>
> $\mathbb{N} = \{0, 1, 2, ...\}$;
>
> 0 is the empty **set**;
>
> 1 is the **set** $\{0\}$;
>
> 2 is the **set** $\{0, 1\}$; etc. etc.

We shall see that *all* mathematical notions can be reduced to the notion of *set*.

Besides this foundational role, set theory is also the mathematical study of the *infinite*. There are infinite sets like $\mathbb{N}$, $\mathbb{Q}$, $\mathbb{R}$ which can be subjected to the constructions and analyses of set theory; there are various degrees of infinity which lead to a rich theory of infinitary combinatorics.

In this course, we shall first apply set theory to obtain the standard foundation of mathematics and then turn towards "pure" set theory.

## 2 The Language of Set Theory

If $m$ is an *element* of $M$ one writes $m \in M$. If all mathematical objects are reducible to sets, *both sides* of these relation have to be sets. This means that set theory studies the $\in$-relation $m \in M$ for arbitrary *sets* $m$ and $M$. As it turns out, this is sufficient for the purposes of set theory and mathematics. In set theory variables range over the class of all sets, the $\in$-relation is the only undefined structural component, every other notion will be defined from the $\in$-relation. Basically, set theoretical statement will thus be of the form

$$...\forall x...\exists y......x \in y...u \equiv v...,$$

belonging to the first-order predicate language with the only given predicate $\in$.

To deal with the complexities of set theory and mathematics one develops a comprehensive and intuitive language of abbreviations and definitions which, eventually, allows to write familiar statements like

$$e^{i\pi} = -1$$

and to view them as statements within set theory.

The language of set theory may be seen as a low-level, internal language. The language of mathematics possesses high-level "macro" expressions which abbreviate low-level statements in an efficient and intuitive way.

## 3 RUSSELL's Paradox

CANTOR's naive description of the notion of set suggests that for any mathematical statement $\varphi(x)$ in one free variable $x$ there is a *set* $y$ such that

$$x \in y \leftrightarrow \varphi(x),$$

i.e., $y$ is the collection of all sets $x$ which satisfy $\varphi$.

This axiom is a basic principle in GOTTLOB FREGE's *Grundgesetze der Arithmetik, 1893*, Grundgesetz V, Grundgesetz der Wertverläufe.

BERTRAND RUSSELL noted in 1902 that setting $\varphi(x)$ to be $x \notin x$ this becomes

$$x \in y \leftrightarrow x \notin x,$$

and in particular for $x = y$:

$$y \in y \leftrightarrow y \notin y.$$

Contradiction.

This contradiction is usually called RUSSELL's paradox, antinomy, contradiction. It was also discovered slightly earlier by ERNST ZERMELO. The paradox shows that the formation of sets as collections of sets by *arbitrary* formulas is not consistent.

## 4 The ZERMELO-FRAENKEL Axioms

The difficulties around RUSSELL's paradox and also around the axiom of choice lead ZERMELO to the formulation of axioms for set theory in the spirit of the axiomatics of DAVID HILBERT of whom ZERMELO was an assistant at the time.

ZERMELO's main idea was to restrict FREGE's Axiom V to formulas which correspond to mathematically important formations of collections, but to avoid arbitrary formulas which can lead to paradoxes like the one exhibited by RUSSELL.

The original axiom system of ZERMELO was extended and detailed by ABRAHAM FRAENKEL (1922), DMITRY MIRIMANOFF (1917/20), and THORALF SKOLEM.

We shall discuss the axioms one by one and simultaneously introduce the logical language and useful conventions.

## 4.1 Set Existence

The *set existence axiom*

$$\exists x \forall y \neg y \in x,$$

like all axioms, is expressed in a language with quantifiers $\exists$ (“there exists”) and $\forall$ (“for all”), which is familiar from the $\epsilon$-$\delta$-statements in analysis. The *language of set theory* uses variables $x$, $y$, ... which may satisfy the binary relations $\in$ or $=$: $x \in y$ (“$x$ is an *element of* $y$”) or $x = y$. These elementary *formulas* may be connected by the *propositional connectives* $\wedge$ (“and”), $\vee$ (“or”), $\rightarrow$ (“implies”), $\leftrightarrow$ (“is equivalent”), and $\neg$ (“not”). The use of this language will be demonstrated by the subsequent axioms.

The axiom expresses the existence of a set which has no elements, i.e., the existence of the *empty set*.

## 4.2 Extensionality

The *axiom of extensionality*

$$\forall x \forall x' (\forall y (y \in x \leftrightarrow y \in x') \rightarrow x = x')$$

expresses that a set is exactly determined by the collection of its elements. This allows to prove that there is exactly one empty set.

**Lemma 1.** $\forall x \forall x' (\forall y \neg y \in x \wedge \forall y \neg y \in x' \rightarrow x = x')$.

**Proof.** Consider $x, x'$ such that $\forall y \neg y \in x \wedge \forall y \neg y \in x'$. Consider $y$. Then $\neg y \in x$ and $\neg y \in x'$. This implies $\forall y (y \in x \leftrightarrow y \in x')$. The axiom of extensionality implies $x = x'$. □

Note that this proof is a usual mathematical argument, and it is also a *formal proof* in the sense of mathematical logic. The sentences of the proof can be derived from earlier ones by purely formal deduction rules. The rules of natural deduction correspond to common sense figures of argumentation which treat hypothetical objects as if they would concretely exist.

## 4.3 Pairing

The *pairing axiom*

$$\forall x \forall y \exists z \forall u (u \in z \leftrightarrow u = x \vee u = y)$$

postulates that for all sets $x, y$ there is set $z$ which may be denoted as

$$z = \{x, y\}.$$

This formula, including the new notation, is equivalent to the formula

$$\forall u (u \in z \leftrightarrow u = x \vee u = y).$$

In the sequel we shall extend the small language of set theory by hundreds of symbols and conventions, in order to get to the ordinary language of mathematics with notations like

$$\mathbb{N}, \mathbb{R}, \sqrt{385}, \pi, \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \int_a^b f'(x) \mathrm{dx} = f(b) - f(a), \text{etc.}$$

Such notations are chosen for intuitive, pragmatic, or historical reasons.

Using the notation for unordered pairs, the pairing axiom may be written as

$$\forall x \forall y \exists z \, z = \{x, y\}.$$

By the axiom of extensionality, the term-like notation has the expected behaviour. E.g.:

**Lemma 2.** $\forall x \forall y \forall z \forall z' (z = \{x, y\} \wedge z' = \{x, y\} \rightarrow z = z')$.

**Proof.** Exercise. □

Note that we implicitly use several notational conventions: variables have to be chosen in a reasonable way, for example the symbols $z$ and $z'$ in the lemma have to be taken different and different from $x$ and $y$. We also assume some operator priorities to reduce the number of brackets: we let $\wedge$ bind stronger than $\vee$, and $\vee$ stronger than $\rightarrow$ and $\leftrightarrow$.

We used the "term" $\{x, y\}$ to occur within set theoretical formulas. This abbreviation is than to be expanded in a natural way, so that officially all mathematical formulas are formulas in the "pure" $\in$-language. We want to see the notation $\{x, y\}$ as an example of a *class term*. We define uniform notations and convention for such abbreviation terms.

## 4.4 Class Terms

The extended language of set theory contains class terms and notations for them. There are axioms for class terms that fix how extended formulas can be reduced to formulas in the unextended $\in$-language of set theory.

**Definition 3.** *A* class term *is of the form* $\{x|\varphi\}$ *where* $x$ *is a variable and* $\varphi \in L^{\in}$*. The usage of these class terms is defined recursively by the following axioms: If* $\{x|\varphi\}$ *and* $\{y|\psi\}$ *are class terms then*

- $u \in \{x|\varphi\} \leftrightarrow \varphi\frac{u}{x}$, *where* $\varphi\frac{u}{x}$ *is obtained from* $\varphi$ *by (resonably) substituting the variable* $x$ *by the variable* $u$ *;*
- $u = \{x|\varphi\} \leftrightarrow \forall v\, (v \in u \leftrightarrow \varphi\frac{v}{x})$*;*
- $\{x|\varphi\} = u \leftrightarrow \forall v\, (\varphi\frac{v}{x} \leftrightarrow v \in u)$*;*
- $\{x|\varphi\} = \{y|\psi\} \leftrightarrow \forall v\, (\varphi\frac{v}{x} \leftrightarrow \psi\frac{v}{y})$*;*
- $\{x|\varphi\} \in u \leftrightarrow \exists v(v \in u \wedge v = \{x|\varphi\}$*;*
- $\{x|\varphi\} \in \{y|\psi\} \leftrightarrow \exists v(\psi\frac{v}{y} \wedge v = \{x|\varphi\})$*.*

*A* term *is either a variable or a class term.*

**Definition 4.**

*a)* $\emptyset := \{x|x \neq x\}$ *is the* empty set*;*

*b)* $V := \{x|x = x\}$ *is the* universe *(of all sets);*

*c)* $\{x, y\} := \{u|u = x \vee u = y\}$ *is the* unordered pair *of* $x$ *and* $y$ *.*

**Lemma 5.**

*a)* $\emptyset \in V$*.*

*b)* $\forall x, y\ \{x, y\} \in V$*.*

**Proof.** a) By the axioms for the reduction of abstraction terms, $\emptyset \in V$ is equivalent to the following formulas

$$\exists v(v = v \wedge v = \emptyset)$$
$$\exists v\, v = \emptyset$$
$$\exists v\, \forall w\, (w \in v \leftrightarrow w \neq w)$$
$$\exists v \forall w\, w \notin v$$

which is equivalent to the axiom of set existence. So $\emptyset \in V$ is another way to write the axiom of set existence.
b) $\forall x, y\ \{x, y\} \in V$ abbreviates the formula

$$\forall x, y \exists z(z = z \wedge z = \{x, y\}).$$

This can be expanded equivalently to the pairing axiom

$$\forall x, y \exists z \forall u(u \in z \leftrightarrow u = x \vee u = y).$$

□

So a) and b) are concise equivalent formulations of the axiom Ex and Pair.

We also introduce *bounded quantifiers* to simplify notation.

**Definition 6.** *Let $A$ be a term. Then* $\forall x \in A\, \varphi \leftrightarrow \forall x (x \in A \rightarrow \varphi)$ *and* $\exists x \in A\, \varphi \leftrightarrow \exists x\, (x \in A \wedge \varphi)$.

**Definition 7.** *Let $x, y, z, \ldots$ be variables and $X, Y, Z, \ldots$ be class terms. Define*

*a)* $X \subseteq Y \leftrightarrow \forall x \in X\, x \in Y$, *$X$ is a* subclass *of $Y$;*

*b)* $X \cup Y := \{x | x \in X \vee x \in Y\}$ *is the* union *of $X$ and $Y$;*

*c)* $X \cap Y := \{x | x \in X \wedge x \in Y\}$ *is the* intersection *of $X$ and $Y$;*

*d)* $X \setminus Y := \{x | x \in X \wedge x \notin Y\}$ *is the* difference *of $X$ and $Y$;*

*e)* $\bigcup X := \{x | \exists y \in X\, x \in y\}$ *is the* union *of $X$;*

*f)* $\bigcap X := \{x | \forall y \in X\, x \in y\}$ *is the* intersection *of $X$;*

*g)* $\mathcal{P}(X) := \{x | x \subseteq X\}$ *is the* power class *of $X$;*

*h)* $\{X\} := \{x | x = X\}$ *is the* singleton set *of $X$;*

*i)* $\{X, Y\} := \{x | x = X \vee x = Y\}$ *is the* (unordered) pair *of $X$ and $Y$;*

*j)* $\{X_0, \ldots, X_{n-1}\} := \{x | x = X_0 \vee \ldots \vee x = X_{n-1}\}$.

One can prove the well-known boolean properties for these operations. We only give a few examples.

**Proposition 8.** $X \subseteq Y \wedge Y \subseteq X \rightarrow X = Y$.

**Proposition 9.** $\bigcup \{x, y\} = x \cup y$.

**Proof.** We show the equality by two inclusions:
($\subseteq$). Let $u \in \bigcup \{x, y\}$. $\exists v (v \in \{x, y\} \wedge u \in v)$. Let $v \in \{x, y\} \wedge u \in v$. $(v = x \vee v = y) \wedge u \in v$.
*Case 1.* $v = x$. Then $u \in x$. $u \in x \vee u \in y$. Hence $u \in x \cup y$.
*Case 2.* $v = y$. Then $u \in y$. $u \in x \vee u \in y$. Hence $u \in x \cup y$.

Conversely let $u \in x \cup y$. $u \in x \vee u \in y$.
*Case 1.* $u \in x$. Then $x \in \{x, y\} \wedge u \in x$. $\exists v (v \in \{x, y\} \wedge u \in v)$ and $u \in \bigcup \{x, y\}$.
*Case 2.* $u \in y$. Then $x \in \{x, y\} \wedge u \in x$. $\exists v (v \in \{x, y\} \wedge u \in v)$ and $u \in \bigcup \{x, y\}$. □

**Exercise 1.** Show: a) $\bigcup V = V$. b) $\bigcap V = \emptyset$. c) $\bigcup \emptyset = \emptyset$. d) $\bigcap \emptyset = V$.

## 4.5 Ordered Pairs

Combining objects into ordered pairs $(x, y)$ is taken as an undefined fundamental operation of mathematics. We cannot use the unordered pair $\{x, y\}$ for this purpose, since it does not respect the order of entries:

$$\{x, y\} = \{y, x\}.$$

We have to introduce some asymmetry between $x$ and $y$ to make them distinguishable. Following KURATOWSKI and WIENER we define:

**Definition 10.** $(x, y) := \{\{x\}, \{x, y\}\}$ *is the* ordered pair *of $x$ and $y$.*

The definition involves substituting class terms within class terms. We shall see in the following how these class terms are eliminated to yield pure $\in$-formulas.

**Lemma 11.** $\forall x \forall y \exists z\, z = (x, y)$.

**Proof.** Consider sets $x$ and $y$. By the pairing axiom choose $u$ and $v$ such that $u = \{x\}$ and $v = \{x, y\}$. Again by pairing choose $z$ such that $z = \{u, v\}$. We argue that $z = (x, y)$. Note that

$(x, y) = \{\{x\}, \{x, y\}\} = \{w | w = \{x\} \vee w = \{x, y\}\}$.

Then $z = (x, y)$ is equivalent to

$\forall w (w \in z \leftrightarrow w = \{x\} \vee w = \{x, y\})$,

$\forall w (w = u \vee w = v \leftrightarrow (w = \{x\} \vee w = \{x, y\})$,
and this is true by the choice of $u$ and $v$. □

The KURATOWSKI-pair satisfies the fundamental property of ordered pairs:

**Lemma 12.** $(x, y) = (x', y') \rightarrow x = x' \wedge y = y'$.

**Proof.** Assume $(x, y) = (x', y')$, i.e.,
(1) $\{\{x\}, \{x, y\}\} = \{\{x'\}, \{x', y'\}\}$.
*Case 1.* $x = y$. Then
$\{x\} = \{x, y\}$,
$\{\{x\}, \{x, y\}\} = \{\{x\}, \{x\}\} = \{\{x\}\}$,
$\{\{x\}\} = \{\{x'\}, \{x', y'\}\}$,
$\{x\} = \{x'\}$ and $x = x'$,
$\{x\} = \{x', y'\}$ and $y' = x$.
Hence $x = x'$ and $y = x = y'$ as required.
*Case 2.* $x \neq y$. (1) implies
$\{x'\} = \{x\}$ or $\{x'\} = \{x, y\}$.
The right-hand side would imply $x = x' = y$, contradicting the case assumption. Hence
$\{x'\} = \{x\}$ and $x' = x$.
Then (1) implies
$\{x, y\} = \{x', y'\} = \{x, y'\}$ and $y = y'$. □

**Exercise 2.**

a) Show that $\langle x, y \rangle := \{\{x, \emptyset\}, \{y, \{\emptyset\}\}\}$ also satisfies the fundamental property of ordered pairs (F. HAUSDORFF).
b) Can $\{x, \{y, \emptyset\}\}$ be used as an ordered pair?

**Exercise 3.** Give a set-theoretical formalization of an ordered-*triple* operation.

## 4.6 Relations and Functions

Ordered pairs allow to introduce *relations* and *functions* in the usual way. One has to distinguish between *sets* which are relations and functions, and *class terms* which are relations and functions.

**Definition 13.** *A term $R$ is a* relation *if all elements of $R$ are ordered pairs, i.e., $R \subseteq V \times V$. Also write $Rxy$ or $x R y$ instead of $(x, y) \in R$. If $A$ is a term and $R \subseteq A \times A$ then $R$ is a* relation *on $A$.*

Note that this definition is really an *infinite schema* of definitions, with instances for all terms $R$ and $A$. The subsequent extensions of our language are also infinite definition schemas. We extend the term language by parametrized collections of terms.

**Definition 14.** *Let $t(\vec{x})$ be a term in the variables $\vec{x}$ and let $\varphi$ be an $\in$-formula. Then $\{t(\vec{x}) | \varphi\}$ stands for $\{z | \exists \vec{x} (\varphi \wedge z = t(\vec{x}))\}$.*

**Definition 15.** *Let $R, S, A$ be terms.*

*a) The* domain *of $R$ is* $\mathrm{dom}(R) := \{x | \exists y\, x R y\}$.
*b) The* range *of $R$ is* $\mathrm{ran}(R) := \{y | \exists x\, x R y\}$.
*c) The* field *of $R$ is* $\mathrm{field}(R) := \mathrm{dom}(R) \cup \mathrm{ran}(R)$.
*d) The* restriction *of $R$ to $A$ is* $R \upharpoonright A := \{(x, y) | x R y \wedge x \in A\}$.
*e) The* image *of $A$ under $R$ is* $R[A] := R''A := \{y | \exists x \in A\, x R y\}$.
*f) The* preimage *of $A$ under $R$ is* $R^{-1}[A] := \{x | \exists y \in A\, x R y\}$.
*g) The* composition *of $S$ and $R$ ("$S$ after $R$") is* $S \circ R := \{(x, z) | \exists y\, (x R y \wedge y S z)\}$.
*h) The* inverse *of $R$ is* $R^{-1} := \{(y, x) | x R y\}$.

Relations can play different roles in mathematics.

**Definition 16.** *Let $R$ be a relation.*

a) *$R$ is* reflexive *iff* $\forall x \in \text{field}(R)\ x R x$ .

b) *$R$ is* irreflexive *iff* $\forall x \in \text{field}(R)\ \neg x R x$ .

c) *$R$ is* symmetric *iff* $\forall x, y\, (x R y \rightarrow y R x)$.

d) *$R$ is* antisymmetric *iff* $\forall x, y\, (x R y \wedge y R x \rightarrow x = y)$.

e) *$R$ is* transitive *iff* $\forall x, y, z\ (x R y \wedge y R z \rightarrow x R z)$.

f) *$R$ is* connex *iff* $\forall x, y \in \text{field}(R)\, (x R y \vee y R x \vee x = y)$.

g) *$R$ is an* equivalence relation *iff $R$ is reflexive, symmetric and transitive.*

h) *Let $R$ be an equivalence relation. Then* $[x]_R := \{y | y R x\}$ *is the* equivalence class *of $x$ modulo $R$.*

It is possible that an equivalence class $[x]_R$ is not a set: $[x]_R \notin V$ . Then the formation of the collection of all equivalence classes modulo $R$ may lead to contradictions. Another important family of relations is given by *order relations*.

**Definition 17.** *Let $R$ be a relation.*

a) *$R$ is a* partial order *iff $R$ is reflexive, transitive and antisymmetric.*

b) *$R$ is a* linear order *iff $R$ is a connex partial order.*

c) *Let $A$ be a term. Then $R$ is a* partial order on $A$ *iff $R$ is a partial order and* $\text{field}(R) = A$ .

d) *$R$ is a* strict partial order *iff $R$ is transitive and irreflexive.*

e) *$R$ is a* strict linear order *iff $R$ is a connex strict partial order.*

Partial orders are often denoted by symbols like $\leqslant$, and strict partial orders by $<$. A common notation in the context of (strict) partial orders $R$ is to write

$$\exists p R q \varphi \text{ and } \forall p R q \varphi \text{ for } \exists p (p R q \wedge \varphi) \text{ and } \forall p (p R q \rightarrow \varphi) \text{ resp.}$$

One of the most important notions in mathematics is that of a *function*.

**Definition 18.** *Let $F$ be a term. Then $F$ is a* function *if it is a relation which satisfies*

$$\forall x, y, y'\ (x F y \wedge x F y' \rightarrow y = y').$$

*If $F$ is a function then*

$$F(x) := \{u | \forall y\, (x F y \rightarrow u \in y)\}$$

*is the* value *of $F$ at $x$.*

If $F$ is a function and $x F y$ then $y = F(x)$. If there is no $y$ such that $x F y$ then $F(x) = V$; the "value" $V$ at $x$ may be read as "undefined". A function can also be considered as the (indexed) sequence of its values, and we also write

$$(F(x))_{x \in A} \text{ or } (F_x)_{x \in A} \text{ instead of } F: A \rightarrow V.$$

We define further notions associated with functions.

**Definition 19.** *Let $F, A, B$ be terms.*

a) *$F$ is a* function from $A$ to $B$, *or* $F: A \rightarrow B$, *iff $F$ is a function,* $\text{dom}(F) = A$, *and* $\text{range}(F) \subseteq B$ .

b) *$F$ is a* partial function from $A$ to $B$, *or* $F: A \rightharpoonup B$, *iff $F$ is a function,* $\text{dom}(F) \subseteq A$, *and* $\text{range}(F) \subseteq B$ .

c) *$F$ is a* surjective *function from $A$ to $B$ iff $F: A \rightarrow B$ and* $\text{range}(F) = B$.

d) *F is an* injective *function from A to B iff* $F: A \to B$ *and*

$$\forall x, x' \in A\ (x \neq x' \to F(x) \neq F(x'))$$

e) *F is a* bijective *function from A* to *B, or* $F: A \leftrightarrow B$*, iff* $F: A \to B$ *is surjective and injective.*

f) ${}^A B := \{f \mid f: A \to B\}$ *is the class of all functions from A to B.*

One can check that these functional notions are consistent and agree with common usage:

**Exercise 4.** Define a relation $\sim$ on $V$ by

$$x \sim y \longleftrightarrow \exists f\, f: x \leftrightarrow y.$$

One say that $x$ and $y$ are *equinumerous* or *equipollent*. Show that $\sim$ is an equivalence relation on $V$. What is the equivalence class of $\emptyset$? What is the equivalence class of $\{\emptyset\}$?

**Exercise 5.** Consider functions $F: A \to B$ and $F': A \to B$. Show that

$$F = F' \text{ iff } \forall a \in A\ F(a) = F'(a).$$

## 4.7 Unions

The *union axiom* reads

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w)).$$

**Lemma 20.** *The union axiom is equivalent to* $\forall x \bigcup x \in V$.

**Proof.** Observe the following equivalences:
$\forall x \bigcup x \in V$
$\leftrightarrow \forall x \exists y\, (y = y \wedge y = \bigcup x)$
$\leftrightarrow \forall x \exists y \forall z (z \in y \leftrightarrow z \in \bigcup x)$
$\leftrightarrow \forall x \exists y \forall z (z \in y \leftrightarrow \exists w \in x\ z \in w)$
which is equivalent to the union axiom. □

Note that the union of $x$ is usually viewed as the union of all *elements* of $x$:

$$\bigcup x = \bigcup_{w \in x} w,$$

where we define

$$\bigcup_{a \in A} t(a) = \{z \mid \exists a \in A\ z \in t(a)\}.$$

Graphically $\bigcup x$ can be illustrated like this:
Combining the axioms of pairing and unions we obtain:

**Lemma 21.** $\forall x_0, ..., x_{n-1}\, \{x_0, ..., x_{n-1}\} \in V$.

Note that this is a *schema* of lemmas, one for each ordinary natural number $n$. We prove the schema by complete induction on $n$.

**Proof.** For $n = 0, 1, 2$ the lemma states that $\emptyset \in V$, $\forall x\ \{x\} \in V$, and $\forall x, y\ \{x, y\} \in V$ resp., and these are true by previous axioms and lemmas. For the induction step assume that the lemma holds for $n$, $n \geqslant 1$. Consider sets $x_0, ..., x_n$. Then

$$\{x_0, ..., x_n\} = \{x_0, ..., x_{n-1}\} \cup \{x_n\}.$$

The right-hand side exists in $V$ by the inductive hypothesis and the union axiom. □

## 4.8 Separation

It is common to form a subset of a given set consisting of all elements which satisfy some condition. This is codified by the *separation schema*. For every $\in$-formula $\varphi(z, x_1, ..., x_n)$ postulate:

$$\forall x_1 ... \forall x_n \forall x \exists y \forall z\, (z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, ..., x_n)).$$

Using class terms the schema can be reformulated as: for every term $A$ postulate

$$\forall x A \cap x \in V.$$

The crucial point is the restriction to the given set $x$. The unrestricted, Fregean version $A \in V$ for every term $A$ leads to the Russell antinomy. We turn the antinomy into a consequence of the separation schema:

**Theorem 22.** *$V \notin V$.*

**Proof.** Assume that $V \in V$. Then $\exists x x = V$. Take $x$ such that $x = V$. Let $R$ be the Russellian class:

$$R := \{x | x \notin x\}.$$

By separation, $y := R \cap x \in V$. Note that $R \cap x = R \cap V = R$. Then

$$y \in y \leftrightarrow y \in R \leftrightarrow y \notin y,$$

contradiction. □

This simple but crucial theorem leads to the distinction:

**Definition 23.** *Let $A$ be a term. Then $A$ is a* proper class *iff $A \notin V$.*

Set theory deals with sets and proper classes. Sets are the favoured objects of set theory, the axiom mainly state favorable properties of sets and set existence. Sometimes one says that a term $A$ *exists* if $A \in V$. The intention of set theory is to construe important mathematical classes like the collection of natural and real numbers as sets so that they can be treated set-theoretically. Zermelo observed that this is possible by requiring some set existences together with the *restricted* separation principle.

**Exercise 6.** Show that the class $\{\{x\} | x \in V\}$ of *singletons* is a proper class.

## 4.9 Power Sets

The *power set axiom* in class term notation is

$$\forall x \mathcal{P}(x) \in V.$$

The power set axiom yields the existence of function spaces.

**Definition 24.** *Let $A, B$ be terms. Then*

$$A \times B := \{(a, b) | a \in A \wedge b \in B\}$$

*is the* cartesian product *of $A$ and $B$.*

**Exercise 7.**

By the specific implementation of Kuratowski ordered pairs:

**Lemma 25.** *$A \times B \subseteq \mathcal{P}(\mathcal{P}(A \cup B))$.*

**Proof.** Let $(a, b) \in A \times B$. Then

$$\begin{aligned} a, b &\in A \cup B \\ \{a\}, \{a, b\} &\subseteq A \cup B \\ \{a\}, \{a, b\} &\in \mathcal{P}(A \cup B) \\ (a, b) = \{\{a\}, \{a, b\}\} &\subseteq \mathcal{P}(A \cup B) \\ (a, b) = \{\{a\}, \{a, b\}\} &\in \mathcal{P}(\mathcal{P}(A \cup B)) \end{aligned}$$

□

**Theorem 26.**

*a)* $\forall x, y\ x \times y \in V$.

*b)* $\forall x, y\ {}^x y \in V$.

**Proof.** Let $x, y$ be sets. a) Using the axioms of pairing, union, and power sets, $\mathcal{P}(\mathcal{P}(x \cup y)) \in V$. By the previous lemma and the axiom schema of separation,

$$x \times y = (x \times y) \cap \mathcal{P}(\mathcal{P}(x \cup y)) \in V.$$

b) ${}^x y \subseteq \mathcal{P}(x \times y)$ since a function $f: x \to y$ is a subset of $x \times y$. By the separation schema,

$$^x y = ^x y \cap \mathcal{P}(x \times y) \in V.$$

□

Note that to "find" the sets in this theorem one has to apply the power set operation repeatedly. We shall see that the universe of all sets can be obtained by iterating the power set operation.

The power set axiom leads to higher *cardinalities*. The theory of cardinalities will be developed later, but we can already prove CANTOR's theorem:

**Theorem 27.** *Let $x \in V$.*

*a) There is an injective map $f: x \to \mathcal{P}(x)$.*

*b) There does not exist an injective map $g: \mathcal{P}(x) \to x$.*

**Proof.** a) Define the map $f: x \to \mathcal{P}(x)$ by $u \mapsto \{u\}$. This is a set since

$$f = \{(u, \{u\}) | u \in x\} \subseteq x \times \mathcal{P}(x) \in V.$$

$f$ is injective: let $u, u' \in x$, $u \neq u'$. By extensionality,

$$f(u) = \{u\} \neq \{u'\} = f(u').$$

b) Assume there were an injective map $g: \mathcal{P}(x) \to x$. Define the CANTORean set

$$c = \{u | u \in x \wedge u \notin g^{-1}(u)\} \in P(x)$$

similar to the class $R$ in RUSSELL's paradox.

Let $u_0 = g(c)$. Then $g^{-1}(u_0) = c$ and

$$u_0 \in c \leftrightarrow u_0 \notin g^{-1}(u_0) = c.$$

Contradiction. □

## 4.10 Replacement

If every element of a set is definably *replaced* by another set, the result is a set again. The *schema of replacement* postulates for every term $F$:

$$F \text{ is a function} \to \forall x F[x] \in V.$$

**Lemma 28.** *The replacement schema implies the separation schema.*

**Proof.** Let $A$ be a term and $x \in V$.
*Case 1.* $A \cap x = \emptyset$. Then $A \cap x \in V$ by the axiom of set existence.
*Case 2.* $A \cap x \neq \emptyset$. Take $u_0 \in A \cap x$. Define a map $F: x \to x$ by

$$F(u) = \begin{cases} u, \text{ if } u \in A \cap x \\ u_0, \text{ else} \end{cases}$$

Then by replacement

$$A \cap x = F[x] \in V$$

as required. □

## 4.11 Infinity

All the axioms so far can be realized in a domain of finite sets, see exercise 12. The true power of set theory is set free by postulating the existence of *one* infinite set and continuing to assume the axioms. The *axiom of infinity* expresses that the set of "natural numbers" exists. To this end, some "number-theoretic" notions are defined.

**Definition 29.**

*a)* $0:=\emptyset$ *is the number* zero*.*

*b)* *For any term* $t$*,* $t+1:=t\cup\{t\}$ *is the* successor *of* $t$*.*

These notions are reasonable in the later formalization of the natural numbers. The axiom of infinity postulates the existence of a set which contains 0 and is closed under successors

$$\exists x\,(0\in x\wedge\forall n\in x\;n+1\in x).$$

Intuitively this says that there is a set which contains all natural numbers. Let us define set-theoretic analogues of the standard natural numbers:

**Definition 30.** *Define*

*a)* $1:=0+1$*;*

*b)* $2:=1+1$*;*

*c)* $3:=2+1$*; ...*

From the context it will be clear, whether "3", say, is meant to be the standard number "three" or the set theoretical object

$$\begin{aligned}3 &= 2\cup\{2\}\\ &= (1+1)\cup\{1+1\}\\ &= (\{\emptyset\}\cup\{\{\emptyset\}\})\cup\{\{\emptyset\}\cup\{\{\emptyset\}\}\}\\ &= \{\emptyset,\{\emptyset\},\{\emptyset\}\cup\{\{\emptyset\}\}\}.\end{aligned}$$

The set-theoretic axioms will ensure that this interpretation of "three" has the important number-theoretic properties of "three".

## 4.12 Foundation

The *axiom schema of foundation* provides structural information about the set theoretic universe $V$. It can be reformulated by postulating, for any term $A$:

$$A\neq\emptyset\rightarrow\exists x\in A\;A\cap x=\emptyset.$$

Viewing $\in$ as some kind of order relation this means that every non-empty class has an $\in$-minimal element $x\in A$ such that the $\in$-predecessors of $x$ are not in $A$. Foundation excludes circles in the $\in$-relation:

**Lemma 31.** *Let* $n$ *be a natural number* $\geqslant 1$ *. Then there are* no $x_0,\ldots,x_{n-1}$ *such that*

$$x_0\in x_1\in\ldots\in x_{n-1}\in x_0\,.$$

**Proof.** Assume not and let $x_0\in x_1\in\ldots\in x_{n-1}\in x_0$. Let

$$A=\{x_0,\ldots,x_{n-1}\}.$$

$A\neq\emptyset$ since $n\geqslant 1$. By foundation take $x\in A$ such that $A\cap x=\emptyset$.
*Case 1.* $x=x_0$. Then $x_{n-1}\in A\cap x=\emptyset$, contradiction.
*Case 2.* $x=x_i$, $i>0$. Then $x_{i-1}\in A\cap x=\emptyset$, contradiction. □

**Exercise 8.** Show that $x\neq x+1$.

**Exercise 9.** Show that the successor function $x \mapsto x+1$ is injective.

**Exercise 10.** Show that the term $\{x, \{x, y\}\}$ may be taken as an ordered pair of $x$ and $y$.

**Theorem 32.** *The foundation scheme is equivalent to the following,* PEANO-*type, induction scheme: for every term $B$ postulate*

$$\forall x\,(x \subseteq B \rightarrow x \in B) \rightarrow B = V.$$

*This says that if a "property" $B$ is inherited by $x$ if all elements of $x$ have the property $B$, then every set has the property $B$.*

**Proof.** $(\rightarrow)$ Assume $B$ were a term which did not satisfy the induction principle:

$$\forall x\,(x \subseteq B \rightarrow x \in B) \text{ and } B \neq V.$$

Set $A = V \setminus B \neq \emptyset$. By foundation take $x \in A$ such that $A \cap x = \emptyset$. Then

$$u \in x \rightarrow u \notin A \rightarrow u \in B,$$

i.e., $x \subseteq B$. By assumption, $B$ is inherited by $x$: $x \in B$. But then $x \notin A$, contradiction.
$(\leftarrow)$ Assume $A$ were a term which did not satisfy the foundation scheme:

$$A \neq \emptyset \text{ and } \forall x \in A\; A \cap x \neq \emptyset.$$

Set $B = V \setminus A$. Consider $x \subseteq B$. Then $A \cap x = \emptyset$. By assumption, $x \notin A$ and $x \in B$. Thus $\forall x\,(x \subseteq B \rightarrow x \in B)$. The induction principle implies that $B = V$. Then $A = \emptyset$, contradiction. □

This proof shows, that the induction principle is basically an equivalent formulation of the foundation principle. The $\in$-relation is taken as some binary relation without reference to specific properties of this relation. This leads to:

**Exercise 11.** A relation $R$ on a domain $D$ is called *wellfounded*, iff for all terms $A$

$$\emptyset \neq A \wedge A \subseteq D \rightarrow \exists x \in A\; A \cap \{y \mid yRx\} = \emptyset.$$

Formulate and prove a principle for $R$-induction on $D$ which coressponds to the assumption that $R$ is wellfounded on $D$.

## 4.13 Set Theoretic Axiom Schemas

Note that the axiom system introduced is an infinite informal *set* of axioms. It seems unavoidable that we have to go back to some previously given set notions to be able to define the collection of set theoretical axioms - another example of the frequent circularity in foundational theories.

**Definition 33.** *The system* ZF *of the* ZERMELO-FRAENKEL axioms of set theory *consists of the following axioms:*

a) *The* set existence axiom (Ex)*:*

$$\exists x \forall y \neg y \in x$$

- *there is a set without elements, the* empty set*.*

b) *The* axiom of extensionality (Ext)*:*

$$\forall x \forall y (\forall z (z \in x \leftrightarrow z \in y) \rightarrow x = y)$$

- *a set is determined by its elements, sets having the same elements are identical.*

c) *The* pairing axiom (Pair)*:*

$$\forall x \forall y \exists z \forall w\,(u \in z \leftrightarrow u = x \vee u = y).$$

- *$z$ is the unordered pair of $x$ and $y$.*

d) *The* union axiom (Union)*:*

$$\forall x \exists y \forall z (z \in y \leftrightarrow \exists w (w \in x \wedge z \in w))$$

*- $y$ is the union of all elements of $x$.*

*e)* *The* separation schema (Sep) *postulates for every $\in$-formula $\varphi(z, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n \forall x \exists y \forall z\, (z \in y \leftrightarrow z \in x \wedge \varphi(z, x_1, ..., x_n))$$

*- this is an infinite scheme of axioms, the set $z$ consists of all elements of $x$ which satisfy $\varphi$.*

*f)* *The* powerset axiom (Pow)*:*

$$\forall x \exists y \forall z (z \in y \leftrightarrow \forall w (w \in z \rightarrow w \in x))$$

*- $y$ consists of all subsets of $x$.*

*g)* *The* replacement schema (Rep) *postulates for every $\in$-formula $\varphi(x, y, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n (\forall x \forall y \forall y' ((\varphi(x, y, x_1, ..., x_n) \wedge \varphi(x, y', x_1, ..., x_n)) \rightarrow y = y') \rightarrow \\ \forall u \exists v \forall y\, (y \in v \leftrightarrow \exists x (x \in u \wedge \varphi(x, y, x_1, ..., x_n))))$$

*- $v$ is the image of $u$ under the map defined by $\varphi$.*

*h)* *The* axiom of infinity (Inf)*:*

$$\exists x (\exists y\, (y \in x \wedge \forall z \neg z \in y) \wedge \forall y (y \in x \rightarrow \exists z (z \in x \wedge \forall w (w \in z \leftrightarrow w \in y \vee w = y))))$$

*- by the closure properties of $x$, $x$ has to be infinite.*

*i)* *The* foundation schema (Found) *postulates for every $\in$-formula $\varphi(x, x_1, ..., x_n)$:*

$$\forall x_1 ... \forall x_n (\exists x \varphi(x, x_1, ..., x_n) \rightarrow \exists x (\varphi(x, x_1, ..., x_n) \wedge \forall x' (x' \in x \rightarrow \neg \varphi(x', x_1, ..., x_n))))$$

*- if $\varphi$ is satisfiable then there are $\in$-minimal elements satisfying $\varphi$.*

## 4.14 ZF in Class Notation

Using class terms, the ZF can be formulated concisely:

**Theorem 34.** *The* ZF *axioms are equivalent to the following system; we take all free variables of the axioms to be universally quantified:*

*a)* *Ex: $\emptyset \in V$.*

*b)* *Ext: $x \subseteq y \wedge y \subseteq x \rightarrow x = y$.*

*c)* *Pair: $\{x, y\} \in V$.*

*d)* *Union: $\bigcup x \in V$.*

*e)* *Sep: $A \cap x \in V$.*

*f)* *Pow: $\mathcal{P}(x) \in V$.*

*g)* *Rep: $F$ is a function $\rightarrow F[x] \in V$.*

*h)* *Inf: $\exists x\, (0 \in x \wedge \forall n \in x\; n + 1 \in x)$.*

*i)* *Found: $A \neq \emptyset \rightarrow \exists x \in A\; A \cap x = \emptyset$.*

This axiom system can be used as a foundation for all of mathematics. Axiomatic set theory considers various axiom systems of set theory.

**Definition 35.** *The axiom system* $\mathrm{ZF}^-$ *consists of the* ZF*-axioms except the power set axiom. The system* EML *("elementary set theory") consists of the axioms* Ex, Ext, Pair*, and* Union.

**Exercise 12.** Consider the axiom system HF consisting of the axioms of EML together with the induction principle: for every term $B$ postulate

$$\forall x, y\, (x \subseteq B \wedge y \in B \rightarrow x \cup \{y\} \in B) \rightarrow B = V.$$

Show that every axiom of ZF except Inf is provable in HF, and that HF proves the *negation* of Inf (HF axiomatizes the **h**eriditarily **f**inite sets, i.e., those sets such that the set itself and all its iterated elements are finite).

# 5 Ordinal Numbers

We had defined some "natural numbers" in set theory. Recall that

$$\begin{aligned}
0 &= \emptyset \\
1 &= 0+1=0\cup\{0\} = \{0\} \\
2 &= 1+1=1\cup\{1\} = \{0,1\} \\
3 &= 2+1=2\cup\{2\} = \{0,1,2\} \\
&\vdots
\end{aligned}$$

We would then like to have $\mathbb{N} = \{0, 1, 2, 3, ...\}$. To obtain a set theoretic formalization of numbers we note some properties of the informal presentation:

1. "Numbers" are ordered by the $\in$-relation:
$$m<n \text{ iff } m\in n.$$
E.g., $1\in 3$ but not $3\in 1$.
2. On each "number", the $\in$-relation is a *strict linear order*: $3 = \{0, 1, 2\}$ is strictly linearly ordered by $\in$.
3. "Numbers" are "complete" with respect to smaller "numbers"
$$i<j<m\rightarrow i\in m.$$
This can be written with the $\in$-relation as
$$i\in j\in m\rightarrow i\in m.$$

**Definition 36.**

*a)* *A is* transitive*,* Trans($A$)*, iff* $\forall y\in A\forall x\in y\, x\in A$ *.*

*b)* *x is an* ordinal (number)*,* Ord($x$)*, if* $\text{Trans}(x)\wedge\forall y\in x\,\text{Trans}(y)$*.*

*c)* *Let* $\text{Ord}:=\{x|\text{Ord}(x)\}$ *be the class of all ordinal numbers.*

We shall use small greek letter $\alpha, \beta, ...$ as variables for ordinals. So $\exists\alpha\varphi$ stands for $\exists\alpha\in\text{Ord}\ \varphi$, and $\{\alpha|\varphi\}$ for $\{\alpha|\text{Ord}(\alpha)\wedge\varphi\}$.

**Exercise 13.** Show that arbitrary unions and intersections of transitive sets are again transitive.

We shall see that the ordinals extend the standard natural numbers. Ordinals are particularly adequate for enumerating infinite sets.

**Theorem 37.**

*a)* $0\in\text{Ord}$.

*b)* $\forall\alpha\ \alpha+1\in\text{Ord}$ *.*

**Proof.** a) Trans($\emptyset$) since formulas of the form $\forall y\in\emptyset...$ are tautologously true. Similarly $\forall y\in\emptyset\,\text{Trans}(y)$.
b) Assume $\alpha\in\text{Ord}$.
(1) $\text{Trans}(\alpha+1)$.
*Proof*. Let $u\in v\in\alpha+1=\alpha\cup\{\alpha\}$.
*Case 1*. $v\in\alpha$. Then $u\in\alpha\subseteq\alpha+1$, since $\alpha$ is transitive.
*Case 2*. $v=\alpha$. Then $u\in\alpha\subseteq\alpha+1$. *qed*(1)
(2) $\forall y\in\alpha+1\,\text{Trans}(y)$.
*Proof*. Let $y\in\alpha+1=\alpha\cup\{\alpha\}$.
*Case 1*. $y\in\alpha$. Then Trans($y$) since $\alpha$ is an ordinal.
*Case 2*. $y=\alpha$. Then Trans($y$) since $\alpha$ is an ordinal. □

**Exercise 14.**

a) Let $A \subseteq \mathrm{Ord}$ be a term, $A \neq \emptyset$. Then $\bigcap A \in \mathrm{Ord}$.

b) Let $x \subseteq \mathrm{Ord}$ be a set. Then $\bigcup x \in \mathrm{Ord}$.

**Theorem 38.** Trans(Ord).

**Proof.** This follows immediately from the transitivity definition of Ord. □

**Exercise 15.** Show that Ord is a proper class. (Hint: if $\mathrm{Ord} \in V$ then $\mathrm{Ord} \in \mathrm{Ord}$.)

**Theorem 39.** *The class* Ord *is strictly linearly ordered by* $\in$, *i.e.,*

*a)* $\forall \alpha, \beta, \gamma\, (\alpha \in \beta \wedge \beta \in \gamma \rightarrow \alpha \in \gamma)$.

*b)* $\forall \alpha\, \alpha \notin \alpha$.

*c)* $\forall \alpha, \beta\, (\alpha \in \beta \vee \alpha = \beta \vee \beta \in \alpha)$.

**Proof.** a) Let $\alpha, \beta, \gamma \in \mathrm{Ord}$ and $\alpha \in \beta \wedge \beta \in \gamma$. Then $\gamma$ is transitive, and so $\alpha \in \gamma$.
b) follows immediately from the non-circularity of the $\in$-relation.
c) Assume that there are "incomparable" ordinals. By the foundation schema choose $\alpha_0 \in \mathrm{Ord}$ $\in$-minimal such that $\exists \beta \neg(\alpha_0 \in \beta \vee \alpha_0 = \beta \vee \beta \in \alpha_0)$. Again, choose $\beta_0 \in \mathrm{Ord}$ $\in$-minimal such that $\neg(\alpha_0 \in \beta_0 \vee \alpha_0 = \beta_0 \vee \beta_0 \in \alpha_0)$. We obtain a contradiction by showing that $\alpha_0 = \beta_0$:

Let $\alpha \in \alpha_0$. By the $\in$-minimality of $\alpha_0$, $\alpha$ is comparable with $\beta_0$: $\alpha \in \beta_0 \vee \alpha = \beta_0 \vee \beta_0 \in \alpha$. If $\alpha = \beta_0$ then $\beta_0 \in \alpha_0$ and $\alpha_0, \beta_0$ would be comparable, contradiction. If $\beta_0 \in \alpha$ then $\beta_0 \in \alpha_0$ by the transitivity of $\alpha_0$ and again $\alpha_0, \beta_0$ would be comparable, contradiction. Hence $\alpha \in \beta_0$.

For the converse let $\beta \in \beta_0$. By the $\in$-minimality of $\beta_0$, $\beta$ is comparable with $\alpha_0$: $\beta \in \alpha_0 \vee \beta = \alpha_0 \vee \alpha_0 \in \beta$. If $\beta = \alpha_0$ then $\alpha_0 \in \beta_0$ and $\alpha_0, \beta_0$ would be comparable, contradiction. If $\alpha_0 \in \beta$ then $\alpha_0 \in \beta_0$ by the transitivity of $\beta_0$ and again $\alpha_0$, $\beta_0$ would be comparable, contradiction. Hence $\beta \in \alpha_0$.

But then $\alpha_0 = \beta_0$ contrary to the choice of $\beta_0$. □

**Definition 40.** *Let* $<: = \in \cap (\mathrm{Ord} \times \mathrm{Ord}) = \{(\alpha, \beta) | \alpha \in \beta\}$ *be the natural strict linear ordering of* Ord *by the* $\in$*-relation.*

**Theorem 41.** *Let* $\alpha \in \mathrm{Ord}$. *Then* $\alpha + 1$ *is the immediate successor of* $\alpha$ *in the* $\in$*-relation:*

*a)* $\alpha < \alpha + 1$;

*b) if* $\beta < \alpha + 1$, *then* $\beta = \alpha$ *or* $\beta < \alpha$.

**Definition 42.** *Let* $\alpha$ *be an ordinal.* $\alpha$ *is a* successor ordinal, $\mathrm{Succ}(\alpha)$, *iff* $\exists \beta\ \alpha = \beta + 1$. $\alpha$ *is a limit ordinal,* $\mathrm{Lim}(\alpha)$, *iff* $\alpha \neq 0$ *and* $\alpha$ *is not a successor ordinal. Also let*

$$\mathrm{Succ}: = \{\alpha | \mathrm{Succ}(\alpha)\} \text{ and } \mathrm{Lim} := \{\alpha | \mathrm{Lim}(\alpha)\}.$$

The existence of limit ordinals will be discussed together with the formalization of the natural numbers.

## 5.1 Induction

Ordinals satisfy an *induction theorem* which generalizes *complete induction* on the integers:

**Theorem 43.** *Let* $\varphi(x, v_0, \ldots, v_{n-1})$ *be an* $\in$*-formula and* $x_0, \ldots, x_{n-1} \in V$. *Assume that the property* $\varphi(x, x_0, \ldots, x_{n-1})$ *is* inductive, *i.e.,*

$$\forall \alpha (\forall \beta \in \alpha\ \varphi(\beta, x_0, \ldots, x_{n-1}) \rightarrow \varphi(\alpha, x_0, \ldots, x_{n-1})).$$

*Then* $\varphi$ *holds for all ordinals:*

$$\forall \alpha\, \varphi(\alpha, x_0, \ldots, x_{n-1}).$$

**Proof.** It suffices to show that

$$B = \{x | x \in \mathrm{Ord} \to \varphi(x, x_0, \ldots, x_{n-1})\} = V.$$

Theorem 32 implies

$$\forall x\, (x \subseteq B \to x \in B) \to B = V$$

and it suffices to show

$$\forall x\, (x \subseteq B \to x \in B).$$

Consider $x \subseteq B$. If $x \notin \mathrm{Ord}$ then $x \in B$. So assume $x \in \mathrm{Ord}$. For $\beta \in x$ we have $\beta \in B$, $\beta \in \mathrm{Ord}$, and so $\varphi(\beta, x_0, \ldots, x_{n-1})$. By the inductivity of $\varphi$ we get $\varphi(x, x_0, \ldots, x_{n-1})$ and again $x \in B$. □

Induction can be formulated in various forms:

**Exercise 16.** Prove the following transfinite induction principle: Let $\varphi(x) = \varphi(x, v_0, \ldots, v_{n-1})$ be an $\in$-formula and $x_0, \ldots, x_{n-1} \in V$. Assume

a) $\varphi(0)$ (the initial case),

b) $\forall \alpha\, (\varphi(\alpha) \to \varphi(\alpha + 1))$ (the successor step),

c) $\forall \lambda \in \mathrm{Lim}\ (\forall \alpha < \lambda\, \varphi(\alpha) \to \varphi(\lambda))$ (the limit step).

Then $\forall \alpha\, \varphi(\alpha)$.

## 5.2 Natural Numbers

We have 0, 1, ... $\in$ Ord. We shall now define and study the set of *natural numbers/integers* within set theory. Recall the axiom of infinity:

$$\exists x\ (0 \in x \wedge \forall u \in x\ u + 1 \in x).$$

The set of natural numbers should be the $\subseteq$-smallest such $x$.

**Definition 44.** *Let* $\omega = \bigcap \{x | 0 \in x \wedge \forall u \in x\ u + 1 \in x\}$ *be the* set of natural numbers. *Sometimes we write* $\mathbb{N}$ *instead of* $\omega$.

**Theorem 45.**

*a)* $\omega \in V$.

*b)* $\omega \subseteq \mathrm{Ord}$.

*c)* $(\omega, 0, +1)$ *satisfy the* second order Peano axiom*, i.e.,*

$$\forall x \subseteq \omega\, (0 \in x \wedge \forall n \in x\ n + 1 \in x \to x = \omega).$$

*d)* $\omega \in \mathrm{Ord}$.

*e)* $\omega$ *is a limit ordinal.*

**Proof.** a) By the axiom of infinity take a set $x_0$ such that

$$0 \in x_0 \wedge \forall u \in x_0\ u + 1 \in x_0.$$

Then

$$\omega = \bigcap \{x | 0 \in x \wedge \forall u \in x\ u + 1 \in x\} = x_0 \cap \bigcap \{x | 0 \in x \wedge \forall u \in x\ u + 1 \in x\} \in V$$

by the separation schema.

b) By a), $\omega \cap \mathrm{Ord} \in V$. Obviously $0 \in \omega \cap \mathrm{Ord} \wedge \forall u \in \omega \cap \mathrm{Ord}\ u + 1 \in \omega \cap \mathrm{Ord}$. So $\omega \cap \mathrm{Ord}$ is one factor of the intersection in the definition of $\omega$ and so $\omega \subseteq \omega \cap \mathrm{Ord}$. Hence $\omega \subseteq \mathrm{Ord}$.

c) Let $x \subseteq \omega$ and $0 \in x \wedge \forall u \in x\ u + 1 \in x$. Then $x$ is one factor of the intersection in the definition of $\omega$ and so $\omega \subseteq x$. This implies $x = \omega$.

d) By b), every element of $\omega$ is transitive and it suffices to show that $\omega$ is transitive. Let

$$x = \{n | n \in \omega \wedge \forall m \in n\ m \in \omega\} \subseteq \omega.$$

We show that the hypothesis of c) holds for $x$. $0\in x$ is trivial. Let $u\in x$. Then $u+1\in\omega$. Let $m\in u+1$. If $m\in u$ then $m\in\omega$ by the assumption that $u\in x$. If $m=u$ then $m\in x\subseteq\omega$. Hence $u+1\in x$ and $\forall u\in x\, u+1\in x$. By b), $x=\omega$. So $\forall n\in\omega\, n\in x$, i.e.,

$$\forall n\in\omega\forall m\in n\; m\in\omega.$$

e) Of course $\omega\neq 0$. Assume for a contradiction that $\omega$ is a successor ordinal, say $\omega=\alpha+1$. Then $\alpha\in\omega$. Since $\omega$ is closed under the +1-operation, $\omega=\alpha+1\in\omega$. Contradiction. □

Thus the axiom of infinity implies the existence of the set of natural numbers, which is also the smallest limit ordinal. The axiom of infinity can now be reformulated equivalently as:

h) Inf: $\omega\in V$.

## 5.3 Recursion

*Recursion*, often called induction, over the natural numbers is a ubiquitous method for defining mathematical object. We prove the following *recursion theorem* for ordinals.

**Theorem 46.** *Let $G\colon V\to V$. Then there is a canonical class term $F$, given by the subsequent proof, such that*

$$F\colon \mathrm{Ord}\to V \text{ and } \forall\alpha\; F(\alpha)=G(F\restriction\alpha).$$

*We then say that $F$ is defined* recursively *(over the ordinals) by the* recursion rule *$G$. $F$ is unique in the sense that if another term $F'$ satisfies*

$$F'\colon \mathrm{Ord}\to V \text{ and } \forall\alpha\; F'(\alpha)=G(F'\restriction\alpha)$$

*then $F=F'$.*

**Proof.** We say that $H\colon \mathrm{dom}(H)\to V$ is *$G$-recursive* if

$$\mathrm{dom}(H)\subseteq\mathrm{Ord}, \mathrm{dom}(H) \text{ is transitive, and } \forall\alpha\in\mathrm{dom}(H)\; H(\alpha)=G(H\restriction\alpha).$$

(1) Let $H, H'$ be $G$-recursive. Then $H, H'$ are *compatible*, i.e., $\forall\alpha\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\; H(\alpha)=H'(\alpha)$.

*Proof*. We want to show that

$$\forall\alpha\in\mathrm{Ord}\;(\alpha\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\to H(\alpha)=H'(\alpha)).$$

By the induction theorem it suffices to show that $\alpha\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\to H(\alpha)=H'(\alpha)$ is inductive, i.e.,

$$\forall\alpha\in\mathrm{Ord}\,(\forall y\in\alpha\;(y\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\to H(y)=H'(y))\to(\alpha\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\to H(\alpha)=H'(\alpha))).$$

So let $\alpha\in\mathrm{Ord}$ and $\forall y\in\alpha\;(y\in\mathrm{dom}(H)\cap\mathrm{dom}(H')\to H(y)=H'(y))$. Let $\alpha\in\mathrm{dom}(H)\cap\mathrm{dom}(H')$. Since $\mathrm{dom}(H)$ and $\mathrm{dom}(H')$ are transitive, $\alpha\subseteq\mathrm{dom}(H)$ and $\alpha\subseteq\mathrm{dom}(H')$. By assumption

$$\forall y\in\alpha\; H(y)=H'(y).$$

Hence $H\restriction\alpha=H'\restriction\alpha$. Then

$$H(\alpha)=G(H\restriction\alpha)=G(H'\restriction\alpha)=H'(\alpha).$$

*qed*(1)

Let

$$F:=\bigcup\,\{f\,|\,f \text{ is } G\text{-recursive}\}.$$

be the union of the class of all *approximations* to the desired function $F$.

(2) $F$ is $G$-recursive.

*Proof*. By (1), $F$ is a function. Its domain $\mathrm{dom}(F)$ is the union of transitive classes of ordinals and hence $\mathrm{dom}(F)\subseteq\mathrm{Ord}$ is transitive.

Let $\alpha \in \mathrm{dom}(F)$. Take some $G$-recursive function$f$ such that $\alpha \in \mathrm{dom}(f)$. Since $\mathrm{dom}(f)$ is transitive, we have

$$\alpha \subseteq \mathrm{dom}(f) \subseteq \mathrm{dom}(F).$$

Moreover

$$F(\alpha) = f(\alpha) = G(f \restriction \alpha) = G(F \restriction \alpha).$$

*qed*(2)
(3) $\forall \alpha\ \alpha \in \mathrm{dom}(F)$.
*Proof*. By induction on the ordinals. We have to show that $\alpha \in \mathrm{dom}(F)$ is inductive in the variable $\alpha$. So let $\alpha \in \mathrm{Ord}$ and $\forall y \in \alpha\ y \in \mathrm{dom}(F)$. Hence $\alpha \subseteq \mathrm{dom}(F)$. Let

$$f = F \restriction \alpha \cup \{(\alpha, G(F \restriction \alpha))\}.$$

$f$ is a function with $\mathrm{dom}(f) = \alpha + 1 \in \mathrm{Ord}$. Let $\alpha' < \alpha + 1$. If $\alpha' < \alpha$ then

$$f(\alpha') = F(\alpha') = G(F \restriction \alpha') = G(f \restriction \alpha').$$

if $\alpha' = \alpha$ then also

$$f(\alpha') = f(\alpha) = G(F \restriction \alpha) = G(f \restriction \alpha) = G(f \restriction \alpha').$$

Hence $f$ is $G$-recursive and $\alpha \in \mathrm{dom}(f) \subseteq \mathrm{dom}(F)$. *qed*(3)
The extensional uniqueness of $F$ follows from (1) □

**Theorem 47.** *Let* $a_0 \in V$, $G_{\mathrm{succ}}: \mathrm{Ord} \times V \to V$, *and* $G_{\mathrm{lim}}: \mathrm{Ord} \times V \to V$. *Then there is a canonically defined class term* $F: \mathrm{Ord} \to V$ *such that*

*a)* $F(0) = a_0$ *;*

*b)* $\forall \alpha\, F(\alpha + 1) = G_{\mathrm{succ}}(\alpha, F(\alpha))$*;*

*c)* $\forall \lambda \in \mathrm{Lim}\ F(\lambda) = G_{\mathrm{lim}}(\lambda, F \restriction \lambda)$.

*Again F is unique in the sense that if some $F'$ also satisfies a)-c) then $F = F'$.*
*We say that F is* recursively defined *by the properties a)-c).*

**Proof.** We incorporate $a_0$, $G_{\mathrm{succ}}$, and $G_{\mathrm{lim}}$ into a single recursion rule $G: V \to V$,

$$G(f) = \begin{cases} a_0 \text{ , if } f = \emptyset, \\ G_{\mathrm{succ}}(\alpha, f(\alpha)) \text{ , if } f: \alpha + 1 \to V, \\ G_{\mathrm{lim}}(\lambda, f) \text{ , if } f: \lambda \to V \text{ and } \mathrm{Lim}(\lambda), \\ \emptyset \text{ , else.} \end{cases}$$

Then the term $F: \mathrm{Ord} \to V$ defined recursively by the recursion rule $G$ satisfies the theorem. □

In many cases, the *limit rule* will just require to form the union of the previous values so that

$$F(\lambda) = \bigcup_{\alpha < \lambda} F(\alpha).$$

Such recursions are called *continuous* (at limits).

## 5.4 Ordinal Arithmetic

We extend the recursion rules of standard integer arithmetic continuously to obtain transfinite version of the arithmetic operations. The initial operation of ordinal arithmetic is the +1-operation defined before. Ordinal arithmetic satisfies some but not all laws of integer arithmetic.

**Definition 48.** *Define* ordinal addition $+: \mathrm{Ord} \times \mathrm{Ord} \to \mathrm{Ord}$ *recursively by*

$$\begin{aligned} \delta + 0 &= \delta \\ \delta + (\alpha + 1) &= (\delta + \alpha) + 1 \\ \delta + \lambda &= \bigcup_{\alpha < \lambda} (\delta + \alpha) \text{ , for limit ordinals } \lambda \end{aligned}$$

**Definition 49.** *Define* ordinal multiplication $\cdot : \mathrm{Ord} \times \mathrm{Ord} \rightarrow \mathrm{Ord}$ *recursively by*

$$\begin{aligned}\delta \cdot 0 &= 0\\ \delta \cdot (\alpha + 1) &= (\delta \cdot \alpha) + \delta\\ \delta \cdot \lambda &= \bigcup_{\alpha < \lambda} (\delta \cdot \alpha) \text{ , for limit ordinals } \lambda\end{aligned}$$

**Definition 50.** *Define* ordinal exponentiation $\_^{-} : \mathrm{Ord} \times \mathrm{Ord} \rightarrow \mathrm{Ord}$ *recursively by*

$$\begin{aligned}\delta^0 &= 1\\ \delta^{\alpha+1} &= \delta^\alpha \cdot \delta\\ \delta^\lambda &= \bigcup_{\alpha < \lambda} \delta^\alpha \text{ , for limit ordinals } \lambda\end{aligned}$$

**Exercise 17.** Explore which of the standard *ring axioms* hold for the ordinals with addition and multiplication. Give proofs and counterexamples.

**Exercise 18.** Show that for any ordinal $\alpha$ , $\alpha + \omega$ is a limit ordinal. Use this to show that the class Lim of all limit ordinals is a proper class.

# 6 Number Systems

We are now able to give set-theoretic formalizations of the standard number systems with their arithmetic operations.

## 6.1 Natural Numbers

**Definition 51.** *The structure*

$$\mathbb{N} := (\omega, +\upharpoonright(\omega \times \omega), \cdot\upharpoonright(\omega \times \omega), <\upharpoonright(\omega \times \omega), 0, 1)$$

*is called the structure of* natural numbers, *or* arithmetic. *We sometimes denote this structure by*

$$\mathbb{N} := (\omega, +, \cdot, <, 0, 1).$$

$\mathbb{N}$ is an adequate formalization of arithmetic within set theory since $\mathbb{N}$ satisfies all standard arithmetical axioms.

**Exercise 19.** Prove:

a) $+[\omega \times \omega] := \{m + n | m \in \omega \wedge n \in \omega\} \subseteq \omega$ .

b) $\cdot [\omega \times \omega] := \{m \cdot n | m \in \omega \wedge n \in \omega\} \subseteq \omega$ .

c) Addition and multiplication are commutative on $\omega$ .

d) Addition and multiplication satisfy the usual monotonicity laws with respect to $<$.

**Definition 52.** *We define the structure*

$$\mathbb{Z} := (\mathbb{Z}, +^{\mathbb{Z}}, \cdot^{\mathbb{Z}}, <^{\mathbb{Z}}, 0^{\mathbb{Z}}, 1^{\mathbb{Z}})$$

*of* integers *as follows:*

*a) Define an equivalence relation* $\approx$ *on* $\mathbb{N} \times \mathbb{N}$ *by*

$$(a, b) \approx (a', b') \text{ iff } a + b' = a' + b.$$

*b) Let* $a - b := [(a, b)]_\approx$ *be the equivalence class of* $(a, b)$ *in* $\approx$*. Note that every* $a - b$ *is a set.*

*c) Let* $\mathbb{Z} := \{a - b | a \in \mathbb{N} \wedge b \in \mathbb{N}\}$ *be the set of* integers.

*d) Define the* integer addition $+^{\mathbb{Z}}: \mathbb{Z} \times \mathbb{Z} \rightarrow \mathbb{Z}$ *by*

$$(a - b) +^{\mathbb{Z}} (a' - b') := (a + a') - (b + b').$$

e) *Define the* integer multiplication $\cdot^{\mathbb{Z}}: \mathbb{Z} \times \mathbb{Z} \to \mathbb{Z}$ *by*

$$(a-b)\cdot^{\mathbb{Z}}(a'-b') := (a\cdot a' + b\cdot b') - (a\cdot b' + a'\cdot b).$$

f) *Define the strict linear order* $<^{\mathbb{Z}}$ *on* $\mathbb{Z}$ *by*

$$(a-b) <^{\mathbb{Z}} (a'-b') \text{ iff } a + b' < a' + b.$$

g) *Let* $0^{\mathbb{Z}} := 0 - 0$ *and* $1^{\mathbb{Z}} := 1 - 0$.

**Exercise 20.** Check that the above definitions are *sound*, i.e., that they do not depend on the choice of representatives of equivalence classes.

**Exercise 21.** Check that $\mathbb{Z}$ satisfies (a sufficient number) of the standard axioms for rings.

The structure $\mathbb{Z}$ extends the structure $\mathbb{N}$ in a natural and familiar way: define an injective map $e: \mathbb{N} \to \mathbb{Z}$ by

$$n \mapsto n - 0.$$

The embedding $e$ is a *homomorphism*:

a) $e(0) = 0 - 0 = 0^{\mathbb{Z}}$ and $e(1) = 1 - 0 = 1^{\mathbb{Z}}$;

b) $e(m+n) = (m+n) - 0 = (m+n) - (0+0) = (m-0) +^{\mathbb{Z}} (n-0) = e(m) +^{\mathbb{Z}} e(n)$;

c) $e(m\cdot n) = (m\cdot n) - 0 = (m\cdot n + 0\cdot 0) - (m\cdot 0 + n\cdot 0) = (m-0)\cdot^{\mathbb{Z}}(n-0) = e(m)\cdot^{\mathbb{Z}} e(n)$;

d) $m < n \leftrightarrow m + 0 < n + 0 \leftrightarrow (m-0) <^{\mathbb{Z}} (n-0) \leftrightarrow e(m) <^{\mathbb{Z}} e(n)$.

By this injective homomorphism, one may consider $\mathbb{N}$ as a *substructure* of $\mathbb{Z}$: $\mathbb{N} \subseteq \mathbb{Z}$.

## 6.2 Rational Numbers

**Definition 53.** *We define the structure*

$$\mathbb{Q}_0^+ := (\mathbb{Q}_0^+, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, <^{\mathbb{Q}}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}})$$

*of* non-negative rational numbers *as follows:*

a) *Define an equivalence relation* $\simeq$ *on* $\mathbb{N} \times (\mathbb{N} \setminus \{0\})$ *by*

$$(a,b) \simeq (a',b') \text{ iff } a\cdot b' = a'\cdot b.$$

b) *Let* $\frac{a}{b} := [(a,b)]_{\simeq}$ *be the equivalence class of* $(a,b)$ *in* $\simeq$. *Note that* $\frac{a}{b}$ *is a set.*

c) *Let* $\mathbb{Q}_0^+ := \{\frac{a}{b} | a \in \mathbb{N} \wedge b \in (\mathbb{N} \setminus \{0\})\}$ *be the set of* non-negative rationals.

d) *Define the* rational addition $+^{\mathbb{Q}}: \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \to \mathbb{Q}_0^+$ *by*

$$\frac{a}{b} +^{\mathbb{Q}} \frac{a'}{b'} := \frac{a\cdot b' + a'\cdot b}{b\cdot b'}.$$

e) *Define the* rational multiplication $\cdot^{\mathbb{Q}}: \mathbb{Q}_0^+ \times \mathbb{Q}_0^+ \to \mathbb{Q}_0^+$ *by*

$$\frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'} := \frac{a\cdot a'}{b\cdot b'}.$$

f) *Define the strict linear order* $<^{\mathbb{Q}}$ *on* $\mathbb{Q}_0^+$ *by*

$$\frac{a}{b} <^{\mathbb{Q}} \frac{a'}{b'} \text{ iff } a\cdot b' < a'\cdot b.$$

g) *Let* $0^{\mathbb{Q}} := \frac{0}{1}$ *and* $1^{\mathbb{Q}} := \frac{1}{1}$.

Again one can check the soundness of the definitions and the well-known laws of standard non-negative rational numbers. Also one may assume $\mathbb{N}$ to be embedded into $\mathbb{Q}_0^+$ as a substructure. The transfer from non-negative to *all* rationals, including negative rationals can be performed in analogy to the transfer from $\mathbb{N}$ to $\mathbb{Z}$.

**Definition 54.** *We define the structure*

$$\mathbb{Q} := (\mathbb{Q}, +^{\mathbb{Q}}, \cdot^{\mathbb{Q}}, <^{\mathbb{Q}}, 0^{\mathbb{Q}}, 1^{\mathbb{Q}})$$

*of* rational numbers *as follows:*

*a*) *Define an equivalence relation* $\approx$ *on* $\mathbb{Q}_0^+ \times \mathbb{Q}_0^+$ *by*

$$(p, q) \approx (p', q') \text{ iff } p + q' = p' + q.$$

*b*) *Let* $p - q := [(p, q)]_{\approx}$ *be the equivalence class of* $(p, q)$ *in* $\approx$.

*c*) *Let* $\mathbb{Q} := \{p - q | p \in \mathbb{Q}_0^+ \wedge p \in \mathbb{Q}_0^+\}$ *be the set of* rationals.

**Exercise 22.** Continue the definition of the structure $\mathbb{Q}$ and prove the relevant properties.

## 6.3 Real Numbers

**Definition 55.** $r \subseteq \mathbb{Q}_0^+$ *is a* positive real number *if*

*a*) $\forall p \in r \forall q \in \mathbb{Q}_0^+ (q <^{\mathbb{Q}} p \rightarrow q \in r)$, *i.e., r is an initial segment of* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$*;*

*b*) $\forall p \in r \exists q \in r\, p <^{\mathbb{Q}} q$, *i.e., r is right-open in* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$*;*

*c*) $0 \in r \neq \mathbb{Q}_0^+$, *i.e., r is nonempty and bounded in* $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$.

**Definition 56.** *We define the structure*

$$\mathbb{R}^+ := (\mathbb{R}^+, +^{\mathbb{R}}, \cdot^{\mathbb{R}}, <^{\mathbb{R}}, 1^{\mathbb{R}})$$

*of* positive real numbers *as follows:*

*a*) *Let* $\mathbb{R}^+$ *be the set of* positive reals.

*b*) *Define the* real addition $+^{\mathbb{R}}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *by*

$$r +^{\mathbb{R}} r' = \{p +^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\}.$$

*c*) *Define the* real multiplication $\cdot^{\mathbb{R}}: \mathbb{R}^+ \times \mathbb{R}^+ \rightarrow \mathbb{R}^+$ *by*

$$r \cdot^{\mathbb{R}} r' = \{p \cdot^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\}.$$

*d*) *Define the strict linear order* $<^{\mathbb{R}}$ *on* $\mathbb{R}^+$ *by*

$$r <^{\mathbb{R}} r' \text{ iff } r \subseteq r' \wedge r \neq r'.$$

*e*) *Let* $1^{\mathbb{R}} := \{p \in \mathbb{Q}_0^+ | q <^{\mathbb{Q}} 1\}$.

We justify some details of the definition.

**Lemma 57.**

*a*) $\mathbb{R}^+ \in V$.

*b*) *If* $r, r' \in \mathbb{R}^+$ *then* $r +^{\mathbb{R}} r'$, $r \cdot^{\mathbb{R}} r' \in \mathbb{R}^+$.

*c*) $<^{\mathbb{R}}$ *is a strict linear order on* $\mathbb{R}^+$.

**Proof.** a) If $r \in \mathbb{R}^+$ then $r \subseteq \mathbb{Q}_0^+$ and $r \in \mathcal{P}(\mathbb{Q}_0^+)$. Thus $\mathbb{R}^+ \subseteq \mathcal{P}(\mathbb{Q}_0^+)$, and $\mathbb{R}^+$ is a set by the power set axiom and separation.
b) Let $r, r' \in \mathbb{R}^+$. We show that

$$r \cdot^{\mathbb{R}} r' = \{p \cdot^{\mathbb{Q}} p' | p \in r \wedge p' \in r'\} \in \mathbb{R}^+.$$

Obviously $r \cdot^{\mathbb{R}} r' \subseteq \mathbb{Q}_0^+$ is a non-empty bounded initial segment of $(\mathbb{Q}_0^+, <^{\mathbb{Q}})$.

Consider $p \in r \cdot^{\mathbb{R}} r'$, $q \in \mathbb{Q}_0^+$, $q <^{\mathbb{Q}} p$. Let $p = \frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'}$ where $\frac{a}{b} \in r$ and $\frac{a'}{b'} \in r'$. Let $q = \frac{c}{d}$. Then $\frac{c}{d} = \frac{c \cdot b'}{d \cdot a'} \cdot^{\mathbb{Q}} \frac{a'}{b'}$, where

$$\frac{c \cdot b'}{d \cdot a'} = q \cdot^{\mathbb{Q}} \frac{b'}{a'} <^{\mathbb{Q}} p \cdot^{\mathbb{Q}} \frac{b'}{a'} = \frac{a}{b} \cdot^{\mathbb{Q}} \frac{a'}{b'} \cdot^{\mathbb{Q}} \frac{b'}{a'} = \frac{a}{b} \in r.$$

Hence $\frac{c \cdot b'}{d \cdot a'} \in r$ and

$$\frac{c}{d} = \frac{c \cdot b'}{d \cdot a'} \cdot^{\mathbb{Q}} \frac{a'}{b'} \in r \cdot^{\mathbb{R}} r'.$$

Similarly one can show that $r \cdot^{\mathbb{R}} r'$ is open on the right-hand side.
c) The transitivity of $<^{\mathbb{R}}$ follows from the transitivity of the relation $\subsetneq$. To show that $<^{\mathbb{R}}$ is connex, consider $r, r' \in \mathbb{R}^+$, $r \neq r'$. Then $r$ and $r'$ are different subsets of $\mathbb{Q}_0^+$. Without loss of generality we may assume that there is some $p \in r' \setminus r$. We show that then $r <^{\mathbb{R}} r'$, i.e., $r \subsetneq r'$. Consider $q \in r$. Since $p \notin r$ we have $p \not<^{\mathbb{Q}} q$ and $q \leqslant^{\mathbb{Q}} p$. Since $r'$ is an initial segment of $\mathbb{Q}_0^+$, $q \in r'$. □

**Exercise 23.** Show that $(\mathbb{R}^+, \cdot^{\mathbb{R}}, 1^{\mathbb{R}})$ is a multiplicative group.

We can now construct the complete real line $\mathbb{R}$ from $\mathbb{R}^+$ just like we constructed $\mathbb{Z}$ from $\mathbb{N}$. Details are left to the reader. We can also proceed to define the structure $\mathbb{C}$ of complex numbers from $\mathbb{R}$.

**Exercise 24.** Formalize the structure $\mathbb{C}$ of complex numbers such that $\mathbb{R} \subseteq \mathbb{C}$.

## 6.4 Discussion

The constructions carried out in the previous subsections contained many arbitrary choices. One could, e.g., define rational numbers as *reduced* fractions instead of equivalence classes of fractions, ensure that the canonical embeddings of number systems are inclusions, etc. If such choices have been made in reasonable ways we obtain the following theorem, which contains everything one wants to know about the number systems. So the statements of the following theorem can be seen as first- and second-order axioms for these systems.

**Theorem 58.** *There are structures $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}$, and $\mathbb{C}$ with the following properties:*

a) *the domains of these structures which are also denoted by $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$, resp., satisfy*

$$\omega = \mathbb{N} \subseteq \mathbb{Z} \subseteq \mathbb{Q} \subseteq \mathbb{R} \subseteq \mathbb{C};$$

b) *there are functions $+: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ and $\cdot: \mathbb{C} \times \mathbb{C} \to \mathbb{C}$ on $\mathbb{C}$ which are usually written as binary infix operations;*

c) *$(\mathbb{C}, +, \cdot, 0, 1)$ is a field; for $a, b \in \mathbb{C}$ write $a - b$ for the unique element $z$ such that $a = b + z$; for $a, b \in \mathbb{C}$ with $b \neq 0$ write $\frac{a}{b}$ for the unique element $z$ such that $a = b \cdot z$;*

d) *there is a constant $i$, the* imaginary unit*, such that $i \cdot i + 1 = 0$ and*

$$\mathbb{C} = \{x + i \cdot y \mid x, y \in \mathbb{R}\};$$

e) *there is a strict linear order $<$ on $\mathbb{R}$ such that $(\mathbb{R}, <, + \upharpoonright \mathbb{R}^2, \cdot \upharpoonright \mathbb{R}^2, 0, 1)$ is an ordered field.*

f) *$(\mathbb{R}, <)$ is* complete*, i.e., bounded subsets of $\mathbb{R}$ possess suprema:*

$$\forall X \subseteq \mathbb{R} \, (X \neq \emptyset \wedge \exists b \in \mathbb{R} \forall x \in X x < b \longrightarrow \exists b \in \mathbb{R} \, (\forall x \in X x < b \wedge \neg \exists b' < b \forall x \in X x < b'))$$

g) *$\mathbb{Q}$ is* dense *in $(\mathbb{R}, <)$:*

$$\forall r, s \in \mathbb{R} \, (r < s \longrightarrow \exists a, b, c \in \mathbb{Q} \; a < r < b < s < c);$$

h) *$(\mathbb{Q}, + \upharpoonright \mathbb{Q}^2, \cdot \upharpoonright \mathbb{Q}^2, 0, 1)$ is a field; moreover*

$$\mathbb{Q} = \left\{ \frac{a}{b} \mid a \in \mathbb{Z}, b \in \mathbb{Z} \setminus \{0\} \right\};$$

i) *$(\mathbb{Z}, + \upharpoonright \mathbb{Z}^2, \cdot \upharpoonright \mathbb{Z}^2, 0, 1)$ is a ring with a unit; moreover*

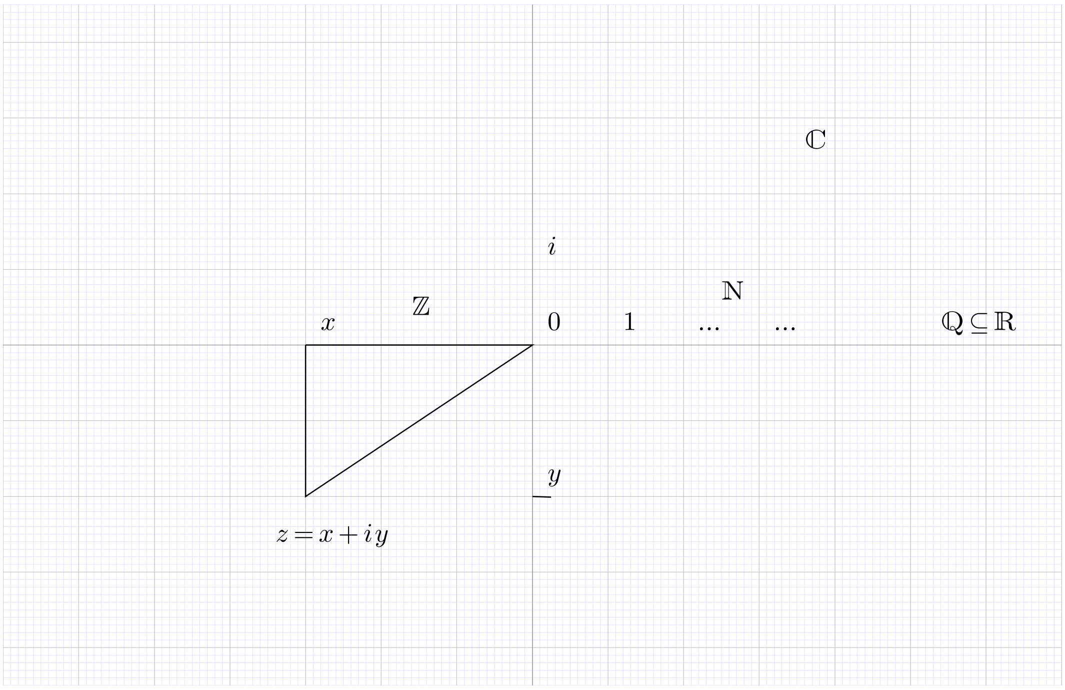
$$\mathbb{Z} = \{a - b \mid a, b \in \mathbb{N}\};$$

j) *$+ \upharpoonright \mathbb{N}^2$ agrees with ordinal addition on $\omega$; $\cdot \upharpoonright \mathbb{N}^2$ agrees with ordinal multiplication on $\omega$;*

*k)* $(\mathbb{N}, +1, 0)$ *satisfies the second-order* PEANO *axioms, i.e., the successor function* $n \mapsto n+1$ *is injective,* $0$ *is not in the image of the successor function, and*

$$\forall X \subseteq \mathbb{N}\,(0 \in X \wedge \forall n \in X n+1 \in X \longrightarrow X = \mathbb{N}).$$

This theorem is all we require from the number systems. The details of the previous construction will not be used again. So we have the standard complex plane, possibly with the identification of $\mathbb{N}$ and $\omega$.




**Remark 59.** In set theory the set $\mathbb{R}$ of reals is often identified with the sets ${}^{\omega}\omega$ or ${}^{\omega}2$, basically because all these sets have the same cardinality. We shall come back to this in the context of cardinality theory.

# 7 Sequences

The notion of a *sequence* is crucial in many contexts.

**Definition 60.**

*a)* *A set* $w$ *is an* $\alpha$-sequence *iff* $w\colon \alpha \to V$*; then* $\alpha$ *is called the* length *of the* $\alpha$*-sequence* $w$ *and is denoted by* $|\alpha|$. $w$ *is a* sequence *iff it is an* $\alpha$*-sequence for some* $\alpha$. *A sequence* $w$ *is called* finite *iff* $|w| < \omega$.

*b)* *A finite sequence* $w\colon n \to V$ *may be denoted by its enumeration* $w_0, \ldots, w_{n-1}$ *where we write* $w_i$ *instead of* $w(i)$*. One also writes* $w_0 \ldots w_{n-1}$ *instead of* $w_0, \ldots, w_{n-1}$*, in particular if* $w$ *is considered to be a* word *formed out of the* symbols $w_0, \ldots, w_{n-1}$.

*c)* *An* $\omega$*-sequence* $w\colon \omega \to V$ *may be denoted by* $w_0, w_1, \ldots$ *where* $w_0, w_1, \ldots$ *suggests a definition of* $w$.

*d)* *Let* $w\colon \alpha \to V$ *and* $w'\colon \alpha' \to V$ *be sequences. Then the* concatenation $w\hat{\ }w'\colon \alpha + \alpha' \to V$ *is defined by*

$$(w\hat{\ }w') \upharpoonright \alpha = w \upharpoonright \alpha \text{ and } \forall i < \alpha'\; w\hat{\ }w'(\alpha + i) = w'(i).$$

*e)* *Let* $w\colon \alpha \to V$ *and* $x \in V$*. Then the* adjunction $w x$ *of* $w$ *by* $x$ *is defined as*

$$w x = w\hat{\ }\{(0, x)\}.$$

Sequences and the concatenation operation satisfy the algebraic laws of a *monoid* with cancellation rules.

**Proposition 61.** *Let $w, w', w''$ be sequences. Then*

a) $(w\hat{}w')\hat{}w'' = w\hat{}(w'\hat{}w'')$.

b) $\emptyset\hat{}w = w\hat{}\emptyset = w$.

c) $w\hat{}w' = w\hat{}w'' \rightarrow w' = w''$.

There are many other operations on sequences. One can *permute* sequences, substitute elements of a sequence, etc.

## 7.1 ($\omega$-)Sequences of Reals

$\omega$-sequences are particularly prominent in analysis. One may now define properties like

$$\lim_{i\to\infty} w_i = z \text{ iff } \forall\varepsilon\in\mathbb{R}^+\exists m<\omega\forall i<\omega\ (i\geqslant m\rightarrow(z-\varepsilon<w_i\wedge w_i<z+\varepsilon))$$

or

$$\forall x\colon\omega\rightarrow\mathbb{R}\,(\lim_{i\to\infty} x_i = a \rightarrow \lim_{i\to\infty} f(x_i) = f(a)).$$

If $x_0, x_1, \ldots$ is given then the partial sums

$$\sum_{i=0}^{n} x_i$$

are defined recursively as

$$\sum_{i=0}^{0} x_i = 0 \quad\text{and}\quad \sum_{i=0}^{n+1} x_i = (\sum_{i=0}^{n} x_i) + x_n\,.$$

The map $\varphi\colon {}^{\omega}2\rightarrow\mathbb{R}$ defined by

$$\varphi((x_i)_{i<\omega}) = \sum_{i=0}^{\infty}\frac{x_i}{2^{i+1}} = \lim_{n\to\infty}\sum_{i=0}^{n}\frac{x_i}{2^{i+1}}\,.$$

maps the function space ${}^{\omega}2$ surjectively onto the real interval

$$[0,1] = \{r\in\mathbb{R}\,|\,0\leqslant r\leqslant 1\}.$$

Such maps are the reason that one often identifies ${}^{\omega}2$ with $\mathbb{R}$ in set theory.

## 7.2 Symbols and Words

Languages are mathematical objects of growing importance. Mathematical logic takes terms and formulas as mathematical material. Terms and formulas are finite sequences of symbols from some alphabet. We represent the standard symbols $=$, $\in$, etc. by some set-theoretical terms $\dot{=}$, $\dot{\in}$, etc. Note that details of such a formalization are highly arbitrary. One really only has to *fix* certain sets to denote certain symbols.

**Definition 62.** *Formalize the* basic set-theoretical symbols *by*

a) $\dot{=} = 0,\ \dot{\in} = 1,\ \dot{\wedge} = 2,\ \dot{\vee} = 3,\ \dot{\rightarrow} = 4,\ \dot{\leftrightarrow} = 5,\ \dot{\neg} = 6,\ \dot{(} = 7,\ \dot{)} = 8,\ \dot{\exists} = 9,\ \dot{\forall} = 10.$

b) *Variables* $\dot{v}_n = (1,n)$ *for* $n<\omega$.

c) *Let* $L_\in = \{\dot{=},\dot{\in},\dot{\wedge},\dot{\vee},\dot{\rightarrow},\dot{\leftrightarrow},\dot{\neg},\dot{(},\dot{)},\dot{\exists},\dot{\forall}\}\cup\{(1,n)|n<\omega\}$ *be the* alphabet of set theory.

d) *A* word *over* $L_\in$ *is a finite sequence with values in* $L_\in$.

e) *Let* $L_\in^* = \{w\,|\,\exists n<\omega\ w\colon n\rightarrow L_\in\}$ *be the set of all words over* $L_\in$.

f) *If* $\varphi$ *is a standard set-theoretical formula, we let* $\dot{\varphi}\in L_\in^*$ *denote the formalization of* $\varphi$. *E.g.,* $\dot{\mathrm{Ex}} = \dot{\exists}\dot{v}_0\dot{\forall}\dot{v}_1\dot{\neg}\dot{v}_1\dot{\in}\dot{v}_0$ *is the formalization of the set existence axiom. If the intention is clear, one often omits the formalization dots and simply writes* $\dot{\mathrm{Ex}} = \exists v_0\forall v_1\neg v_1\in v_0$.

This formalization can be developed much further, so that the notions and theorems of first-order logic are available in the theory ZF. By carrying out the definition of the axiom system ZF *within* set theory, one obtains a term $\dot{\mathrm{ZF}}$ which represents ZF within ZF. This (quasi) self-referentiality is the basis for limiting results like the GÖDEL incompleteness theorems.

## 8 The von Neumann Hierarchy

We use ordinal recursion to obtain more information on the universe of all sets.

**Definition 63.** *Define the* von Neumann Hierarchy $(V_\alpha)_{\alpha\in\mathrm{Ord}}$ *by recursion:*

*a)* $V_0=\emptyset$ *;*

*b)* $V_{\alpha+1}=\mathcal{P}(V_\alpha)$ *;*

*c)* $V_\lambda=\bigcup_{\alpha<\lambda}V_\alpha$ *for limit ordinals* $\lambda$ *.*

We show that the von Neumann hierarchy is indeed a (fast-growing) hierarchy

**Lemma 64.** *Let* $\beta<\alpha\in\mathrm{Ord}$*. Then*

*a)* $V_\beta\in V_\alpha$

*b)* $V_\beta\subseteq V_\alpha$

*c)* $V_\alpha$ *is transitive*

**Proof.** We conduct the proof by a simultaneous induction on $\alpha$.
$\alpha=0$: $\emptyset$ is transitive, thus a)-c) hold at 0.
For the *successor case* assume that a)-c) hold at $\alpha$. Let $\beta<\alpha+1$. By the inductive assumption, $V_\beta\subseteq V_\alpha$ and $V_\beta\in\mathcal{P}(V_\alpha)=V_{\alpha+1}$. Thus a) holds at $\alpha+1$. Consider $x\in V_\alpha$. By the inductive assumption, $x\subseteq V_\alpha$ and $x\in V_{\alpha+1}$. Thus $V_\alpha\subseteq V_{\alpha+1}$. Then b) at $\alpha+1$ follows by the inductive assumption. Now consider $x\in V_{\alpha+1}=\mathcal{P}(V_\alpha)$. Then $x\subseteq V_\alpha\subseteq V_{\alpha+1}$ and $V_{\alpha+1}$ is transitive.
For the *limit case* assume that $\alpha$ is a limit ordinal and that a)-c) hold at all $\gamma<\alpha$. Let $\beta<\alpha$. Then $V_\beta\in V_{\beta+1}\subseteq\bigcup_{\gamma<\alpha}V_\gamma=V_\alpha$ hence a) holds at $\alpha$. b) is trivial for limit $\alpha$. $V_\alpha$ is transitive as a union of transitive sets. □

The $V_\alpha$ are nicely related to the ordinal $\alpha$.

**Lemma 65.** *For every* $\alpha$*,* $V_\alpha\cap\mathrm{Ord}=\alpha$*.*

**Proof.** Induction on $\alpha$. $V_0\cap\mathrm{Ord}=\emptyset\cap\mathrm{Ord}=\emptyset=0$.
For the *successor case* assume that $V_\alpha\cap\mathrm{Ord}=\alpha$. $V_{\alpha+1}\cap\mathrm{Ord}$ is transitive, and every element of $V_{\alpha+1}\cap\mathrm{Ord}$ is transitive. Hence $V_{\alpha+1}\cap\mathrm{Ord}$ is an ordinal, say $\delta=V_{\alpha+1}\cap\mathrm{Ord}$. $\alpha=V_\alpha\cap\mathrm{Ord}$ implies that $\alpha\in V_{\alpha+1}\cap\mathrm{Ord}=\delta$ and $\alpha+1\leqslant\delta$. Assume for a contradiction that $\alpha+1<\delta$. Then $\alpha+1\in V_{\alpha+1}$ and $\alpha+1\subseteq V_\alpha\cap\mathrm{Ord}=\alpha$, contradiction. Thus $\alpha+1=\delta=V_{\alpha+1}\cap\mathrm{Ord}$.
For the *limit case* assume that $\alpha$ is a limit ordinal and that $V_\beta\cap\mathrm{Ord}=\beta$ holds for all $\beta<\alpha$. Then

$$V_\alpha\cap\mathrm{Ord}=(\bigcup_{\beta<\alpha}V_\beta)\cap\mathrm{Ord}=\bigcup_{\beta<\alpha}(V_\beta\cap\mathrm{Ord})=\bigcup_{\beta<\alpha}\beta=\alpha.$$

□

The foundation schema implies that the $V_\alpha$-hierarchy exhausts the universe $V$.

**Theorem 66.**

*a)* $\forall x\subseteq\bigcup_{\alpha\in\mathrm{Ord}}V_\alpha\,\exists\beta\,x\subseteq V_\beta$.

*b)* $V=\bigcup_{\alpha\in\mathrm{Ord}}V_\alpha$.

**Proof.** a) Let $x\subseteq\bigcup_{\alpha\in\mathrm{Ord}}$. Define a function $f\colon x\to\mathrm{Ord}$ by

$$f(u)=\min\{\gamma\,|\,u\in V_\gamma\}.$$

By the axioms of replacement and union, $\beta = \bigcup \{f(u)+1 | u \in x\} \in V$ and $\beta \in \mathrm{Ord}$. Let $u \in x$. Then $f(u) < f(u)+1 \leqslant \beta$ and $u \in V_{f(u)} \subseteq V_\beta$. Thus $x \subseteq V_\beta$.
b) Let $B = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. By the schema of $\in$-induction it suffices to show that

$$\forall x\,(x \subseteq B \rightarrow x \in B).$$

So let $x \subseteq B = \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha$. By a) take $\beta$ such that $x \subseteq V_\beta$. Then $x \in V_{\beta+1} \subseteq \bigcup_{\alpha \in \mathrm{Ord}} V_\alpha = B$. □

The $V_\alpha$-hierarchy ranks the elements of $V$ into levels.

**Definition 67.** *Define the* rank (function) $\mathrm{rk}: V \rightarrow \mathrm{Ord}$ *by*

$$x \in V_{\mathrm{rk}(x)+1} \setminus V_{\mathrm{rk}(x)}.$$

The rank function satisfies a recursive law.

**Lemma 68.** $\forall x\ \mathrm{rk}(x) = \bigcup_{y \in x} \mathrm{rk}(y)+1$.

**Proof.** Let us prove the statement

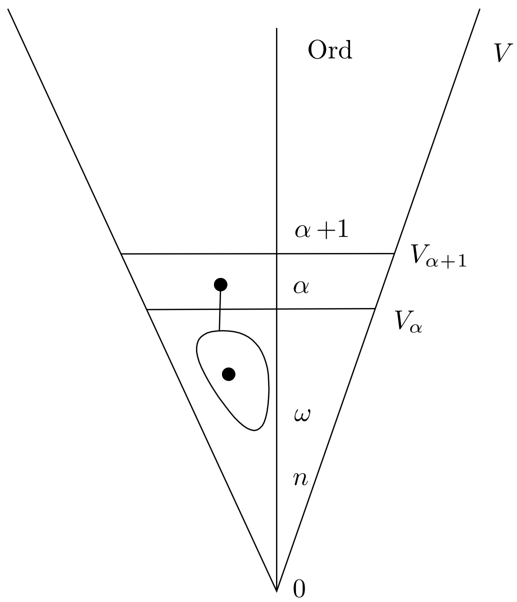
$$\forall x \in V_\alpha\ \mathrm{rk}(x) = \bigcup_{y \in x} \mathrm{rk}(y)+1$$

by induction on $\alpha$. The case $\alpha = 0$ is trivial. The limit case is obvious since $V_\lambda = \bigcup_{\alpha<\lambda} V_\alpha$ for limit $\lambda$.

For the successor case assume that the statement holds for $\alpha$. Consider $x \in V_{\alpha+1}$. If $x \in V_\alpha$ the statement holds by the inductive assumption. So assume that $x \in V_{\alpha+1} \setminus V_\alpha$. Then $\mathrm{rk}(x) = \alpha$. Let $y \in x \subseteq V_\alpha$. Then $y \in V_{\beta+1} \setminus V_\beta$ for some $\beta = \mathrm{rk}(y) < \alpha$. $\mathrm{rk}(y)+1 \subseteq \alpha$. Thus $\bigcup_{y \in x} \mathrm{rk}(y)+1 \subseteq \alpha$. Assume that $\gamma = \bigcup_{y \in x} \mathrm{rk}(y)+1 < \alpha$. Let $y \in x$. Then $\mathrm{rk}(y)+1 \leqslant \gamma$ and $y \in V_{\mathrm{rk}(y)+1} \subseteq V_\gamma$. Thus $x \subseteq V_\gamma$, $x \in V_{\gamma+1} \subseteq V_\alpha$, contradicting the assumption that $x \in V_{\alpha+1} \setminus V_\alpha$. □

**Lemma 69.** *Let $A$ be a term. Then $A \in V$ iff $\exists \alpha\ A \subseteq V_\alpha$.*

The previous analysis of the $V_\alpha$-hierarchy suggest the following picture of the universe $V$.



# 9 The Axiom of Choice

Natural numbers $n \in \mathbb{N}$ are used to enumerate finite sets $a$ as

$$a = \{a_0, a_1, \ldots, a_{n-1}\}.$$

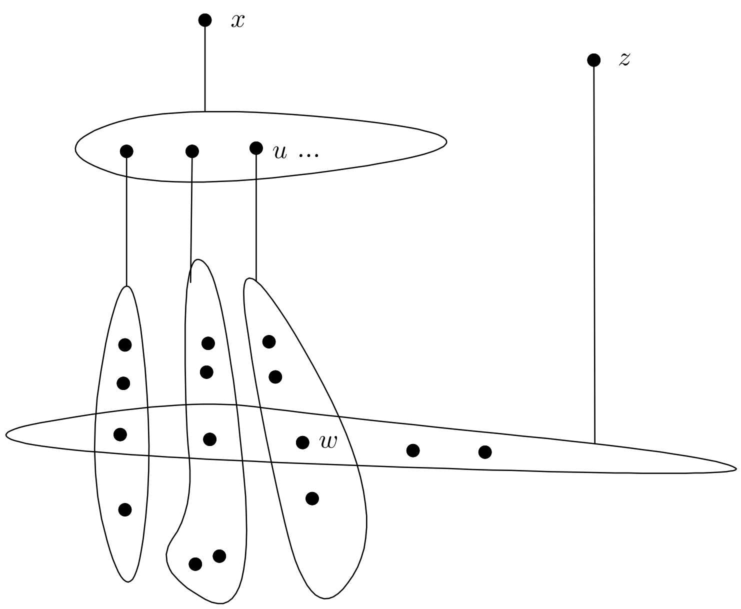
Assuming the *axiom of choice*, one can use ordinals to enumerate any set $a$ as

$$a = \{a_i \mid i < \alpha\}.$$

**Definition 70.** *The* Axiom of Choice*,* AC *is the statement*

$$\forall x (\emptyset \notin x \wedge \forall u, v \in x (u \neq v \rightarrow u \cap v = \emptyset) \rightarrow \exists z \forall u \in x \exists w\, u \cap z = \{w\}).$$

*The axiom expresses that for every set $x$ consisting of nonempty pairwise disjoint elements there exists a* choice set *$z$, i.e., for every element $u \in x$ the intersection $u \cap z$ consists exactly of one element. Thus $z$ "chooses" one element out of every element of $x$.*




It seems intuitively clear that such choices are possible. On the other hand we shall see that the axiom of choice has unintuitive, paradoxical consequences.

**Theorem 71.** *The following statements are equivalent:*

*a)* AC*;*

*b)* $\forall x \exists g$ *($g$ is a function with domain $x \wedge \forall u \in x$ $(u \neq \emptyset \rightarrow g(u) \in u)$); such a function $g$ is called a* choice function *for $x$;*

*c)* $\forall x \exists \alpha \exists f f: \alpha \leftrightarrow x$.

**Proof.** *a)* $\rightarrow$ *b)* Assume AC. Let $x$ be a set. We may assume that every element of $x$ is nonempty. The class

$$x' = \{\{u\} \times u \mid u \in x\}$$

is the image of $x$ under the set valued map $u \mapsto \{u\} \times u$, and thus a set by replacement. The elements $\{u\} \times u$ of $x'$ are nonempty and pairwise disjoint. By AC, take a choice set $z$ for $x'$. Define a choice function $g: x \to V$ by letting $g(u)$ be the unique element of $u$ such that

$$(\{u\} \times u) \cap z = \{(u, g(u))\}.$$

*b*) → *c*) Assume *b*). Let $x$ be a set and let $g: \mathcal{P}(x) \setminus \{\emptyset\} \to V$ be a choice function for $\mathcal{P}(x) \setminus \{\emptyset\}$. Define a function $F: \mathrm{Ord} \to x \cup \{x\}$ by ordinal recursion such that

$$F(\alpha) = \begin{cases} g(x \setminus F[\alpha]), \text{ if } x \setminus F[\alpha] \neq \emptyset; \\ x, \text{ if } x \setminus F[\alpha] = \emptyset. \end{cases}$$

At "time" $\alpha$, the function $F$ chooses an element $F(\alpha) \in x$ which has not been chosen before. If all elements of $x$ have been chosen, this is signaled by $F$ by the value $x$ which is not an element of $x$.
(1) Let $\alpha < \beta$ and $F(\beta) \neq x$. Then $F(\alpha), F(\beta) \in x$ and $F(\alpha) \neq F(\beta)$.
*Proof*. $F(\beta) \neq x$ implies that $x \setminus F[\beta] \neq \emptyset$ and hence $F(\beta) = g(x \setminus F[\beta]) \in x \setminus F[\beta]$. Since $\alpha \in \beta$, $x \setminus F[\alpha] \neq \emptyset$ and $F(\alpha) = g(x \setminus F[\alpha]) \in x \setminus F[\alpha]$. $F(\alpha) \neq F(\beta)$ follows from $F(\beta) \in x \setminus F[\beta]$. *qed*(1)
(2) There is $\alpha \in \mathrm{Ord}$ such that $F(\alpha) = x$.
*Proof*. Assume not. Then by (1), $F: \mathrm{Ord} \to x$ is injective. Hence $F^{-1}$ is a function and $\mathrm{Ord} = F^{-1}[x]$. By replacement, Ord is a set, but this is a contradiction. *qed*(2)
By (2) let $\alpha$ be minimal such that $F(\alpha) = x$. Let $f = F \upharpoonright \alpha: \alpha \to x$. By the definition of $F$, $x \setminus F[\alpha] = \emptyset$, i.e., $F[\alpha] = x$ and $f$ is surjective. By (1), $f$ is also injective, i.e., $f: \alpha \leftrightarrow x$.

*c*) → *a*) Assume *c*). Let the set $x$ consist of nonempty pairwise disjoint elements. Apply *c*) to $\bigcup x$. Take an ordinal $\alpha$ and a function $f: \alpha \to \bigcup x$. Define a choice set $z$ for $x$ by setting

$$z = \{f(\xi) | \exists u \in x\, (f(\xi) \in u \wedge \forall \zeta < \xi\, f(\zeta) \notin u)\}.$$

So $z$ chooses for every $u \in x$ that $f(\xi) \in u$ with $\xi$ minimal. □

We shall later use the enumeration property *c*) to define the cardinality of a set. ZORN's Lemma is an important existence principle which is also equivalent to AC.

**Definition 72.** *Let $(P, \leqslant)$ be a partial order.*

a) *$X \subseteq P$ is a* chain *in $(P, \leqslant)$ if $(X, \leqslant)$ is a linear order where $(X, \leqslant)$ is a short notation for the structure $(X, \leqslant \cap X^2)$.*
b) *An element $p \in P$ is an* upper bound *for $X \subseteq P$ iff $\forall x \in X\, x \leqslant p$.*
c) *$(P, \leqslant)$ is* inductive *iff every chain in $(P, \leqslant)$ possesses an upper bound.*
d) *An element $p \in P$ is a* maximal element *of $(P, \leqslant)$ iff $\forall q \in P\, (q \geqslant p \to q = p)$.*

**Theorem 73.** *The axiom of choice is equivalent to the following principle, called Zorn's Lemma: every inductive partial order $(P, \leqslant) \in V$ possesses a maximal element.*

**Proof.** Assume AC and let $(P, \leqslant) \in V$ be an inductive partial order. Let $g: \mathcal{P}(P) \setminus \{\emptyset\} \to V$ be a choice function for $\mathcal{P}(P) \setminus \{\emptyset\}$. Define a function $F: \mathrm{Ord} \to P \cup \{P\}$ by ordinal recursion; if there is an upper bound for $F[\alpha]$ which is not an element of $F[\alpha]$ let

$$F(\alpha) = g(\{p \in P \setminus F[\alpha] \mid p \text{ is an upper bound for } F[\alpha]\});$$

otherwise set

$$F(\alpha) = P.$$

At "time" $\alpha$, the function $F$ chooses a strict upper bound of $F[\alpha]$ if possible. If this is not possible, this is signaled by $F$ by the value $P$.
The definition of $F$ implies immediately:
(1) Let $\alpha < \beta$ and $F(\beta) \neq P$. Then $F(\alpha) < F(\beta)$.
(2) There is $\alpha \in \mathrm{Ord}$ such that $F(\alpha) = P$.
*Proof*. Assume not. Then by (1), $F: \mathrm{Ord} \to P \in V$ is injective, and we get the same contradiction as in the proof of Theorem 71. *qed*(2)

By (2) let $\alpha$ be minimal such that $F(\alpha) = P$. By (1), $F[\alpha]$ is a chain in $(P, \leqslant)$. Since the partial order is inductive, take an upper bound $p$ of $F[\alpha]$. We claim that $p$ is a maximal element of $(P, \leqslant)$. Assume not and let $q \in P$, $q > p$. Then $q$ is a strict upper bound of $F[\alpha]$ and $q \notin F[\alpha]$. But then the definition of $F$ yields $F(\alpha) \neq P$, contradiction.

For the converse assume Zorn's Lemma and consider a set $x$ consisting of nonempty pairwise disjoint elements. Define the set of "partial choice sets" which have empty or singleton intersection with every element of $x$:

$$P = \left\{z \subseteq \bigcup x \mid \forall u \in x (u \cap z = \emptyset \vee \exists w\, u \cap z = \{w\})\right\}.$$

$P$ is partially ordered by $\subseteq$. If $X$ is a chain in $(X, \subseteq)$ then $\bigcup X$ is an upper bound for $X$. Hence $(X, \subseteq)$ is inductive.

By Zorn's Lemma let $z$ be a maximal element of $(X, \subseteq)$. We claim that $z$ is a "total" choice set for $x$:
(3) $\forall u \in x \exists w\, u \cap z = \{w\}$.
*Proof*. If not, take $u \in x$ such that $u \cap z = \emptyset$. Take $w \in u$ and let $z' = z \cup \{w\}$. Then $z' \in P$, contrary to the the $\subseteq$-maximality of $z$. □

**Theorem 74.** *Every vector space $U \in V$ has a basis $B$, which is linearly independent and spans $U$.*

**Proof.** Let $U$ be a vector space with scalar field $K$. Let

$$P = \{b \subseteq U \mid b \text{ is linearly independent in } U\}.$$

We shall apply Zorn's lemma to the partial order $(P, \subseteq)$.
(1) $(P, \subseteq)$ is inductive.
*Proof*. Let $X \subseteq P$ be a chain. Let $c = \bigcup X \subseteq U$. We show that $c$ is linearly independent. Consider a linear combination

$$k_0 \cdot v_0 + \ldots + k_{n-1} \cdot v_{n-1} = 0,$$

where $v_0, \ldots, v_{n-1} \in c$ and $k_0, \ldots, k_{n-1} \in K$. Take $b_0, \ldots, b_{n-1} \in X$ such that $v_0 \in b_0, \ldots, v_{n-1} \in b_{n-1}$. Since $X$ is a chain there is some $b_i$, $i < n$ such that $b_0, \ldots, b_{n-1} \subseteq b_i$. Then $v_0, \ldots, v_{n-1} \in b_i$. Since $b_i \in P$ is linearly independent, $k_0 = \ldots = k_{n-1} = 0$. *qed*(1)

By Zorn's lemma, $(P, \subseteq)$ has a maximal element, say $B$. $B$ is linearly independent since $B \in P$.
(2) $B$ spans $U$.
*Proof*. Let $v \in U$. If $v \in B$ then $v$ is in the span of $B$. So consider the case that $v \notin B$. Then $B \cup \{v\}$ is a proper superset of $B$. By the $\subseteq$-maximality of $B$, $B \cup \{v\}$ is linearly dependent. So there is a non-trivial linear combination

$$k_0 \cdot v_0 + \ldots + k_{n-1} \cdot v_{n-1} + k \cdot v = 0,$$

where $v_0, \ldots, v_{n-1} \in B$ and at least one of the coefficients $k_0, \ldots, k_{n-1}, k \in K$ is non-zero. If $k = 0$,

$$k_0 \cdot v_0 + \ldots + k_{n-1} \cdot v_{n-1} = 0$$

would be a non-trivial representation of 0, contradicting that $B$ is linearly independent. Hence $k \neq 0$ and

$$v = -\frac{k_0}{k} \cdot v_0 - \ldots - \frac{k_{n-1}}{k} \cdot v_{n-1}.$$

So $v$ is in the span of $B$. □

Actually one can show the converse of this Theorem: if every vector space has a basis, then AC holds.

As another application of Zorn's lemma we consider *filters* which are collections of "large" subsets of some domain.

**Definition 75.** *Let $Z$ be a set. We say that $F$ is a* filter *on $Z$ if*

*a)* $F \subseteq \mathcal{P}(Z)$*;*

*b*) $\emptyset \notin F$;

*c*) $X \in F$ *and* $X \subseteq Y \subseteq Z$ *implies that* $Y \in F$;

*d*) $X, Y \in F$ *implies that* $X \cap Y \in F$.

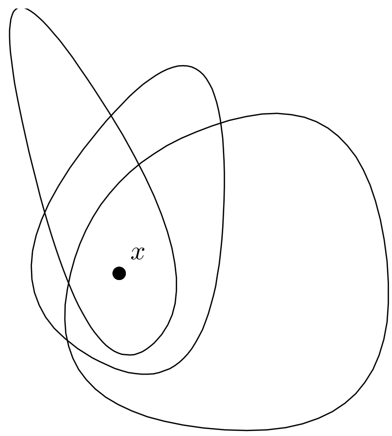
*If moreover*

$$X \subseteq Z \rightarrow X \in F \vee (Z \setminus X) \in F$$

*we call* $F$ *an* ultrafilter *on* $Z$.

Important examples of filters are *neighbourhood filters* $N_x$ of points $x$ in some topological space $(Z, \mathcal{T})$:

$$N_x = \{U \subseteq Z \mid U \text{ is a neighbourhood of } x\}.$$




A combinatorial example is the *Frechet filter* on $\omega$:

$$F = \{X \subseteq \omega \mid \exists n \in \omega \forall m \in \omega\, (m > n \rightarrow m \in X)\}.$$

The expression "$A(n)$ holds for almost all $n \in \omega$" is equivalent to

$$\{n \in \omega \mid A(n)\} \in F.$$

**Theorem 76.** *Let* $F$ *be a filter on the set* $Z$. *Then there is an extension* $G \supseteq F$ *such that* $G$ *is an ultrafilter on* $Z$.

**Proof.** Let

$$P = \{H \subseteq \mathcal{P}(Z) | H \text{ is a filter on } U \text{ and } H \supseteq F\}.$$

We shall apply Zorn's lemma to the partial order $(P, \subseteq)$.
(1) $(P, \subseteq)$ is inductive.
*Proof*. Let $C \subseteq P$ be a chain. Let $H' = \bigcup X \subseteq \mathcal{P}(Z)$. We show that $H'$ is a filter on $Z$. Trivially $\emptyset \notin H'$. Consider $X \in H'$ and $X \subseteq Y \subseteq Z$. Then $X \in H$ for some $H \in C$. Since $H$ is a filter, $X \in H$ and so $Y \in H \subseteq H'$.

For the closure under intersections consider $X, Y \in H'$. Then $X \in H_0$ for some $H_0 \in C$, and $Y \in H_1$ for some $H_1 \in C$. Since $C$ is a chain, we have, wlog, that $H_0 \subseteq H_1$. Then $X, Y \in H_1$, and $X \cap Y \in H_1 \subseteq H'$. *qed*(1)

By Zorn's lemma, let $G \in P$ be a maximal element. Then $G$ is a filter which extends $F$.
(2) $G$ is an ultrafilter on $Z$.
*Proof*. Consider $X_0 \subseteq Z$. Assume for a contradiction that $X_0 \notin G$ and $Z \setminus X_0 \notin G$.
*Case 1*. $X \cap X_0 \neq \emptyset$ for every $X \in G$. Define

$$G' = \{Y \subseteq Z \mid \exists X \in G\ Y \supseteq X \cap X_0\}.$$

$G'$ is a filter on $Z$; we only check Definition 75, *d*): let $Y_1, Y_2 \in G'$ with $Y_1 \supseteq X_1 \cap X_0$ and $Y_2 \supseteq X_2 \cap X_0$ where $X_1, X_2 \in G$. Then $Y_1 \cap Y_2 \supseteq (X_1 \cap X_2) \cap X_0$ where $X_1 \cap X_2 \in G$, and so $Y_1 \cap Y_2 \in G'$.

Obviously $G' \supseteq G \supseteq F$ and $G' \neq G$ since $X_0 \in G'$ and $X_0 \notin G$. This contradicts the maximality of $G$ in $(P, \subseteq)$.
*Case 2.* $X_1 \cap X_0 = \emptyset$ for some $X_1 \in G$. Then $X_1 \subseteq Z \setminus X_0$. For $X \in G$ we have

$$X \cap (Z \setminus X_0) \supseteq X \cap X_1 \neq \emptyset$$

since $X \cap X_1 \in G$. So we can carry out the argument of *Case 1* with $Z \setminus X_0$ in place of $X_0$ and also get the desired contradiction. □

**Definition 77.** *The axiom system* ZFC *consists of the* ZF*-axioms together with the axiom of choice* AC*.*

The system ZFC is usually taken as the foundation of mathematics. The ZF axioms have a good intuitive motivation. The axiom of choice is more controversial; AC has desirable consequences like Zorn's Lemma and its applications, but on the other hand AC has some paradoxical and problematic consequences. The status of AC within set theory can be compared to the parallel axiom in geometry. Similar to the situation in (non-)euclidean geometry one can show that if there is a model of the ZF axioms then there is a model of ZFC.

**Exercise 25.** Show that in the theory ZF the axiom of choice is equivalent to the *Hausdorff Maximality Principle* which says: for every partial order $(P, \leqslant) \in V$ there is an inclusion maximal chain $X$ in $(P, \leqslant)$, i.e., if $Y \supseteq X$ is a chain in $(P, \leqslant)$ then $Y = X$. [Hausdorff, Grundzüge der Mengenlehre, p. 141: *Wir haben damit für eine teilweise geordnete Menge A die Existenz größter geordneter Teilmengen B bewiesen; natürlich kann es deren verschiedene geben.*]

## 10 Wellfounded Relations

The axiom schema of foundation yields an induction theorem for the $\in$-relation, and in the previous section we have seen a recursive law for the rank-function. We generalize these techniques to *wellfounded* relations.

**Definition 78.** *Let $R$ be a relation on a domain $D$.*

*a)* *$R$ is* wellfounded*, iff for all terms $A$*

$$\emptyset \neq A \wedge A \subseteq D \rightarrow \exists x \in A\; A \cap \{y \mid yRx\} = \emptyset.$$

*b)* *$R$ is* strongly wellfounded *iff it is wellfounded and*

$$\forall x \in D\; \{y \in D \mid yRx\} \in V.$$

*c)* *$R$ is a* wellorder *iff $R$ is a wellfounded strict linear order.*

*d)* *$R$ is a* strong wellorder *iff $R$ is a strongly wellfounded wellorder.*

By the scheme of foundation, the $\in$-relation is strongly wellfounded. The ordinals are strongly wellordered by $<$. There are wellfounded relations which are *not* strongly wellfounded: e.g., let $R \subseteq \mathrm{Ord} \times \mathrm{Ord}$,

$$x R y \text{ iff } (x \neq 0 \wedge y \neq 0 \wedge x < y) \vee (y = 0 \wedge x \neq 0),$$

be a rearrangement of $(\mathrm{Ord}, <)$ with 0 put on top of all the other ordinals.

For strongly wellfounded relations, every element is contained in a *set-sized* initial segment of the relation.

**Lemma 79.** *Let $R$ be a strongly wellfounded relation on $D$. Then*

$$\forall x \subseteq D \exists z\, (z \subseteq D \wedge x \subseteq z \wedge \forall u \in z \forall v R u\; v \in z).$$

*Moreover for all $x \subseteq D$, the $R$-*transitive closure

$$\mathrm{TC}_R(x) = \bigcap \{z \mid z \subseteq D \wedge x \subseteq z \wedge \forall u \in z \forall v R u\; v \in z\}$$

*of $x$ is a set. In case $R$ is the $\in$-relation, we write* $\mathrm{TC}(x)$ *instead of* $\mathrm{TC}_\in(x)$*.*

**Proof.** We prove by $R$-induction that

$$\forall x\in D\ \mathrm{TC}_R(\{x\})\in V.$$

So let $x\in D$ and $\forall yRx\ \mathrm{TC}_R(\{y\})\in V$. Then

$$z=\{x\}\cup\bigcup_{yRx}\mathrm{TC}_R(\{y\})\in V$$

by replacement. $z$ is a subset of $D$ and includes $\{x\}$. $z$ is $R$-closed, i.e., closed with respect to $R$-predecessors: each $\mathrm{TC}_R(\{y\})$ is $R$-closed, and if $yRx$ then $y\in\{y\}\subseteq\mathrm{TC}_R(\{y\})\subseteq z$ . So $\mathrm{TC}_R(\{x\})$ is the intersection of a non-empty class, hence a set.

Finally observe that we may set

$$\mathrm{TC}_R(x)=\bigcup_{y\in x}\mathrm{TC}_R(\{y\}).$$

□

**Exercise 26.** Show that for an ordinal $\alpha$ , $\mathrm{TC}(\alpha)=\alpha$ and $\mathrm{TC}(\{\alpha\})=\alpha+1$ .

For *strongly* wellfounded relations, the following recursion theorem holds:

**Theorem 80.** *Let $R$ be a strongly wellfounded relation on $D$ . Let $G\colon V\to V$. Then there is a canonical class term $F$, given by the subsequent proof, such that*

$$F\colon D\to V \text{ and } \forall x\in D\ F(x)=G(F\upharpoonright\{y\,|\,yRx\}).$$

*We then say that $F$ is defined by $R$-recursion with the* recursion rule *$G$. $F$ is unique in the sense that if another term $F'$ satisfies*

$$F'\colon D\to V \text{ and } \forall\alpha\in D\ F'(x)=G(F'\upharpoonright\{y\,|\,yRx\})$$

*then $F=F'$.*

**Proof.** We proceed as in the ordinal recursion theorem. Let

$$\tilde{F}:=\{f\,|\,\exists z\subseteq D\,(\forall x\in z\ \{y\,|\,yRx\}\subseteq z,\ f\colon z\to V \text{ and } \forall x\in z\ f(x)=G(f\upharpoonright\{y\,|\,yRx\}))\}$$

be the class of all *approximations* to the desired function $F$.

(1) Let $f,g\in\tilde{F}$. Then $f,g$ are *compatible*, i.e., $\forall x\in\mathrm{dom}(f)\cap\mathrm{dom}(g)\ f(x)=g(x)$.

*Proof*. By induction on $R$. Let $x\in\mathrm{dom}(f)\cap\mathrm{dom}(g)$ and assume that $\forall yRx\ f(y)=g(y)$. Then $f\upharpoonright\{y\,|\,yRx\}=g\upharpoonright\{y\,|\,yRx\}$

$$f(x)=G(f\upharpoonright\{y\,|\,yRx\})=G(g\upharpoonright\{y\,|\,yRx\})=g(x).$$

*qed*(1)

By the compatibility of the approximation functions the union

$$F=\bigcup\tilde{F}$$

is a function defined on $\mathrm{dom}(F)\subseteq D$ . $\mathrm{dom}(F)$ is $R$-closed since the domain of every approximation is $R$-closed.

(2) $\forall x\in\mathrm{dom}(F)\,(\{y\,|\,yRx\}\subseteq\mathrm{dom}(F)\wedge F(x)=G(F\upharpoonright\{y\,|\,yRx\}))$.

*Proof*. Let $x\in\mathrm{dom}(F)$. Take some approximation $f\in\tilde{F}$ such that $x\in\mathrm{dom}(f)$. Then $\{y\,|\,yRx\}\subseteq\mathrm{dom}(f)\subseteq\mathrm{dom}(F)$ and

$$F(x)=f(x)=G(f\upharpoonright\{y\,|\,yRx\})=G(F\upharpoonright\{y\,|\,yRx\}).$$

*qed*(2)

(3) $D=\mathrm{dom}(F)$.

*Proof*. We show by $R$-induction that $\forall x\in D\ x\in\mathrm{dom}(F)$. Let $x\in D$ and assume that $\forall yRx\ y\in\mathrm{dom}(F)$. $\mathrm{TC}_R(\{y\,|\,yRx\})\subseteq\mathrm{dom}(F)$ since $\mathrm{dom}(F)$ is $R$-closed. Then

$$f=(F\upharpoonright\mathrm{TC}_R(\{y\,|\,yRx\}))\cup\{(x,G(F\upharpoonright\{y\,|\,yRx\}))\}$$

is an approximation with $x\in\mathrm{dom}(f)$, and so $x\in\mathrm{dom}(F)$. □

**Exercise 27.** Define set theoretic operations

$$x + y = x \cup \{x + z | z \in y\}$$

and

$$x \cdot y = \bigcup_{z \in y} (x \cdot z + x)$$

and study their arithmetic/algebraic properties. Show that they extend ordinal arithmetic.

**Theorem 81.** *Let $R$ be a strongly wellfounded relation on $D$ and suppose that $R$ is* extensional*, i.e., $\forall x,\ y \in D\ (\forall u\ (u R x \leftrightarrow u R y) \rightarrow x = y)$. Then there is a transitive class $\bar{D}$ and an isomorphism $\pi\colon (D, R) \leftrightarrow (\bar{D}, \in)$. $\bar{D}$ and $\pi$ are uniquely determined by $R$ and $D$, they are called the* MOSTOWSKI*-collapse of $R$ and $D$.*

**Proof.** Define $\pi\colon D \rightarrow V$ by $R$-recursion with

$$\pi(x) = \{\pi(y) | y R x\}.$$

Let $\bar{D} = \mathrm{rng}(\pi)$.
(1) $\bar{D}$ is transitive.
*Proof*. Let $\pi(x) \in \bar{D}$ and $u \in \pi(x) = \{\pi(y) | y R x\}$. Let $u = \pi(y),\ y R x$ . Then $u \in \mathrm{rng}(\pi) = \bar{D}$ . *qed*(1)
(2) $\pi$ is injective.
*Proof*. We prove by $\in$-induction that every $z \in \bar{D}$ has exactly one preimage under $\pi$ . So let $z \in \bar{D}$ and let this property be true for all elements of $z$ . Assume that $x, y \in D$ and $\pi(x) = \pi(y) = z$ . Let $u R x$ . Then $\pi(u) \in \pi(x) = \pi(y) = \{\pi(v) | v R y\}$. Take $v R y$ such that $\pi(u) = \pi(v)$. By the inductive assumption, $u = v$, and $u R y$ . Thus $\forall u\ (u R x \rightarrow u R y)$. By symmetry, $\forall u\ (u R x \leftrightarrow u R y)$. Since $R$ is extensional, $x = y$ . So $z$ has exactly one preimage under $\pi$ . *qed*(2)
(3) $\pi$ is an *isomorphism*, i.e., $\pi$ is bijective and $\forall x, y \in D\ (x R y \leftrightarrow \pi(x) \in \pi(y))$.
*Proof*. Let $x,\ y \in D$. If $x R y$ then $\pi(x) \in \{\pi(u) | u R y\} = \pi(y)$. Conversely, if $\pi(x) \in \{\pi(u) | u R y\} = \pi(y)$ then let $\pi(x) = \pi(u)$ for some $u R y$ . Since $\pi$ is injective, $x = u$ and $x R y$ . *qed*(3)
Uniqueness of the collapse $\bar{D}$ and $\pi$ is given by the next theorem. □

**Theorem 82.** *Let $X$ and $Y$ be transitive and let $\sigma\colon X \leftrightarrow Y$ be an* $\in$-$\in$-isomorphism *between $X$ and $Y$, i.e., $\forall x, y \in X\ (x \in y \leftrightarrow \sigma(x) \in \sigma(y))$. Then $\sigma = \mathrm{id} \upharpoonright X$ and $X = Y$.*

**Proof.** We show that $\sigma(x) = x$ by $\in$-induction over $X$. Let $x \in X$ and assume that $\forall y \in x\ \sigma(y) = y$ .

Let $y \in x$. By induction assumption, $y = \sigma(y) \in \sigma(x)$. Thus $x \subseteq \sigma(x)$.

Conversely, let $v \in \sigma(x)$. Since $Y = \mathrm{rng}(\sigma)$ is transitive take $u \in X$ such that $v = \sigma(u)$. Since $\sigma$ is an isomorphism, $u \in x$. By induction assumption, $v = \sigma(u) = u \in x$. Thus $\sigma(x) \subseteq x$. □

If $R$ is a well-order on $D$ then $R$ is obviously extensional. We study the Mostowski collapse of strongly well-ordered relations.

**Theorem 83.** *Let $R$ be a strongly well-ordered relation on $D$. Let $\pi\colon (D,\ R) \leftrightarrow (\bar{D},\ \in)$ be the* MOSTOWSKI*-collapse of $R$ and $D$. If $D$ is a proper class then $\bar{D} = \mathrm{Ord}$. If $D$ is a set then $\bar{D}$ is an ordinal which is called the* ordertype *of $(D, R)$. We then write $\bar{D} = \mathrm{otp}(D, R)$.*

**Proof.** $\bar{D}$ is transitive since it is a Mostowski collapse.
(1) Every element of $\bar{D}$ is transitive.
*Proof*. Let $x \in y \in z \in \bar{D}$. Since $\bar{D}$ is transitive, $x,\ y,\ z \in \bar{D}$ and there are $a, b, c \in D$ such that $x = \pi(a),\ y = \pi(b)$, and $z = \pi(c)$. Since $\pi$ is an order-isomorphism, $a R b R c$ . Since $R$ is a transitive relation, $a R c$ . This implies $x \in z$ . *qed*(1)
(2) Every element of $\bar{D}$ is an ordinal.
*Proof*. Let $z \in \bar{D}$. $z$ is transitive, and it remains to show that every element of $z$ is transitive. Let $y \in z$ . Then $y \in \bar{D}$ and so $y$ is transitive by (1). *qed*(2)

Consider the case that $D$ is a proper class. Then $\bar{D}$ is a proper class of ordinals. $\bar{D}$ must be unbounded in the ordinals, since it would be a set otherwise. By transitivity, every ordinal which is smaller than some element of $\bar{D}$ is an element of $\bar{D}$. Hence $\bar{D} = \mathrm{Ord}$.

If $D$ is a set, then $\bar{D}$ is a transitive set, and by (1), $\bar{D} \in \mathrm{Ord}$. □

By Lemma 82, any order-isomorphism $\sigma: (\alpha, <) \leftrightarrow (\beta, <)$ between ordinals must be the identity. So the ordertype of a set-sized well-order $(D, R)$ is the *unique* ordinal, to which it is order-isomorphic.

**Lemma 84.** *Let $x \subseteq \alpha \in \mathrm{Ord}$. Then $(x, <)$ is a well-order. Let $\pi: (x, <) \leftrightarrow (\mathrm{opt}(x, <), <)$ be the Mostowski collapse of $(x, <)$. Then $\forall \xi \in x\ \xi \geqslant \pi(\xi)$ and $\mathrm{otp}(x, <) \leqslant \alpha$.*

**Proof.** By induction on $\xi \in x$. Let $\delta \in \pi(\xi) = \{\pi(\zeta) | \zeta \in x \wedge \zeta < \xi\}$. Let $\delta = \pi(\zeta)$ with $\zeta \in x \wedge \zeta < \xi$. By induction $\delta = \pi(\zeta) \leqslant \zeta < \xi$. Thus $\pi(\xi) \subseteq \xi$ and $\pi(\xi) \leqslant \xi$.

Similarly consider $\delta \in \mathrm{otp}(x, <) = \{\pi(\zeta) | \zeta \in x\}$. Let $\delta = \pi(\zeta)$ with $\zeta \in x$. Then $\delta = \pi(\zeta) \leqslant \zeta < \alpha$. Thus $\mathrm{otp}(x, <) \subseteq \alpha$. □

# 11 Cardinalities

Apart from its foundational role, set theory is mainly concerned with the study of arbitrary infinite sets and in particular with the question of their size. Cantor's approach to infinite sizes follows naive intuitions familiar from finite sets of objects.

**Definition 85.**

a) *$x$ and $y$ are* equipollent*, or* equipotent*, or have* the same cardinality*, written $x \sim y$, if $\exists f f: x \leftrightarrow y$.*

b) *$x$ has* cardinality at most that of *$y$, written $x \preccurlyeq y$, if $\exists f f: x \rightarrow y$ is injective.*

c) *We write $x \prec y$ for $x \preccurlyeq y$ and $x \nsim y$.*

These relations are easily shown to satisfy

**Lemma 86.** *Assume* ZF*. Then*

a) *$\sim$ is an equivalence relation on V.*

b) *$x \sim y \rightarrow x \preccurlyeq y \wedge y \preccurlyeq x$.*

c) *$x \preccurlyeq x$.*

d) *$x \preccurlyeq y \wedge y \preccurlyeq z \rightarrow x \preccurlyeq z$.*

e) *$x \subseteq y \rightarrow x \preccurlyeq y$.*

The converse of *b*) is also true and proved in an exercise.

**Theorem 87.** (Cantor - Bernstein) $x \preccurlyeq y \wedge y \preccurlyeq x \rightarrow x \sim y$.

Assuming the axiom of choice, every set is equipollent with an ordinal (Theorem 71 c). One can take the minimal such ordinal as the canonical representative of the equivalence class with respect to $\sim$.

**Definition 88.**

a) $\mathrm{card}(x) = \min\ \{\alpha |\ \exists f f: \alpha \leftrightarrow x\}$ *is the* cardinality *of the set $x$. One also writes $\bar{\bar{x}} = \mathrm{card}(x)$.*

b) *An ordinal $\kappa$ is a* cardinal *iff it $\kappa = \mathrm{card}(x)$ for some set $x$.*

c) *Let $\mathrm{Cd} = \{\kappa \in \mathrm{Ord} \mid \kappa$ is a cardinal$\}$ be the class of all cardinals, and let $\mathrm{Card} = \{\kappa \geqslant \omega \mid \kappa$ is a cardinal$\}$ be the class of infinite cardinals.*

Let us assume AC until further notice. Then Cantor's two approaches to cardinality agree.

**Theorem 89.**

*a)* $x\preccurlyeq y\leftrightarrow \mathrm{card}(x)\leqslant \mathrm{card}(y)$.

*b)* $x\sim y\leftrightarrow \mathrm{card}(x)=\mathrm{card}(y)$.

**Proof.** *a)* Let $x\preccurlyeq y$ and let $f: x\rightarrow y$ be injective. Further let $f_x: \mathrm{card}(x)\leftrightarrow x$ and $f_y: \mathrm{card}(y)\leftrightarrow y$. Then $f_y^{-1}\circ f\circ f_x: \mathrm{card}(x)\rightarrow \mathrm{card}(y)$ is injective. Let $z=f_y^{-1}\circ f\circ f_x[\mathrm{card}(x)]\subseteq \mathrm{card}(y)$. Then $\mathrm{card}(x)=\mathrm{card}(z)\leqslant \mathrm{otp}(z)\leqslant \mathrm{card}(y)$.

Conversely, let $\mathrm{card}(x)\leqslant \mathrm{card}(y)$ with $f_x: \mathrm{card}(x)\leftrightarrow x$ and $f_y: \mathrm{card}(y)\leftrightarrow y$ as above. Then $f_y\circ f_x^{-1}: x\rightarrow y$ is injective and $x\preccurlyeq y$.

*b)* is trivial. □

As an immediate corollary we get the Cantor–Schröder–Bernstein theorem with AC.

**Theorem 90.** (ZFC) *Let $a\preccurlyeq b$ and $b\preccurlyeq a$. Then $a\sim b$.*

We shall now explore "small" cardinals. Below $\omega$, the notions of natural number, ordinal number and cardinal number agree.

**Theorem 91.** *For all natural numbers $n<\omega$ holds*

*a)* $\mathrm{card}(n)=n$;

*b)* $n\in \mathrm{Cd}$.

**Proof.** *a)* By complete induction on $n$.

For $n=0$, $\emptyset: 0\leftrightarrow 0$ and hence $\mathrm{card}(0)=0$.

Assume that $\mathrm{card}(n)=n$. We claim that $\mathrm{card}(n+1)=n+1$. Obviously $\mathrm{card}(n+1)\leqslant n+1$. Assume for a contradiction that $m=\mathrm{card}(n+1)<n+1$. Take $f: m\leftrightarrow n+1$. Let $f(i_0)=n$.
*Case 1*: $i_0=m-1$. Then $f\upharpoonright(m-1): (m-1)\leftrightarrow n$ and $\mathrm{card}(n)\leqslant m-1<n$, contradiction.
*Case 2*: $i_0<m-1$. Then define $g: (m-1)\leftrightarrow n$ by

$$g(i)=\begin{cases} f(i)\,, \text{ if } i\neq i_0;\\ f(m-1)\,, \text{ if } i=i_0\,.\end{cases}$$

Hence $\mathrm{card}(n)\leqslant m-1<n$, contradiction.
*b)* follows immediately from *a)*. □

**Theorem 92.**

*a)* $\mathrm{card}(\omega)=\omega$;

*b)* $\omega\in \mathrm{Card}$.

**Proof.** Assume for a contradiction that $n=\mathrm{card}(\omega)<\omega$. Let $f: n\leftrightarrow\omega$. Define $g: (n-1)\rightarrow\omega$ by

$$g(i)=\begin{cases} f(i), \text{ if } f(i)<f(n-1),\\ f(i)-1, \text{ if } f(i)>f(n-1).\end{cases}$$

(1) $g$ is injective.
*Proof.* Let $i<j<n-1$.
*Case 1.* $f(i),f(j)<f(n-1)$. Then $g(i)=f(i)\neq f(j)=g(j)$.
*Case 2.* $f(i)<f(n-1)<f(j)$. Then $g(i)=f(i)<f(n-1)\leqslant f(j)-1=g(j)$.
*Case 3.* $f(j)<f(n-1)<f(i)$. Then $g(j)=f(j)<f(n-1)\leqslant f(i)-1=g(i)$.
*Case 4.* $f(n-1)<f(i),f(j)$. Then $g(i)=f(i)-1\neq f(j)-1=g(j)$. *qed*(1)
(2) $g$ is surjective.
*Proof*. Let $k\in\omega$.
*Case 1.* $k<f(n-1)$. By the bijectivity of $f$ take $i<n-1$ such that $f(i)=k$. Then $g(i)=f(i)=k$.
*Case 2.* $k\geqslant f(n-1)$. By the bijectivity of $f$ take $i<n-1$ such that $f(i)=k+1$. Then $g(i)=f(i)-1=k$. *qed*(2)

But this is a contradiction to the supposed minimality of $n=\mathrm{card}(\omega)$. □

**Lemma 93.**

a) $\operatorname{card}(\omega+1)=\omega$.

b) $\operatorname{card}(\omega+\omega)=\omega$.

c) $\operatorname{card}(\omega\cdot\omega)=\omega$.

**Proof.** *a*) Define $f_a: \omega \leftrightarrow \omega+1$ by

$$f(n)=\begin{cases}\omega\,, \text{ if } n=0\\ n-1\,, \text{ else}\end{cases}$$

*b*) Define $f_b: \omega \leftrightarrow \omega+\omega$ by

$$f(n)=\begin{cases}m\,, \text{ if } n=2\cdot m\\ \omega+m\,, \text{ if } n=2\cdot m+1\end{cases}$$

*c*) Define $f_c: \omega \leftrightarrow \omega\cdot\omega$ by

$$f(n)=\omega\cdot k+l, \text{ if } n=2^k\cdot(2\cdot l+1)-1$$

□

# 12 Finite, countable, uncountable sets

**Definition 94.**

a) *x is* finite *if* $\operatorname{card}(x)<\omega$.

b) *x is* infinite *if x is not finite.*

c) *x is* countable *if* $\operatorname{card}(x)\leqslant\omega$.

d) *x is* countably infinite *if* $\operatorname{card}(x)=\omega$.

e) *x is* uncountable *if x is not countable.*

## 12.1 Finite sets

We have the following closure properties for finite sets:

**Theorem 95.** *Let $a, b$ finite, let $x \in V$.*

a) *Every subset of a finite set is finite.*

b) *$a\cup\{x\}$, $a\cup b$, $a\cap b$, $a\times b$, $a\setminus b$, and $\mathcal{P}(a)$ are finite. We have* $\operatorname{card}(\mathcal{P}(a))=2^{\operatorname{card}(a)}$.

c) *If $a_i$ is finite for $i\in b$ then $\bigcup_{i<b} a_i$ is finite.*

**Proof.** Easy. □

Finite sets can be distinguished by dependencies between injective and surjective maps.

**Theorem 96.** *Let a be finite. Then*

a) $\forall f\left(f: a \xrightarrow{\text{inj.}} a \text{ implies } f: a \xrightarrow{\text{surj.}} a\right)$

b) $\forall f\left(f: a \xrightarrow{\text{surj.}} a \text{ implies } f: a \xrightarrow{\text{inj.}} a\right)$

Using the axiom of choice one can also show the converse.

**Theorem 97.** *Let a be infinite. Then*

a) $\exists f f: \omega \xrightarrow{\text{inj.}} a$.

b) $\exists f\left(f: a \xrightarrow{\text{inj.}} a \text{ and } \neg f: a \xrightarrow{\text{surj.}} a\right)$

c) $\exists f\left(f: a \xrightarrow{\text{surj.}} a \text{ and } \neg f: a \xrightarrow{\text{inj.}} a\right)$

This yields:

**Theorem 98.** *For $a \in V$ the following statements are equivalent:*

*a) $a$ is finite;*

*b) $\forall f \left( f: a \xrightarrow{\text{inj.}} a \text{ implies } f: a \xrightarrow{\text{surj.}} a \right)$;*

*c) $\forall f \left( f: a \xrightarrow{\text{surj.}} a \text{ implies } f: a \xrightarrow{\text{inj.}} a \right)$.*

If one does not assume the axiom of choice, one can use *b)* or *c)* to define the notion of finiteness.

## 12.2 Countable sets

We have the following closure properties for countable sets:

**Theorem 99.** *Let $a, b$ countable, let $x \in V$.*

*a) Every subset of a countable set is countable*

*b) $a \cup \{x\}$, $a \cup b$, $a \cap b$, $a \times b$, $a \setminus b$ are countable*

*c) If $a_n$ is countable for $n < \omega$ then $\bigcup_{n<\omega} a_n$ is countable*

**Proof.** Countability will be shown by exhibiting injections into countable sets. Then *a)* is trivial.
*b)* Let $f_a: a \to \omega$ and $f_b: b \to \omega$ be injective. Then define injective maps:

$$f_0: a \cup \{x\} \to \omega,\ f_0(u) = \begin{cases} f_a(u) + 1, \text{ if } u \in a \\ 0, \text{ else} \end{cases}$$

$$f_1: a \cup b \to \omega,\ f_1(u) = \begin{cases} 2 \cdot f_a(u) + 1, \text{ if } u \in a \\ 2 \cdot f_b(u), \text{ else} \end{cases}$$

$$f_2: a \times b \to \omega,\ f_2(u, v) = 2^{f_a(u)} \cdot (2 \cdot f_b(v) + 1)$$

*c)* By the axiom of choice choose a sequence $(h_n | n < \omega)$ of injections $h_n: a_n \to \omega$. Define

$$f_3: \bigcup_{n<\omega} a_n \to \omega,\ f_3(u) = 2^n \cdot (2 \cdot h_n(u) + 1), \text{ where } n \text{ is minimal such that } u \in a_n.$$

□

## 12.3 Uncountable sets

**Theorem 100.** (Cantor) $x \prec \mathcal{P}(x)$

**Proof.** $\mathrm{card}(x) \leqslant \mathrm{card}(\mathcal{P}(x))$ is clear. Assume that $\mathrm{card}(x) = \mathrm{card}(\mathcal{P}(x))$ and let $f: x \leftrightarrow \mathcal{P}(x)$ be bijective. Define

$$a = \{u \in x | u \notin f(u)\} \subseteq x.$$

Let $a = f(u_0)$. Then

$$u_0 \in f(u_0) \leftrightarrow u_0 \in a \leftrightarrow u_0 \notin f(u_0).$$

Contradiction. Hence $\mathrm{card}(x) < \mathrm{card}(\mathcal{P}(x))$. □

**Theorem 101.** $\aleph := \mathrm{card}(\mathcal{P}(\omega))$ *is an uncountable cardinal.*

Note that by previous exercises or lemmas we have

$$\mathrm{card}(\mathcal{P}(\omega)) = \mathrm{card}(\mathbb{R}) = \mathrm{card}({}^2\omega) = \mathrm{card}({}^\omega\omega)$$

Cantor spent a lot of efforts on determining the size of $\aleph$ and postulated that $\aleph$ is the smallest uncountable cardinal.

## 13 The Alefs

**Theorem 102.** $\forall\alpha\exists\kappa\in\mathrm{Card}\,\kappa>\alpha$. *Hence* Card *is a proper class of ordinals.*

**Proof.** Let $\alpha\geqslant\omega$. Then $\kappa=\mathrm{card}(\mathcal{P}(\alpha))>\mathrm{card}(\alpha)$. And $\kappa>\alpha$ since otherwise $\mathrm{card}(\mathcal{P}(\alpha))\leqslant\alpha$ and $\mathrm{card}(\mathrm{card}(\mathcal{P}(\alpha)))\leqslant\mathrm{card}(\alpha)$. □

**Definition 103.** *For any ordinal $\delta$ let $\delta^+$ be the smallest cardinal $>\delta$.*

**Definition 104.** *Define the alef sequence*

$$(\aleph_\alpha|\alpha\in\mathrm{Ord})$$

*recursively by*

$$\begin{aligned}\aleph_0 &= \omega\\ \aleph_{\alpha+1} &= \aleph_\alpha^+\\ \aleph_\lambda &= \bigcup_{\alpha<\lambda}\aleph_\alpha \text{ for limit ordinals } \lambda\end{aligned}$$

Obviously

$$\mathrm{Card}=\{\aleph_\alpha|\alpha\in\mathrm{Ord}\}$$

is the class of all cardinals.

**Definition 105.** *An infinite cardinal of the form $\aleph_{\alpha+1}$ is a* successor cardinal. *An infinite cardinal of the form $\aleph_\lambda$ with $\lambda$ a limit ordinal is a* limit cardinal.

## 14 Cardinal Arithmetic

For disjoint finite sets $a$ and $b$ natural addition and multiplication satisfies

$$\mathrm{card}(a\cup b)=\mathrm{card}(a)+\mathrm{card}(b) \text{ and } \mathrm{card}(a\times b)=\mathrm{card}(a)\cdot\mathrm{card}(b).$$

This motivates the following extension of natural arithmetic to all cardinals.

**Definition 106.** *Let $\kappa,\lambda$ finite or infinite cardinals. Then let*

*a)* $\kappa+\lambda=\mathrm{card}(a\cup b)$*, where $a,b$ are disjoint sets with $\kappa=\mathrm{card}(a)$ and $\lambda=\mathrm{card}(b)$; $\kappa+\lambda$ is the* (cardinal) sum *of $\kappa$ and $\lambda$.*

*b)* $\kappa\cdot\lambda=\mathrm{card}(\kappa\times\lambda)$*; $\kappa\cdot\lambda$ is the* (cardinal) product *of $\kappa$ and $\lambda$.*

*c)* $\kappa^\lambda=\mathrm{card}({}^\lambda\kappa)$*; $\kappa^\lambda$ is the* (cardinal) power *of $\kappa$ and $\lambda$.*

Note that we are using the same notations as for *ordinal* arithmetic. It will usually be clear from the context whether ordinal or cardinal operations are intended.

The "arithmetic" properties of certain set operations yield usual arithmetic laws for cardinal arithmetic.

**Lemma 107.**

*a) Cardinal addition is associative and commutative with neutral element* 0.

*b) Cardinal multiplication is associative and commutative with neutral element* 1.

c) $\kappa\cdot(\lambda+\mu)=\kappa\cdot\lambda+\kappa\cdot\mu$ .

d) $\kappa^0=1$ , $0^\kappa=0$ *for* $\kappa\neq 0$, $\kappa^1=\kappa$, $1^\kappa=1$, $\kappa^{\lambda+\mu}=\kappa^\lambda\cdot\kappa^\mu$ , $\kappa^{\lambda\cdot\mu}=(\kappa^\lambda)^\mu$.

**Proof.** *c*) Let $a,b$ be disjoint sets with $\lambda=\mathrm{card}(a)$ and $\mu=\mathrm{card}(b)$. Then

$$\begin{aligned}\kappa\cdot(\lambda+\mu) &= \mathrm{card}(\kappa\times(a\cup b))\\ &= \mathrm{card}((\kappa\times a)\cup(\kappa\times b))\\ &= \mathrm{card}((\kappa\times a))+\mathrm{card}((\kappa\times b))\\ &= \kappa\cdot\lambda+\kappa\cdot\mu,\end{aligned}$$

using that $\kappa\times(a\cup b)=(\kappa\times a)\cup(\kappa\times b)$ and that $\kappa\times a$ and $\kappa\times b$ are disjoint.
*d*)

$$\kappa^0=\mathrm{card}({}^0\kappa)=\mathrm{card}(\{\emptyset\})=\mathrm{card}(1)=1.$$

In case $\kappa\neq 0$ we have that ${}^\kappa 0=\{f\,|\,f:\kappa\to\emptyset\}=\emptyset$ and thus

$$0^\kappa=\mathrm{card}({}^\kappa 0)=\mathrm{card}(\emptyset)=0.$$

For $\kappa^1=\kappa$ consider the map $\kappa\leftrightarrow{}^1\kappa$ given by $\alpha\mapsto\{(0,\alpha)\}$.
For $1^\kappa=1$ observe that ${}^\kappa 1=\{\{(\alpha,0)|\alpha<\kappa\}\}$ is a singleton set.

Let $a,b$ be disjoint sets with $\lambda=\mathrm{card}(a)$ and $\mu=\mathrm{card}(b)$. Then

$$\begin{aligned}\kappa^{\lambda+\mu} &= \mathrm{card}({}^{a\cup b}\kappa)\\ &= \mathrm{card}(({}^a\kappa)\times({}^b\kappa))\\ &= \mathrm{card}({}^a\kappa)\cdot\mathrm{card}({}^b\kappa)\\ &= \kappa^\lambda\cdot\kappa^\mu,\end{aligned}$$

using that ${}^{a\cup b}\kappa\sim({}^a\kappa)\times({}^b\kappa)$ via the map $f\mapsto(f\restriction a,f\restriction b)$.

Finally,

$$\begin{aligned}\kappa^{\lambda\cdot\mu} &= \mathrm{card}({}^{\lambda\times\mu}\kappa)\\ &= \mathrm{card}({}^\mu({}^\lambda\kappa))\\ &= \mathrm{card}({}^\lambda\kappa)^\mu\\ &= (\kappa^\lambda)^\mu,\end{aligned}$$

using that ${}^{\lambda\times\mu}\kappa\sim{}^\mu({}^\lambda\kappa)$ via the map

$$f\mapsto(f_\xi|\xi<\mu)$$

where $f_\xi:\lambda\to\kappa$ with $f_\xi(\zeta)=f(\zeta,\xi)$, □

We determine the values of cardinal addition and multiplication for infinite cardinals.

**Definition 108.** *Define the* Gödel ordering $<^2$ *of* $\mathrm{Ord}\times\mathrm{Ord}$ *by*

$$\begin{aligned}(\alpha,\beta)<^2(\alpha',\beta') \quad \textit{iff} \quad & \max(\alpha,\beta)<\max(\alpha',\beta'),\\ \textit{or} \quad & \max(\alpha,\beta)=\max(\alpha',\beta')\wedge\alpha<\alpha',\\ \textit{or} \quad & \max(\alpha,\beta)=\max(\alpha',\beta')\wedge\alpha=\alpha'\wedge\beta<\beta'.\end{aligned}$$

**Lemma 109.** $<^2$ *is a wellordering of* $\mathrm{Ord}\times\mathrm{Ord}$ . *Let* $G$: $(\mathrm{Ord}\times\mathrm{Ord},<^2)\leftrightarrow(\mathrm{Ord},<)$ *be the Mostowski collapse of* $(\mathrm{Ord}\times\mathrm{Ord},<^2)$. $G$ *is the* Gödel pairing function. *Define inverse functions* $G_1:\mathrm{Ord}\to\mathrm{Ord}$ *and* $G_2:\mathrm{Ord}\to\mathrm{Ord}$ *such that*

$$\forall\alpha\ \ G(G_1(\alpha),G_2(\alpha))=\alpha.$$

**Lemma 110.** $G:\aleph_\alpha\times\aleph_\alpha\leftrightarrow\aleph_\alpha$ .

**Proof.** By induction on $\alpha$ .
*Case 1*. $\alpha=0$ . By the definition of $<^2$, $\aleph_0\times\aleph_0$ is an initial segment of $<^2$. Let

$$G[\aleph_0\times\aleph_0]=\delta\in\mathrm{Ord}.$$

We show that $\delta = \aleph_0$. Since $\aleph_0 \times \aleph_0$ is infinite, $\delta \geqslant \aleph_0$. Assume that $\delta > \aleph_0$. Take $m, n \in \omega$ such that $G(m,n) = \omega$. Then $(m,n)$ has infinitely many predecessors in $<^2$. But on the other hand

$$\{(k,l) | (k,l) <^2 (m,n)\} \subseteq (\max(m,n)+1) \times (\max(m,n)+1)$$

is finite. Hence $G[\aleph_0 \times \aleph_0] = \aleph_0$.
*Case 2.* $\alpha > 0$ and the Lemma holds for $\beta < \alpha$. Let

$$G[\aleph_\alpha \times \aleph_\alpha] = \eta \in \mathrm{Ord}.$$

We show that $\eta = \aleph_\alpha$. Since $\mathrm{card}(\aleph_\alpha \times \aleph_\alpha) \geqslant \aleph_\alpha$ we have $\eta \geqslant \aleph_\alpha$. Assume that $\eta > \aleph_\alpha$. Take $(\xi, \zeta) \in \aleph_\alpha \times \aleph_\alpha$ such that $G(\xi, \zeta) = \aleph_\alpha$. Then $G$ witnesses that

$$\{(\xi', \zeta') | (\xi', \zeta') <^2 (\xi, \zeta)\} \sim \aleph_\alpha.$$

On the other hand set $\aleph_\beta = \mathrm{card}(\max(\xi,\zeta)+1) < \aleph_\alpha$. Then, using the inductive hypothesis,

$$\begin{aligned} \mathrm{card}(\{(\xi', \zeta') | (\xi', \zeta') <^2 (\xi, \zeta)\}) &\leqslant \mathrm{card}((\max(\xi,\zeta)+1) \times (\max(\xi,\zeta)+1)) \\ &= \mathrm{card}(\aleph_\beta \times \aleph_\beta) \\ &= \aleph_\beta < \aleph_\alpha, \end{aligned}$$

contradiction. Hence $G[\aleph_\alpha \times \aleph_\alpha] = \aleph_\alpha$. □

**Theorem 111.**

a) *If* $\kappa \in \mathrm{Card}$ *then* $\kappa \cdot \kappa = \kappa$.

b) *If* $\kappa \in \mathrm{Card}$ *and* $\lambda \in \mathrm{Cd}$, $\lambda \neq 0$ *then* $\kappa \cdot \lambda = \max(\kappa, \lambda)$.

c) *If* $\kappa \in \mathrm{Card}$ *and* $\lambda \in \mathrm{Cd}$ *then* $\kappa + \lambda = \max(\kappa, \lambda)$.

**Proof.** *a)* $\kappa \cdot \kappa = \mathrm{card}(\kappa \times \kappa) = \kappa$, by the properties of the Gödel pairing function.
*b)* The map $i \mapsto (i, 0)$ injects $\kappa$ into $\kappa \times \lambda$, and the map $j \mapsto (0, j)$ injects $\lambda$ into $\kappa \times \lambda$. Hence $\kappa$, $\lambda \leqslant \kappa \cdot \lambda$. Thus

$$\max(\kappa, \lambda) \leqslant \kappa \cdot \lambda \leqslant \max(\kappa, \lambda) \cdot \max(\kappa, \lambda) \overset{(a)}{=\!=} \max(\kappa, \lambda).$$

*c)* Obviously $\kappa \sim \{0\} \times \kappa$ and $\lambda \sim \{1\} \times \lambda$. The inclusion

$$(\{0\} \times \kappa) \cup (\{1\} \times \lambda) \subseteq \max(\kappa, \lambda) \times \max(\kappa, \lambda)$$

implies

$$\max(\kappa, \lambda) \leqslant \kappa + \lambda \leqslant \max(\kappa, \lambda) \cdot \max(\kappa, \lambda) \overset{(a)}{=\!=} \max(\kappa, \lambda).$$ □

For infinite cardinal *exponentiation* the situation is very different. Only a few values can be determined explicitely.

**Lemma 112.** *For* $\kappa \in \mathrm{Card}$ *and* $1 \leqslant n < \omega$ *we have* $\kappa^n = \kappa$.

**Proof.** By complete induction. $\kappa^1 = \kappa$ was proved before. And

$$\kappa^{n+1} = (\kappa^n) \cdot \kappa^1 = \kappa \cdot \kappa = \kappa.$$ □

The "next" exponential value $2^{\aleph_0}$ is however very undetermined. It is possible, in a sense to be made precise later, that $2^{\aleph_0}$ is any successor cardinal, like e.g. $\aleph_{13}$.

Cantor's continuum hypothesis is equivalent to the cardinal arithmetic statement

$$2^{\aleph_0} = \aleph_1.$$

**Lemma 113.** *For* $\kappa \in \mathrm{Card}$ *and* $2 \leqslant \lambda \leqslant 2^\kappa$ *we have* $\lambda^\kappa = 2^\kappa$.

**Proof.**

$$2^\kappa \leqslant \lambda^\kappa \leqslant (2^\kappa)^\kappa = 2^{\kappa \cdot \kappa} = 2^\kappa.$$ □

## 15 Cofinality

To get some more information on cardinal exponentiation, we need to measure how "fast" a cardinal can be approximated using smaller cardinals.

**Definition 114.**

*a)* *A set* $x \subseteq \lambda$ *is* cofinal *in the limit ordinal* $\lambda$ *if* $\forall \alpha < \lambda \exists \xi \in x\, \alpha < \xi$.

*b)* *The* cofinality *of a limit ordinal* $\lambda$ *is*

$$\mathrm{cof}(\lambda) = \min\{\mathrm{otp}(x) | x \subseteq \lambda \text{ is cofinal in } \lambda\}.$$

*c)* *A limit ordinal* $\lambda$ *is* regular *if* $\mathrm{cof}(\lambda) = \lambda$*; otherwise* $\lambda$ *is singular.*

These notions are due to Felix Hausdorff, who called these notions "konfinal" and "Konfinalität". Please observe the "ko<u>n</u>final" in German.

**Lemma 115.**

*a)* $\mathrm{cof}(\lambda) = \min\{\mathrm{card}(x) | x \subseteq \lambda \text{ is cofinal in } \lambda\}$

*b)* $\mathrm{cof}(\aleph_0) = \aleph_0$, *i.e.,* $\aleph_0$ *is regular*

*c)* $\mathrm{cof}(\lambda) \leqslant \mathrm{card}(\lambda) \leqslant \lambda$

*d)* $\mathrm{cof}(\lambda) \in \mathrm{Card}$

*e)* $\mathrm{cof}(\lambda)$ *is regular, i.e.,* $\mathrm{cof}(\mathrm{cof}(\lambda)) = \mathrm{cof}(\lambda)$

*f)* *If* $\gamma$ *is a limit ordinal then* $\mathrm{cof}(\aleph_\gamma) = \mathrm{cof}(\gamma)$

*g)* $\mathrm{cof}(\aleph_\omega) = \aleph_0$, *i.e.,* $\aleph_\omega$ *is a singular cardinal*

**Proof.** *a)* $\geqslant$ holds since $\mathrm{otp}(x) \geqslant \mathrm{card}(x)$. Conversely let $x$ have minimal cardinality such that $x$ is cofinal in $\lambda$ and let $f: \mathrm{card}(x) \leftrightarrow x$. Define a weakly increasing map $g: \mathrm{card}(x) \to \lambda$ by

$$g(i) = \bigcup_{j<i} f(j).$$

$g$ is welldefined by the minimality of $x$. $y = g[\mathrm{card}(x)]$ is cofinal in $\lambda$. $y$ is order-isomorphic to

$$\{i < \mathrm{card}(x) | \forall j < i\, g(j) < g(i)\} \subseteq \mathrm{card}(x).$$

Hence

$$\mathrm{otp}(y) = \mathrm{otp}(\{i < \mathrm{card}(x) | \forall j < i\, g(j) < g(i)\}) \leqslant \mathrm{card}(x).$$

Thus

$$\mathrm{cof}(\lambda) \leqslant \mathrm{otp}(y) = \mathrm{card}(x) = \min\{\mathrm{card}(x) | x \subseteq \lambda \text{ is cofinal in } \lambda\}.$$

*b)* – *d)* follow from *a)*.
*e)* Let $x \subseteq \lambda$ be cofinal in in $\lambda$ with $\mathrm{otp}(x) = \mathrm{cof}(\lambda)$ and order-isomorphism $f: \mathrm{cof}(\lambda) \leftrightarrow x$. Let $y \subseteq \mathrm{cof}(\lambda)$ be cofinal with $\mathrm{otp}(y) = \mathrm{cof}(\mathrm{cof}(\lambda))$ and order-isomorphism $g: \mathrm{cof}(\mathrm{cof}(\lambda)) \leftrightarrow y$. Then $z = f \circ g[\mathrm{cof}(\mathrm{cof}(\lambda))]$ is cofinal in $\lambda$ with $\mathrm{otp}(z) = \mathrm{cof}(\mathrm{cof}(\lambda))$. Hence

$$\mathrm{cof}(\lambda) \leqslant \mathrm{otp}(z) = \mathrm{cof}(\mathrm{cof}(\lambda)).$$

The converse inequality follows from *c)*.
*f)* ($\leqslant$) Let $x$ be cofinal in $\gamma$ with $\mathrm{otp}(x) = \mathrm{cof}(\gamma)$. Then $\{\aleph_i | i \in x\}$ is cofinal in $\aleph_\gamma$ with

$$\mathrm{otp}(\{\aleph_i | i \in x\}) = \mathrm{otp}(x) = \mathrm{cof}(\gamma).$$

Hence $\mathrm{cof}(\aleph_\gamma) \leqslant \mathrm{cof}(\gamma)$.
($\geqslant$) Now let $y$ be cofinal in $\aleph_\gamma$ with $\mathrm{otp}(y) = \mathrm{cof}(\aleph_\gamma)$. Define $x = \{i < \gamma | \exists \delta \in y\, \aleph_i \leqslant \delta < \aleph_{i+1}\}$. Then $x$ is cofinal in $\gamma$ with $\mathrm{card}(x) \leqslant \mathrm{card}(y) = \mathrm{cof}(\aleph_\gamma)$. Hence $\mathrm{cof}(\gamma) \leqslant \mathrm{cof}(\aleph_\gamma)$. □

**Theorem 116.** *Every successor cardinal* $\aleph_{\alpha+1}$ *is regular.*

**Proof.** Assume that $\aleph_{\alpha+1}$ is singular. Let $x$ have minimal cardinality such that $x$ is cofinal in $\aleph_{\alpha+1}$ . Then $\mathrm{card}(x) \leqslant \aleph_\alpha$ . Let $f\colon \aleph_\alpha \to x$ be surjective. Using the axiom of choice take a sequence $(g_i|0<i<\aleph_{\alpha+1})$ of surjective functions $g_i\colon \aleph_\alpha \to i$ . Define function $h\colon \aleph_\alpha \times \aleph_\alpha \to \aleph_{\alpha+1}$ by

$$h(\xi,\zeta)=g_{f(\xi)}(\zeta).$$

(1) $h\colon \aleph_\alpha \times \aleph_\alpha \to \aleph_{\alpha+1}$ is surjective.
*Proof*. Let $\nu \in \aleph_{\alpha+1}$ . Take $\xi < \aleph_\alpha$ such that $f(\xi) > \nu$ . $g_{f(\xi)}\colon \aleph_\alpha \to f(\xi)$ is surjective. Take $\zeta < \aleph_\alpha$ such that $g_{f(\xi)}(\zeta)=\nu$. Thus $\nu = h(\xi,\zeta) \in \mathrm{ran}(h)$. *qed*(1)

This implies

$$\aleph_{\alpha+1}=\mathrm{card}(\aleph_{\alpha+1}) \leqslant \mathrm{card}(\aleph_\alpha \times \aleph_\alpha) = \aleph_\alpha \cdot \aleph_\alpha = \aleph_\alpha .$$

Contradiction. □

So $\aleph_0, \aleph_1, \aleph_2, \ldots, \aleph_n, \ldots$ are all regular.

**Question 117.** (Hausdorff) *Are there regular limit cardinals* $>\aleph_0$ *?*

**Definition 118.** *For* $(\kappa_i|i<\delta)$ *a sequence of finite or infinite cardinals define the* sum

$$\sum_{i<\delta} \kappa_i = \mathrm{card}\left(\bigcup_{i<\delta} \kappa_i \times \{i\}\right)$$

*and the* product

$$\prod_{i<\delta} \kappa_i = \mathrm{card}(\mathsf{X}_{i<\delta}\kappa_i)$$

*where*

$$\mathsf{X}_{i<\delta}A_i = \{f\,|\,f\colon \delta \to V \wedge \forall i<\delta\, f(i) \in A_i\}.$$

**Theorem 119.** (König) *If* $(\kappa_i|i<\delta)$ *and* $(\lambda_i|i<\delta)$ *are sequences of cardinals such that* $\forall i<\delta \kappa_i<\lambda_i$ *then*

$$\sum_{i<\delta} \kappa_i < \prod_{i<\delta} \lambda_i$$

**Proof.** Assume for a contradiction that $\sum_{i<\delta} \kappa_i \geqslant \prod_{i<\delta} \lambda_i$ and that $G\colon \bigcup_{i<\delta} \kappa_i \times \{i\} \leftrightarrow \mathsf{X}_{i<\delta}\lambda_i$ were a surjection. For $i<\delta$

$$\mathrm{card}(\{G(\nu,i)(i)|\nu<\kappa_i\}) \leqslant \kappa_i < \lambda_i ,$$

and one can choose $\nu_i \in \lambda_i \setminus \{G(\nu,i)(i)|\nu<\kappa_i\}$. Define $f \in \mathsf{X}_{i<\delta}\lambda_i$ by

$$f(i)=\nu_i .$$

Since $G$ is surjective, take $(\nu_0, i_0) \in \mathrm{dom}(G)$ such that $G(\nu_0,i_0)=f$ . Then

$$G(\nu_0,i_0)(i_0)=f(i_0)=\nu_{i_0} \neq G(\nu,i_0)(i_0)$$

for all $\nu<\kappa_{i_0}$. Contradiction. □

**Theorem 120.** *If* $\kappa,\lambda$ *are cardinals such that* $\kappa \geqslant 2$ *and* $\lambda \geqslant \aleph_0$ *then*

$$\mathrm{cof}(\kappa^\lambda) > \lambda$$

*Hence*

$$\mathrm{cof}(2^{\aleph_0}) \geqslant \aleph_1$$

*and in particular*

$$2^{\aleph_0} \neq \aleph_\omega .$$

**Proof.** Assume that $\mathrm{cof}(\kappa^\lambda) \leqslant \lambda$ . Then there is a function $f\colon \lambda \to \kappa^\lambda$ such that $\mathrm{ran}(f)$ is cofinal in $\kappa^\lambda$. Then $\bigcup_{i<\lambda} f(i) = \kappa^\lambda$ and so

$$\kappa^\lambda = \mathrm{card}\left(\bigcup_{i<\lambda} f(i)\right) \leqslant \mathrm{card}\left(\bigcup_{i<\lambda} f(i) \times \{i\}\right) = \mathrm{card}\left(\bigcup_{i<\lambda} \mathrm{card}(f(i)) \times \{i\}\right) = \sum_{i<\lambda} \mathrm{card}(f(i)).$$

But by König's Theorem,

$$\kappa^\lambda = \kappa^{\lambda\cdot\lambda} = (\kappa^\lambda)^\lambda = \prod_{i<\lambda} \kappa^\lambda > \sum_{i<\lambda} \operatorname{card}(f(i)).$$

□

**Theorem 121.** (The Hausdorff recursion formula)

$$\aleph_{\alpha+1}^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}\,.$$

**Proof.** Distinguish two cases:
*Case 1*: $\aleph_{\alpha+1} \leqslant 2^{\aleph_\beta}$. Then

$$\aleph_{\alpha+1}^{\aleph_\beta} = 2^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}\,.$$

*Case 2*: $2^{\aleph_\beta} < \aleph_{\alpha+1}$: Then $\aleph_\beta < \aleph_{\alpha+1}$. Using the regularity of $\aleph_{\alpha+1}$

$$\begin{aligned}
& \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1} \\
& \leqslant \aleph_{\alpha+1}^{\aleph_\beta} \cdot \aleph_{\alpha+1}^{\aleph_\beta} = \\
= \aleph_{\alpha+1}^{\aleph_\beta} & = \operatorname{card}(\{f \mid f\colon \aleph_\beta \to \aleph_{\alpha+1}\}) \\
& = \operatorname{card}\left(\bigcup_{\nu<\aleph_{\alpha+1}} \{f \mid f\colon \aleph_\beta \to \nu\}\right) \\
& \leqslant \sum_{\nu<\aleph_{\alpha+1}} \operatorname{card}(\{f \mid f\colon \aleph_\beta \to \nu\}) \\
& = \sum_{\nu<\aleph_{\alpha+1}} \operatorname{card}(\{f \mid f\colon \aleph_\beta \to \operatorname{card}(\nu)\}) \\
& = \sum_{\nu<\aleph_{\alpha+1}} \operatorname{card}(\nu)^{\aleph_\beta} \\
& \leqslant \sum_{\nu<\aleph_{\alpha+1}} \aleph_\alpha^{\aleph_\beta} \\
& = \aleph_\alpha^{\aleph_\beta} \cdot \aleph_{\alpha+1}
\end{aligned}$$

□

# 16 Cardinal exponentiation and the Generalized Continuum Hypothesis

The function $\kappa \mapsto 2^\kappa$ is called the *continuum function*, due to the relations between $2^{\aleph_0}$ and the usual continuum of real numbers. The *beth numbers* are defined in analogy with the aleph function, using the $2^\kappa$-operation instead of the cardinal successor function.

**Definition 122.** *Define the sequence*

$$(\beth_\alpha \mid \alpha \in \mathrm{Ord})$$

*of* beth numbers *recursively by*

$$\begin{aligned}
\beth_0 &= \aleph_0 \\
\beth_{\alpha+1} &= 2^{\beth_\alpha} \\
\beth_\lambda &= \bigcup_{\alpha<\lambda} \beth_\alpha \text{ for limit ordinals } \lambda
\end{aligned}$$

Like every continuous ordinal function, there are fixed points $\beth_\alpha = \alpha$ of this sequence.

**Definition 123.**

a) *An* inaccessible cardinal $\kappa$ *is a regular fixed point of the $\aleph_\alpha$-function:*

$$\kappa = \aleph_\kappa \text{ and } \operatorname{cof}(\kappa) = \kappa\,.$$

b) *A* strongly inaccessible cardinal $\kappa$ *is a regular fixed point of the* $\beth_\alpha$*-sequence:*

$$\kappa = \beth_\kappa \text{ and } \operatorname{cof}(\kappa) = \kappa.$$

The existence of inaccessible and strongly inaccessible cardinals can not be shown in ZFC, provided the theory ZFC is consistent.

**Definition 124.** *Define the* gimel function $\gimel: \mathrm{Card} \to \mathrm{Card}$ *by* $\gimel(\kappa) = \kappa^{\operatorname{cof}(\kappa)}$.

By König's theorem, $\gimel(\kappa) > \kappa$. Note that $\aleph$ (Alef), $\beth$ (Beth) and $\gimel$ (Gimel) are the first three letters of the Hebrew alphabet. The gimel function determines all values of the continuum function.

**Definition 125.** *For* $\kappa \in \mathrm{Cd}$ *and* $\lambda \in \mathrm{Card}$ *let*

$$\kappa^{<\lambda} = \bigcup_{\nu < \lambda} \kappa^{\operatorname{card}(\nu)}.$$

**Theorem 126.**

a) *If* $\kappa$ *is regular then* $2^\kappa = \gimel(\kappa)$.

b) *If* $\kappa$ *is a singular cardinal and the continuum function is eventually constant below* $\kappa$ *, i.e.,*

$$\exists \bar{\kappa} < \kappa \forall \lambda \, (\bar{\kappa} \leqslant \lambda < \kappa \rightarrow 2^{\bar{\kappa}} = 2^\lambda),$$

*then* $2^\kappa = 2^{<\kappa}$.

c) *If* $\kappa$ *is a singular cardinal and the continuum function is* not *eventually constant below* $\kappa$ *then* $2^\kappa = \gimel(2^{<\kappa})$.

**Proof.** *a*) If $\kappa$ is regular then

$$2^\kappa = \kappa^\kappa = \kappa^{\operatorname{cof}(\kappa)} = \gimel(\kappa).$$

Now let $\kappa$ be singular and let the sequence $(\kappa_i | i < \operatorname{cof}(\kappa))$ be strictly increasing and cofinal in $\kappa$. For $i < \operatorname{cof}(\kappa)$ choose (AC) an injection $f_i: \mathcal{P}(\kappa_i) \to 2^{<\kappa}$. Define

$$G: \mathcal{P}(\kappa) \to {}^{\operatorname{cof}(\kappa)}(2^{<\kappa})$$

by

$$x \mapsto (f_i(x \cap \kappa_i) | i < \operatorname{cof}(\kappa)).$$

We argue that $G$ is injective: let $x, y \in \mathcal{P}(\kappa)$, $x \neq y$. Then take $i < \operatorname{cof}(\kappa)$ such that $x \cap \kappa_i \neq y \cap \kappa_i$. Since $f_i$ is injective: $f_i(x \cap \kappa_i) \neq f_i(y \cap \kappa_i)$. Then $G(x) \neq G(y)$ because

$$G(x)(i) = f_i(x \cap \kappa_i) \neq f_i(y \cap \kappa_i) = G(y)(i).$$

By the injectivity of $G$

$$2^\kappa \leqslant (2^{<\kappa})^{\operatorname{cof}(\kappa)}.$$

*b*) Let $2^{<\kappa} = 2^{\bar{\kappa}}$ be the eventually constant value of the continuum function below $\kappa$. Then

$$2^{<\kappa} \leqslant 2^\kappa \leqslant (2^{<\kappa})^{\operatorname{cof}(\kappa)} = (2^{\bar{\kappa}})^{\operatorname{cof}(\kappa)} = 2^{\bar{\kappa} \cdot \operatorname{cof}(\kappa)} = 2^{\max(\bar{\kappa}, \operatorname{cof}(\kappa))} = 2^{\bar{\kappa}} = 2^{<\kappa}.$$

*c*) In this case we show that $\operatorname{cof}(\kappa) = \operatorname{cof}(2^{<\kappa})$. The function

$$i \mapsto 2^{\kappa_i}$$

is not eventually constant and thus maps $\operatorname{cof}(\kappa)$ cofinally into $2^{<\kappa}$. Hence $\operatorname{cof}(2^{<\kappa}) \leqslant \operatorname{cof}(\kappa)$. Assume that $\operatorname{cof}(2^{<\kappa}) < \operatorname{cof}(\kappa)$. Let $z \subseteq 2^{<\kappa}$ be cofinal such that $\operatorname{card}(z) < \operatorname{cof}(\kappa)$. Then

$$\bar{z} = \{i | \exists \delta \in z \, 2^{\kappa_i} \leqslant \delta < 2^{\kappa_{i+1}}\} \subseteq \operatorname{cof}(\kappa)$$

is cofinal in $\operatorname{cof}(\kappa)$ and $\operatorname{card}(\bar{z}) \leqslant \operatorname{card}(z) < \operatorname{cof}(\kappa)$, contradiction.

So we obtain

$$\gimel(2^{<\kappa}) = (2^{<\kappa})^{\mathrm{cof}(2^{<\kappa})} = (2^{<\kappa})^{\mathrm{cof}(\kappa)} \leqslant (2^{\kappa})^{\mathrm{cof}(\kappa)} \leqslant 2^{\kappa} \leqslant (2^{<\kappa})^{\mathrm{cof}(\kappa)} = (2^{<\kappa})^{\mathrm{cof}(2^{<\kappa})} = \gimel(2^{<\kappa}).$$

□

The following theorem shows that $\kappa^{\lambda}$ is uniquely determined by the gimel function.

**Theorem 127.** *Let* $\lambda \in \mathrm{Card}$*. Then* $\kappa^{\lambda}$ *is determined by the previous theorem and by recursion on* $\kappa$*:*

*a)* $0^{\lambda} = 0,\ 1^{\lambda} = 1$.

*b) For* $2 \leqslant \kappa \leqslant \lambda$ *we have* $\kappa^{\lambda} = 2^{\lambda}$*.*

*c) If* $\kappa > \lambda$ *and* $\xi < \kappa$ *such that* $\xi^{\lambda} \geqslant \kappa$ *then* $\kappa^{\lambda} = \xi^{\lambda}$*.*

*d) If* $\kappa > \lambda$, $\forall \xi < \kappa\ \xi^{\lambda} < \kappa$*, and* $\mathrm{cof}(\kappa) > \lambda$ *then* $\kappa^{\lambda} = \kappa$*.*

*e) If* $\kappa > \lambda$, $\forall \xi < \kappa\ \xi^{\lambda} < \kappa$*, and* $\mathrm{cof}(\kappa) \leqslant \lambda$ *then* $\kappa^{\lambda} = \gimel(\kappa)$*.*

**Proof.** *a)* and *b)* follow immediately from earlier results.
*c)* $\xi^{\lambda} \leqslant \kappa^{\lambda} \leqslant (\xi^{\lambda})^{\lambda} = \xi^{\lambda}$.
*d)* $\mathrm{cof}(\kappa) > \lambda$ implies that every function from $\lambda$ into $\kappa$ is bounded by some ordinal $\nu < \kappa$. Hence

$$\begin{aligned} \kappa \leqslant \kappa^{\lambda} &= \mathrm{card}\{f \mid f\colon \lambda \to \kappa\} \\ &= \mathrm{card}\left(\bigcup_{\nu<\kappa} \{f \mid f\colon \lambda \to \nu\}\right) \\ &\leqslant \sum_{\nu<\kappa} \mathrm{card}(\{f \mid f\colon \lambda \to \nu\}) \\ &= \sum_{\nu<\kappa} \mathrm{card}(\{f \mid f\colon \lambda \to \mathrm{card}(\nu)\}) \\ &= \sum_{\nu<\kappa} \mathrm{card}(\nu)^{\lambda} \\ &\leqslant \sum_{\nu<\kappa} \kappa \\ &= \kappa. \end{aligned}$$

*e)* Let $(\xi_i \mid i < \mathrm{cof}(\kappa))$ be a strictly increasing sequence which is cofinal in $\kappa$. Define a function

$$G\colon {}^{\lambda}\kappa \to \bigtimes_{i<\mathrm{cof}(\kappa)} {}^{\lambda}\xi_i$$

by

$$f \mapsto (f_i \mid i < \mathrm{cof}(\kappa))$$

where

$$f_i(\alpha) = \begin{cases} f(\alpha), \text{ if } f(\alpha) < \xi_i \\ 0, \text{ else} \end{cases}$$

Then $G$ is injective: Let $f, g \in {}^{\lambda}\kappa$, $f \neq g$. Take $\alpha$ such that $f(\alpha) \neq g(\alpha)$ and take $i$ such that $f(\alpha), g(\alpha) < \xi_i$. Then $f_i(\alpha) = f(\alpha) \neq g(\alpha) = g_i(\alpha)$, $f_i \neq g_i$, and hence $G(f) \neq G(g)$.

Using $G$ we get

$$\begin{aligned} \gimel(\kappa) = \kappa^{\mathrm{cof}(\kappa)} \leqslant \kappa^{\lambda} &\leqslant \prod_{i<\mathrm{cof}(\kappa)} \mathrm{card}(\xi_i)^{\lambda} \\ &\leqslant \prod_{i<\mathrm{cof}(\kappa)} \kappa \\ &= \kappa^{\mathrm{cof}(\kappa)} = \gimel(\kappa) \end{aligned}$$

□

**Definition 128.** (Hausdorff) *The* generalized continuum hypothesis (GCH) *is the statement*

$$\forall \kappa \in \mathrm{Card}\ 2^{\kappa} = \kappa^{+}.$$

This is the “minimal” hypothesis in view of Cantor’s $2^\kappa \geqslant \kappa^+$. The GCH generalizes Cantor’s continuum hypothesis CH and also the hypothesis $2^{\aleph_1} = \aleph_2$ also expressed by Cantor. Since CH is independent of the axioms of set theory, GCH is independent as well. Indeed the continuum function is hardly determined by the axioms of ZFC and one can for example have

$$2^{\aleph_0} = \aleph_{73}, 2^{\aleph_1} = \aleph_{2015}, \ldots$$

Obviously

**Lemma 129.** GCH *implies that* $\forall \kappa \in \mathrm{Card}\ \beth(\kappa) = \kappa^+$.

Thus GCH also determines all values of the $\kappa^\lambda$ function. Axiomatic set theory proves that one can assume GCH without the danger of adding inconsistencies to the system ZFC: a model of the ZFC axioms can be modified into a model of ZFC + GCH. The consequences of GCH for cardinal exponentiation can be readily described.

**Theorem 130.** *Assume* GCH. *Then for* $\kappa, \lambda \in \mathrm{Card}$ $\kappa^\lambda$ *is determined as follows:*

*a)* *For* $\lambda < \mathrm{cof}(\kappa)$*:* $\kappa^\lambda = \kappa$.

*b)* *For* $\mathrm{cof}(\kappa) \leqslant \lambda \leqslant \kappa : \kappa^\lambda = \kappa^+$.

*c)* *For* $\lambda > \kappa : \kappa^\lambda = \lambda^+$.

**Proof.** *a)*

$$\begin{aligned} \kappa &\leqslant \kappa^\lambda \\ &= \mathrm{card}\{f \mid f\colon \lambda \to \kappa\} \\ &= \mathrm{card} \bigcup_{\nu < \kappa} \{f \mid f\colon \lambda \to \nu\} \\ &\leqslant \sum_{\nu < \kappa} \mathrm{card}(\nu)^\lambda \\ &\leqslant \sum_{\nu < \kappa} \kappa \\ &= \kappa. \end{aligned}$$

*b)* By GCH and König’s theorem, $\kappa^{\mathrm{cof}(\kappa)} = \kappa^+$. Thus

$$\kappa^+ = \kappa^{\mathrm{cof}(\kappa)} \leqslant \kappa^\lambda \leqslant \kappa^\kappa = 2^\kappa = \kappa^+.$$

*c)*

$$\lambda^+ = 2^\lambda \leqslant \kappa^\lambda \leqslant \lambda^\lambda = 2^\lambda = \lambda^+.$$

□

**Question 131.** *Is every (infinite) cardinal product* $\prod_{i<\alpha} \kappa_i$ *also determined by* GCH*?*

# 17 Closed unbounded and stationary sets

The continuum function $\kappa \mapsto 2^\kappa$ satisfies the laws

a) $\kappa \leqslant \lambda \to 2^\kappa \leqslant 2^\lambda$

b) $\mathrm{cof}(2^\kappa) > \kappa$

Axiomatic set theory shows that for *regular* cardinals $\kappa$ these are the only laws deducible from ZFC: for (adequate) functions $F\colon \mathrm{Card} \to \mathrm{Card}$ satisfying *a)* and *b)* for regular cardinals there is a model of set theory in which

$$\kappa \text{ regular} \to 2^\kappa = F(\kappa).$$

So there remains the consideration of $2^\kappa$ for *singular* cardinals $\kappa$. Indeed singular cardinal exponentiation satisfies some interesting further laws and is an area of present research. To prove a few of these laws we have to extend the apparatus of uncountable combinatorics.

**Definition 132.** *Let* $\kappa \in \mathrm{Card}$ *and* $C \subseteq \kappa$. *C is* unbounded *in* $\kappa$ *if*

$$\forall \alpha < \kappa \exists \beta \in C \alpha < \beta.$$

*C is* closed *in* $\kappa$ *if*

$$\forall \lambda < \kappa (\lambda \text{ is a limit ordinal} \wedge C \cap \lambda \text{ is unbounded in } \lambda \rightarrow \lambda \in C).$$

*Thus C contains its limit points* $<\kappa$.

*C is* closed unbounded*, or* cub *in* $\kappa$*, if C is unbounded in* $\kappa$ *and closed in* $\kappa$.

**Exercise 28.** Define a topology on $\kappa$ such that the closed sets of the topology are exactly the closed sets in the sense of the previous definition.

**Lemma 133.** *Let* $\kappa \in \mathrm{Card}$, $\mathrm{cof}(\kappa) \geqslant \omega_1$ *and* $C, D$ *be closed unbounded in* $\kappa$. *Then* $C \cap D$ *is cub in* $\kappa$.

**Proof.** $C \cap D$ is closed in $\kappa$: Let $\lambda < \kappa$ be a limit ordinal and a limit point of $C \cap D$. Then $\lambda$ is a limit point of $C$ and $\lambda \in C$. Similarly $\lambda \in D$ and together $\lambda \in C \cap D$.

$C \cap D$ is unbounded in $\kappa$: Let $\alpha < \kappa$. Define a sequence $(\beta_n | n < \omega)$ by recursion:

$$\beta_n = \begin{cases} \text{the least element of } C \text{ which is larger than } \alpha, \beta_0, \ldots, \beta_{n-1} \text{ in case } n \text{ is even} \\ \text{the least element of } D \text{ which is larger than } \beta_0, \ldots, \beta_{n-1} \text{ in case } n \text{ is odd} \end{cases}$$

Let $\beta = \bigcup_{n<\omega} \beta_n$. $\beta$ is a limit ordinal $>\alpha$. $\beta < \kappa$ since $\mathrm{cof}(\kappa) \geqslant \omega_1$. By construction, $\beta$ is a limit point of $C$ and of $D$. Hence $\beta \in C \cap D$. □

**Exercise 29.** Let $\kappa \in \mathrm{Card}$, $\mathrm{cof}(\kappa) \geqslant \omega_1$. Let $(C_i | i < \gamma)$ be a sequence of sets $C_i$ which are closed unbounded in $\kappa$ and let $\gamma < \mathrm{cof}(\kappa)$. Then $\bigcap_{i<\gamma} C_i$ is cub in $\kappa$.

**Definition 134.** *Let* $\kappa \in \mathrm{Card}$, $\mathrm{cof}(\kappa) \geqslant \omega_1$. *The* closed unbounded filter *on* $\kappa$ *is*

$$\mathcal{C}_\kappa = \{X \subseteq \kappa | \text{there is a set } C \subseteq X \text{ which is closed unbounded in } \kappa\}.$$

**Lemma 135.** $\mathcal{C}_\kappa$ *is a* $\mathrm{cof}(\kappa)$*-complete filter on* $\kappa$*, i.e.*

*a)* $\emptyset \neq \mathcal{C}_\kappa \subseteq \mathcal{P}(\kappa)$

*b)* $\emptyset \notin \mathcal{C}_\kappa$

*c)* $X \in \mathcal{C}_\kappa \wedge X \subseteq Y \subseteq \kappa \rightarrow Y \in \mathcal{C}_\kappa$

*d)* $X \in \mathcal{C}_\kappa \wedge Y \in \mathcal{C}_\kappa \rightarrow X \cap Y \in \mathcal{C}_\kappa$

*e)* $\gamma < \mathrm{cof}(\kappa) \wedge \{X_i | i < \gamma\} \subseteq \mathcal{C}_\kappa \rightarrow \bigcap_{i<\gamma} X_i \in \mathcal{C}_\kappa$

**Proof.** *c)* and *d)* follow from Lemma 133 and Exercise 29. □

A filter captures a notion of "large set". Even intersections of large sets are large, so that certain contructions can be continued on large sets. Largeness also yields notions of "small" and of "not small", called "non-stationary" and "stationary".

**Definition 136.** *Let* $\kappa \in \mathrm{Card}$, $\mathrm{cof}(\kappa) \geqslant \omega_1$.

*a)* $X \subseteq \kappa$ *is* non-stationary *in* $\kappa$ *if* $\kappa \setminus X \in \mathcal{C}_\kappa$. *We call*

$$\mathrm{NS}_\kappa = \{X \mid \kappa \setminus X \in \mathcal{C}_\kappa\}$$

*the* non-stationary ideal *on* $\kappa$.

*b)* $X \subseteq \kappa$ *is* stationary *in* $\kappa$ *if* $X \notin \mathrm{NS}_\kappa$.

**Lemma 137.** $X \subseteq \kappa$ *is* stationary *in* $\kappa$ *iff* $X \cap C \neq \emptyset$ *for every cub* $C \subseteq \kappa$.

**Proof.** $X$ is stationary iff $X \notin \mathrm{NS}_\kappa$ iff $\kappa \setminus X \notin \mathcal{C}_\kappa$ iff there is no $C \subseteq \kappa$ cub such that $C \subseteq \kappa \setminus X$ iff for every cub $C \subseteq \kappa$ $C \nsubseteq \kappa \setminus X$ iff for every cub $C \subseteq \kappa$ $X \cap C \neq \emptyset$. □

**Lemma 138.** *Every set in* $\mathcal{C}_\kappa$ *is stationary.*

**Lemma 139.** $\mathrm{NS}_\kappa$ *is a* $\mathrm{cof}(\kappa)$-complete ideal *on* $\kappa$, *i.e.*

*a*) $\emptyset \neq \mathrm{NS}_\kappa \subseteq \mathcal{P}(\kappa)$

*b*) $\kappa \notin \mathrm{NS}_\kappa$

*c*) $X \in \mathrm{NS}_\kappa \wedge Y \subseteq X \subseteq \kappa \rightarrow Y \in \mathrm{NS}_\kappa$

*d*) $X \in \mathrm{NS}_\kappa \wedge Y \in \mathrm{NS}_\kappa \rightarrow X \cap Y \in \mathrm{NS}_\kappa$

*e*) $\gamma < \mathrm{cof}(\kappa) \wedge \{X_i | i < \gamma\} \subseteq \mathrm{NS}_\kappa \rightarrow \bigcup_{i<\gamma} X_i \in \mathrm{NS}_\kappa$

**Proof.** *e*) Let $\gamma < \mathrm{cof}(\kappa) \wedge \{X_i | i < \gamma\} \subseteq \mathrm{NS}_\kappa$. Then $\{\kappa \setminus X_i | i < \gamma\} \subseteq \mathcal{C}_\kappa$. By Lemma 133, $\bigcap_{i<\gamma} (\kappa \setminus X_i) \in \mathcal{C}_\kappa$. Hence

$$\bigcup_{i<\gamma} X_i = \kappa \setminus \bigcap_{i<\gamma} (\kappa \setminus X_i) \in \mathrm{NS}_\kappa .$$

□

For regular uncountable $\kappa$ these filters and ideals have even better completeness properties.

**Definition 140.** *Let* $\kappa$ *be a regular uncountable cardinal. For a sequence* $(X_i)_{i<\kappa}$ *of subsets of* $\kappa$ *define*

*a*) *the* diagonal intersection

$$\bigtriangleup_{i<\kappa} X_i = \{\beta < \kappa \mid \forall i < \beta\ \beta \in X_i\},$$

*b*) *the* diagonal union

$$\bigtriangledown_{i<\kappa} X_i = \{\beta < \kappa \mid \exists i < \beta\ \beta \in X_i\}.$$

**Lemma 141.** *Let* $\kappa$ *be a regular uncountable cardinal. Then*

*a*) $\mathcal{C}_\kappa$ *is closed under diagonal intersections, i.e.,*

$$\{X_i | i < \kappa\} \subseteq \mathcal{C}_\kappa \rightarrow \bigtriangleup_{i<\kappa} X_i \in \mathcal{C}_\kappa ,$$

*b*) $\mathrm{NS}_\kappa$ *is closed under diagonal unions, i.e.,*

$$\{X_i | i < \kappa\} \subseteq \mathrm{NS}_\kappa \rightarrow \bigtriangledown_{i<\kappa} X_i \in \mathrm{NS}_\kappa .$$

**Proof.** *a*) Let $\{X_i | i < \kappa\} \subseteq \mathcal{C}_\kappa$. For $i < \kappa$ choose $C_i \in \mathcal{C}_\kappa$ such that $C_i \subseteq X_i$. Then

$$\bigtriangleup_{i<\kappa} C_i \subseteq \bigtriangleup_{i<\kappa} X_i$$

and it suffices to show that $\bigtriangleup_{i<\kappa} C_i$ is cub in $\kappa$.

$\bigtriangleup_{i<\kappa} C_i$ is closed in $\kappa$: Let $\lambda < \kappa$ be a limit ordinal and a limit point of $\bigtriangleup_{i<\kappa} C_i$. Consider $j < \lambda$. By the definition of the diagonal intersection

$$\left(\bigtriangleup_{i<\kappa} C_i\right) \setminus (j+1) \subseteq C_j .$$

Hence $\lambda$ is a limit point of $C_j$ and $\lambda \in C_j$ by the closure of $C_j$. Thus $\forall j < \lambda\ \lambda \in C_j$ and thus $\lambda \in \bigtriangleup_{i<\kappa} C_i$.

$\bigtriangleup_{i<\kappa} C_i$ is unbounded in $\kappa$: Let $\alpha < \kappa$. Define a sequence $(\beta_n | n < \omega)$ by recursion: set $\beta_0 = \alpha$ and

$$\beta_{n+1} = \text{the least element of} \left(\bigcap_{i<\beta_n} C_i\right) \setminus (\beta_n + 1) .$$

Let $\beta = \bigcup_{n<\omega} \beta_n$. $\beta$ is a limit ordinal $>\alpha$. $\beta < \kappa$ since $\mathrm{cof}(\kappa) \geqslant \omega_1$. We show that $\beta \in \Delta_{i<\kappa} C_i$. Consider $j < \beta$. Take $n < \omega$ such that $j < \beta_n$. Then

$$\{\beta_k \mid n < k < \omega\} \subseteq C_j$$

and $\beta$ is a limit point of $C_j$. $\beta \in C_j$ by the closure of $C_j$. Hence $\forall j < \beta\ \beta \in C_j$. □

Sets in an ideal behave similar to sets of (Lebesgue-)measure 0. Then sets not in the ideal have "positive measure". So stationary sets are *positive* with respect to the non-stationary ideal.

Closure under diagonal intersections corresponds to a surprising canonization property of certain functions.

**Definition 142.** *A function* $f: Z \to \mathrm{Ord}$ *where* $Z \subseteq \mathrm{Ord}$ *is* regressive *if*

$$\alpha \in Z \setminus \{0\} \to f(\alpha) < \alpha.$$

**Exercise 30.** If $\gamma \geqslant 2$ then there is no regressive *injective* function $f: \gamma \to \gamma$.

**Theorem 143.** (Fodor's Lemma) *Let $\kappa$ be a regular uncountable cardinal and let $f: S \to \kappa$ be regressive, where $S$ is stationary in $\kappa$. Then there is a stationary $T \subseteq S$ such that $f \upharpoonright T$ is constant.*

**Proof.** Assume that for every $i < \kappa$ $f^{-1}[\{i\}]$ is not stationary. So for every $i < \kappa$ choose a $C_i$ cub such that $\forall j \in T \cap C_i\ f(j) \neq i$. The set

$$C = \Delta_{i<\kappa} C_i$$

is cub in $\kappa$, and so there is $\alpha \in C \cap T$, $\alpha > 0$. But then for all $i < \alpha$ $\alpha \in T \cap C_i$ and $f(\alpha) \neq i$. But then $f(\alpha) \geqslant \alpha$, contradicting the regressivity of $f$. □

We now give examples of closed unbounded and stationary sets.

**Lemma 144.** *Let $\kappa$ be an uncountable regular cardinal and let $C$ be cub in $\kappa$. The* derivation *$C'$ of $C$ is defined as*

$$C' = \{\alpha \in C \mid \alpha \textit{ is a limit point of } C\}.$$

*Then $C'$ is cub in $\kappa$.*

*On can form* iterated derivations *$C^{(i)}$ for $i < \kappa$ by*

$$\begin{aligned} C^{(0)} &= C \\ C^{(i+1)} &= \left(C^{(i)}\right)' \\ C^{(\lambda)} &= \bigcap_{i<\lambda} C^{(i)} \textit{ for limit ordinals } \lambda < \kappa \end{aligned}$$

*Every $C^{(i)}$ is cub in $\kappa$.*

The lemma implies immediately:

**Lemma 145.** *Let* Lim *be the class of limit ordinals. Let $\kappa$ be an uncountable regular cardinal. Then* $\mathrm{Lim} \cap \kappa$ *is cub in $\kappa$.*

Topologically these derivation correspond to the process of omitting isolated points. Such iterated derivations were first studied by Cantor.

**Example 146.** For $\kappa$ an uncountable regular cardinal let $(\kappa)^{<\omega}$ be the set of all finite sequences from $\kappa$, i.e., $(\kappa)^{<\omega} = \{u \mid \exists n < \omega\ u: n \to \kappa\}$. For $h: (\kappa)^{<\omega} \to \kappa$ let

$$C_h = \{\beta < \kappa \mid h[(\beta)^{<\omega}] \subseteq \beta\}$$

be the set of ordinals $<\kappa$ which are closed under $h$. Then $C_h$ is cub in $\kappa$. Given $\alpha<\kappa$, a closed ordinal $\beta>\alpha$ can be found as $\beta=\bigcup_{n<\omega}\beta_n$ where $\beta_0=\alpha$ and

$$\beta_{n+1}=\left(\bigcup h[(\beta_n)^{<\omega}]\right)+1<\kappa.$$

Conversely, if $C$ is cub in $\kappa$ one can define $g:\kappa\to\kappa$ by

$$g(\alpha)=\text{the smallest element of } C \text{ which is } >\alpha.$$

If we define $C_g$ as above then

$$C_g=\{0\}\cup C'.$$

**Lemma 147.** *Let $\mu<\kappa$ be uncountable regular cardinals. Then*

$$E^\kappa_\mu=\{\alpha<\kappa\mid\operatorname{cof}(\alpha)=\mu\}$$

*is stationary in $\kappa$.*

**Proof.** Let $C$ be cub in $\kappa$. Define a strictly increasing $\mu+1$-sequence $(\alpha_i)_{i\leqslant\mu}$ of elements of $C$ by

$$\alpha_i=\text{the smallest element } \alpha \text{ of } C \text{ such that } \forall j<i\ \alpha_j<\alpha.$$

Then $\alpha_\mu<\kappa$ and $\operatorname{cof}(\alpha_\mu)=\mu$. Hence $C\cap E^\kappa_\mu\neq\emptyset$. □

So $E^{\aleph_2}_\omega$ and $E^{\aleph_2}_{\omega_1}$ are disjoint stationary subsets of $\aleph_2$. Actually one can find a lot of disjoint stationary sets, using Fodor's lemma.

**Theorem 148.** *Let $\kappa$ be a successor cardinal and let $S\subseteq\kappa$ be stationary. Then there is a family $(S_i\mid i<\kappa)$ of pairwise disjoint stationary subsets of $S$.*

**Proof.** Let $\lambda\in\text{Card}$ such that $\kappa=\lambda^+$. For each $\nu<\kappa$, $\nu\neq0$ choose a surjective function $f_\nu$: $\lambda\to\nu$.
(1) For every $\alpha<\kappa$ there is some $i<\lambda$ such that $\{\nu\in S\mid f_\nu(i)\geqslant\alpha\}$ is stationary in $\kappa$.
*Proof*. Assume for a contradiction that there is $\alpha<\kappa$ such that for any $i<\lambda$ the set $\{\nu\in S\mid f_\nu(i)\geqslant\alpha\}$ is non-stationary in $\kappa$. Choose cub sets $C_i$ such that

$$\{\nu\in S\mid f_\nu(i)\geqslant\alpha\}\cap C_i=\emptyset.$$

The set $\bigcap_{i<\lambda}C_i$ is cub in $\kappa$. Let $\nu\in S\cap\bigcap_{i<\lambda}C_i$ and $\nu>\alpha$. Then $f_\nu(i)<\alpha$ for all $i<\lambda$, which contradicts the surjectivity of $f_\nu:\lambda\to\nu$. *qed*(1)
(2) There is some $i_*<\lambda$ such that for every $\alpha<\kappa$ the set $\{\nu\in S\mid f_\nu(i_*)\geqslant\alpha\}$ is stationary in $\kappa$.
*Proof*. By (1), we can find for every $\alpha<\kappa$ some $i_\alpha<\lambda$ such that $\{\nu\in S\mid f_\nu(i_\alpha)\geqslant\alpha\}$ is stationary in $\kappa$. By the pidgeon principle there is an unbounded subset $Z\subseteq\kappa$ and an $i_*<\lambda$ such that $\forall\alpha\in Z\ i_\alpha=i_*$. So for every $\alpha\in Z$ the set $\{\nu\in S\mid f_\nu(i_*)\geqslant\alpha\}$ is stationary in $\kappa$, which proves the claim. *qed*(2)

For $\beta<\kappa$ set $S_\beta=\{\nu\in S\mid f_\nu(i_*)=\beta\}$.
(3) The set of $\beta<\kappa$, where $S_\beta$ is stationary in $\kappa$, is unbounded in $\kappa$.
*Proof*. Assume not and let $\alpha<\kappa$ such that $S_\beta$ is stationary implies $\beta<\alpha$. By (2), $T=\{\nu\in S\mid f_\nu(i_*)\geqslant\alpha\}$ is stationary in $\kappa$. The function $\nu\mapsto f_\nu(i_*)<\nu$ is regressive on $T$. By Fodor's Theorem the function is constant on a stationary subset of $T$. Let $\beta$, $\alpha\leqslant\beta<\kappa$ be the constant value. Then $S_\beta=\{\nu\in S\mid f_\nu(i_*)=\beta\}$ is stationary in $\kappa$, contradiction. *qed*(3)

So there are $\kappa$-many $\beta<\kappa$ such that $S_\beta$ is stationary. Note that these $S_\beta$ are pairwise disjoint subsets of $S$. □

Abstractly this means that every $\mathrm{NS}_\kappa$-positive set can be split into $\kappa$-many $\mathrm{NS}_\kappa$-positive sets. Consider the property: there are $\mu$-many $\mathrm{NS}_\kappa$-positive sets $(S_i\mid i<\mu)$ which are *almost disjoint* with respect to $\mathrm{NS}_\kappa$: $i\neq j\to S_i\cap S_j\in\mathrm{NS}_\kappa$. If this property is false, we say that the ideal $\mathrm{NS}_\kappa$ is *$\mu$-saturated*. The property that $\mathrm{NS}_{\aleph_1}$ is $\aleph_2$-saturated is not decided by ZFC. That property has many consequences and is central in modern set theoretic research.

# 18 Silver's Theorem

The value of $2^\kappa$ for regular cardinals $\kappa$ is hardly determined by the value of $2^\lambda$ at other cardinals. The situation at singular cardinals is different, the first result in this area was proved by Jack Silver. We shall use the notion of *almost disjoint* functions.

**Definition 149.** *Let $\lambda$ be a limit ordinal. Two functions $f, g: \lambda \to V$ are* almost disjoint *if there is $\alpha < \lambda$ such that $\forall \beta\, (\alpha < \beta < \lambda \to f(\beta) \neq g(\beta))$. A set $\mathcal{F} \subseteq {}^\lambda V$ of functions is* almost disjoint *if $f$ and $g$ are almost disjoint for any $f, g \in \mathcal{F}$, $f \neq g$.*

**Lemma 150.** *a) There is no almost disjoint family $\mathcal{F} \subseteq {}^\omega 2$ of size 3.*
*b) There is an almost disjoint family $\mathcal{F} \subseteq {}^\omega \omega$ of size $2^{\aleph_0}$.*

**Proof.** *a)* is obvious. *b)* Take $h: (\omega)^{<\omega} \leftrightarrow \omega$. For $a: \omega \to 2$ define $f_a: \omega \to \omega$ by

$$f_a(n) = h(a \upharpoonright n).$$

Consider functions $a, b: \omega \to 2$, $a \neq b$. We show that $f_a$ and $f_b$ are almost disjoint. Take $n < \omega$ such that $a \upharpoonright n \neq b \upharpoonright n$. Then for $n \leqslant m < \omega$ we have

$$f_a(m) = h(a \upharpoonright m) \neq h(b \upharpoonright m) = f_b(m).$$

Thus $\{f_a \mid a \in {}^\omega 2\}$ is an almost disjoint family of size $2^{\aleph_0}$. □

**Theorem 151.** *(Silver) Let $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$. Let $2^\mu = \mu^+$ for all $\omega \leq \mu \in \kappa \cap \mathrm{Card}$. Then $2^\kappa = \kappa^+$.*

So let us assume that $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$ and $2^\mu = \mu^+$ for all $\omega \leq \mu \in \kappa \cap \mathrm{Card}$. Take a strictly increasing sequence $(\kappa_\alpha \mid \alpha < \lambda)$ which is cofinal in $\kappa$ and continuous, i.e., for any limit ordinal $\delta < \lambda$ we have $\kappa_\delta = \bigcup_{\alpha < \delta} \kappa_\alpha$.

**Lemma 152.** *Let $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$, and assume that $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Let $\mathcal{F} \subseteq \prod_{\alpha < \lambda} A_\alpha$ be almost disjoint, where $S_0 = \{\alpha < \lambda \mid \mathrm{card}(A_\alpha) \leq \kappa_\alpha\}$ is stationary in $\lambda$. Then $\mathrm{card}(\mathcal{F}) \leq \kappa$.*

**Proof.** Assume w.l.o.g. that $A_\alpha \subseteq \kappa_\alpha$ for $\alpha \in S$. For $f \in \mathcal{F}$ define $h_f: S_0 \to \lambda$ by

$$h_f(\alpha) := \text{the least } \beta \text{ such that } f(\alpha) \in \kappa_\beta.$$

Then $h \upharpoonright (S_0 \cap \mathrm{Lim})$ is regressive. So, by Fodor's lemma, one can choose a stationary $S_f \subseteq S_0 \cap \mathrm{Lim}$ such that $h$ is constant on $S_f$. Hence $f$ is, on $S_f$, bounded in $\kappa$. If $f \upharpoonright S_f = g \upharpoonright S_g$, then $f = g$, since $\mathcal{F}$ is almost disjoint. So $f \mapsto f \upharpoonright S_f$ is one-to-one. For a fixed $T \subseteq \lambda$, the set of functions on $T$ that are bounded in $\kappa$ has cardinality $\sup\{\mu^\lambda \mid \mu < \kappa\} = \kappa$ by the cardinality assumption. Also there are $<\kappa$ such $T$, since $\mathrm{card}(\wp(\lambda)) = 2^\lambda < \kappa$. Hence $\mathrm{card}(\mathcal{F}) \leq \kappa \cdot \kappa = \kappa$. □

**Lemma 153.** *Let $\omega < \lambda = \mathrm{cof}(\kappa) < \kappa \in \mathrm{Card}$, and assume that $\mu^\lambda < \kappa$ for all $\mu < \kappa$. Let $\mathcal{F} \subseteq \prod_{\alpha < \lambda} A_\alpha$ be almost disjoint, $\mathrm{card}(A_\alpha) \leq \kappa_\alpha^+$. Then $\mathrm{card}(\mathcal{F}) \leq \kappa^+$.*

**Proof.** Assume w.l.o.g. that $A_\alpha \subseteq \kappa_\alpha^+$. Let $S \subseteq \lambda$ be stationary and $f \in \mathcal{F}$. Let

$$\mathcal{F}_{f,S} = \{g \in \mathcal{F} \mid (\forall \alpha \in S)(g(\alpha) \leq f(\alpha))\}.$$

The map $g \mapsto g \upharpoonright S$ injects $\mathcal{F}_{f,S}$ into $\prod_{\alpha \in S} (f(\alpha) + 1)$ where $\mathrm{card}(f(\alpha) + 1) \leq \kappa_\alpha$. By the previous lemma, $\mathrm{card}(\mathcal{F}_{f,S}) \leq \kappa$.

Define

$$\mathcal{F}_f = \bigcup \{\mathcal{F}_{f,S} \mid S \subseteq \lambda \text{ is stationary}\}.$$

Since there are $<\kappa$ stationary subsets of $\lambda$,
(1) $\mathrm{card}(\mathcal{F}_f) \leq \kappa$.

We construct a sequence $(f_\xi|\ \xi<\delta)$ of functions in $\mathcal{F}$ by induction such that $\mathcal{F}=\bigcup\ \{\mathcal{F}_{f_\xi}|\ \xi<\delta\}$.

Take an arbitrary $f_0\in\mathcal{F}$.

If $(f_\nu|\ \nu<\xi)$ is already defined, choose $f_\xi\notin\bigcup\ \{\mathcal{F}_{f_\nu}|\ v<\xi\}$ if possible. If there is no such $f_\xi$, set $\delta=\xi$ and stop.

(2) $\delta\leq\kappa^+$.

*Proof*. Assume that $f_{\kappa^+}$ is defined. If $\nu<\kappa^+$ then $f_{\kappa^+}\notin\mathcal{F}_{f_\nu}$ . So $\{\alpha|\ f_{\kappa^+}(\alpha)\leq f_v(\alpha)\}$ is non-stationary and $\{\alpha|\ f_\nu(\alpha)\leq f_{\kappa^+}(\alpha)\}\in\mathcal{C}_\lambda$. Thus $f_v\in\mathcal{F}_{\kappa^+}$ for all $v<\xi$ , and $\mathrm{card}(\mathcal{F}_{f_{\kappa^+}})\geqslant\kappa^+$. This contradicts (1). *qed*(2)

Altogether

$$\mathrm{card}(\mathcal{F})=\mathrm{card}(\bigcup\ \{\mathcal{F}_{f_\xi}|\ \xi<\delta\})\leqslant\sum_{\xi<\delta}\mathrm{card}(\mathcal{F}_{f_\xi})\leqslant\sum_{\xi<\delta}\kappa\leqslant\kappa\cdot\kappa^+=\kappa^+.$$

□

**Proof.** (Silver's Theorem) Define a map from $\mathcal{P}(\kappa)$ into $\prod_{\alpha<\lambda}\wp(\kappa_\alpha)$ by

$$X\mapsto f_X=(X\cap\kappa_\alpha\,|\,\alpha<\lambda).$$

If $X\neq Y$ then $f_X$ and $F_Y$ are almost disjoint. So

$$\mathcal{F}=\{f_X\,|\,X\in\mathcal{P}(\kappa)\}\subseteq\prod_{\alpha<\lambda}\wp(\kappa_\alpha)$$

is an almost disjoint family of functions. The GCH below $\kappa$ implies that $\mathrm{card}(\wp(\kappa_\alpha))=2^{\kappa_\alpha}=\kappa_\alpha^+$ . Moreover $\mu^\lambda\leqslant\max\,(\mu,\,\lambda)^{\max(\mu,\lambda)}=\max\,(\mu,\,\lambda)^+<\kappa$ for all $\mu<\kappa$ . By Lemma 153, $\mathrm{card}(\mathcal{F})\leqslant\kappa^+$. Hence

$$\kappa^+\leqslant 2^\kappa=\mathrm{card}(\mathcal{P}(\kappa))\leqslant\mathrm{card}(\mathcal{F})\leqslant\kappa^+$$

□

**Exercise 31.** Use the methods of the proof of Silver's Theorem to show

a) Let $\mathcal{F}\subseteq\prod_{\alpha<\lambda}A_\alpha$ be almost disjoint, $\mathrm{card}(A_\alpha)\leq\kappa_\alpha^{++}$. Then $\mathrm{card}(\mathcal{F})\leq\kappa^{++}$.

b) Let $\omega<\lambda=\mathrm{cof}(\kappa)<\kappa\in\mathrm{Card}$. Let $2^\mu=\mu^{++}$ for all $\omega\leq\mu\in\kappa\cap\mathrm{Card}$. Then $2^\kappa\leqslant\kappa^{++}$.

# 19 The Galvin Hajnal Theorem

The previous exercise indicates the possibility that one may generalize Silver's theorem by a kind of induction on the height of the continuum function below $\kappa$ . This idea leads to the Galvin-Hajnal theorem.

**Theorem 154.** *Let $\kappa=\aleph_\alpha$ be a singular strong limit cardinal (i.e., $\mu<\aleph_\alpha\rightarrow 2^\mu<\aleph_\alpha$) with $\omega<\lambda=\mathrm{cof}(\aleph_\alpha)<\aleph_\alpha$. Then*

$$2^{\aleph_\alpha}<\aleph_\gamma$$

*where* $\gamma=\left(2^{\mathrm{card}(\alpha)}\right)^+$.

Note that if $\alpha=\aleph_\alpha$ the theorem claims that

$$2^\alpha<\aleph_{(2^\alpha)^+}$$

which is obviously true.

So we may fix the cardinal situation with the extra assumption that $\alpha<\aleph_\alpha$. Let us also fix a strictly increasing continuous sequence $(\kappa_\delta\mid\delta<\lambda)$ which is cofinal in $\aleph_\alpha$ . The continuum function below $\aleph_\alpha$ determines a function $\varphi_0{:}\,\lambda\rightarrow\alpha$ by

$$2^{\kappa_\delta}=\kappa_\delta^{+\varphi_0(\delta)},$$

where $\kappa_\delta^{+\varphi_0(\delta)} = \aleph_{\eta+\varphi_0(\delta)}$ if $\kappa_\delta = \aleph_\eta$. The following argument uses induction on functions $\varphi: \lambda \to \alpha$ along appropriate wellfounded relations which we introduce first.

**Definition 155.** *For a stationary set $S \subseteq \lambda$ define a relation $<_S$ on functions $\varphi, \psi: \lambda \to \alpha$ by*

$$\varphi <_S \psi \text{ iff there is a cub } C \subseteq \lambda \text{ such that } \forall \delta \in S \cap C\ \varphi(\delta) < \psi(\delta).$$

Equivalently one can say that the set of $\delta$ where $\varphi$ and $\psi$ behaves differently is very small:

$$\varphi <_S \psi \text{ iff } \{\delta \in S \mid \varphi(\delta) \geqslant \psi(\delta)\} \in \mathrm{NS}_\lambda.$$

**Lemma 156.** *$<_S$ is a strongly wellfounded relation on ${}^\lambda\alpha$.*

**Proof.** Assume not. Then, using AC, there is a strictly descending $\omega$-sequence

$$\psi_0 >_S \psi_1 >_S \psi_2 >_S \ldots.$$

Choose cub sets $C_0, C_1, \ldots \subseteq \lambda$ such that $\forall \delta \in S \cap C_n\ \psi_n(\delta) > \psi_{n+1}(\delta)$. Since $\bigcap_{n<\omega} C_n$ is cub in $\lambda$ one can take $\delta \in S \cap \bigcap_{n<\omega} C_n$. Then

$$\psi_0(\delta) > \psi_1(\delta) > \psi_2(\delta) > \ldots.$$

is a descending $\omega$-sequence of ordinals. Contradiction. $\square$

**Definition 157.** *Define the $<_S$-rank $\|\psi\|_S \in \mathrm{Ord}$ of $\psi \in {}^\lambda\alpha$ by recursion on $<_S$:*

$$\|\psi\|_S = \bigcup \{\|\varphi\|_S + 1 \mid \varphi <_S \psi\}.$$

*We also write $\|\psi\|$ instead of $\|\psi\|_\lambda$.*

We prove some properties of this rank.

**Lemma 158.**

*a) If $S \subseteq T$ are stationary in $\lambda$ then $\varphi <_T \psi$ implies $\varphi <_S \psi$.*

*b) If $S \subseteq T$ are stationary in $\lambda$ then $\|\psi\|_T \leqslant \|\psi\|_S$.*

*c) If $S, T$ are stationary in $\lambda$ then*

$$\|\psi\|_{S\cup T} = \min(\|\psi\|_S, \|\psi\|_T).$$

*d) If $S$ is stationary and $N$ is nonstationary then $\varphi <_{S\cup N} \psi$ iff $\varphi <_S \psi$.*

*e) If $S$ is stationary and $N$ is nonstationary then*

$$\|\psi\|_{S\cup N} = \|\psi\|_S.$$

**Proof.** *a)* Let $\varphi <_T \psi$. Take $C \subseteq \lambda$ cub such that $\forall \delta \in T \cap C\ \varphi(\delta) < \psi(\delta)$. Then $\forall \delta \in S \cap C$ $\varphi(\delta) < \psi(\delta)$ and so $\varphi <_S \psi$.
*b)* By induction on $<_T$.

$$\begin{aligned} \|\psi\|_T &= \bigcup \{\|\varphi\|_T + 1 \mid \varphi <_T \psi\} \\ &\leqslant \bigcup \{\|\varphi\|_S + 1 \mid \varphi <_T \psi\} \text{ by the inductive assumption} \\ &\leqslant \bigcup \{\|\varphi\|_S + 1 \mid \varphi <_S \psi\} \text{ by } a) \\ &= \|\psi\|_S. \end{aligned}$$

*c)* By *b)* $\|\psi\|_{S\cup T} \leqslant \|\psi\|_S, \|\psi\|_T$ and so

$$\|\psi\|_{S\cup T} \leqslant \min(\|\psi\|_S, \|\psi\|_T).$$

Assume that the equality is false and consider $\psi$ $<_{S\cup T}$-minimal such that

$$\|\psi\|_{S\cup T} < \min(\|\psi\|_S, \|\psi\|_T).$$

Since $\|\psi\|_{S\cup T} \in \|\psi\|_S = \bigcup \{\|\varphi\|_S + 1 \mid \varphi <_S \psi\}$ take $\psi_S <_S \psi$ such that $\|\psi\|_{S\cup T} < \|\psi_S\|_S + 1$, i.e., $\|\psi\|_{S\cup T} \leqslant \|\psi_S\|_S$; take $C_S \subseteq \lambda$ cub such that $\forall \delta \in S \cup C_S\ \psi_S(\delta) < \psi(\delta)$. Similarly take $\psi_T <_T \psi$ such that $\|\psi\|_{S\cup T} \leqslant \|\psi_T\|_T$ and some $C_T \subseteq \lambda$ cub such that $\forall \delta \in T \cup C_T\ \psi_T(\delta) < \psi(\delta)$. Define $\varphi: \lambda \to \alpha$,

$$\varphi(\delta) = \begin{cases} \psi_S(\delta) \text{ if } \delta \in S \setminus T \\ \psi_T(\delta) \text{ if } \delta \in T \setminus S \\ \max(\psi_S(\delta), \psi_T(\delta)) \text{ if } \delta \in S \cap T \\ 0 \text{ else} \end{cases}$$

For $\delta \in (S \cup T) \cap (C_S \cap C_T)$ holds $\varphi(\delta) < \psi(\delta)$, thus $\varphi <_{S\cup T} \psi$. Since $\forall \delta \in S\ \psi_S(\delta) \leqslant \varphi(\delta)$ we have $\|\psi_S\|_S \leqslant \|\varphi\|_S$. Similarly $\|\psi_T\|_T \leqslant \|\varphi\|_T$. By the $<_{S\cup T}$-minimality of $\psi$ we have

$$\|\varphi\|_{S\cup T} = \min(\|\varphi\|_S, \|\varphi\|_T) \geqslant \min(\|\psi_S\|_S, \|\psi_T\|_T) \geqslant \|\psi\|_{S\cup T}$$

contradicting $\varphi <_{S\cup T} \psi$.

*d*) $\varphi <_{S\cup N} \psi$ iff $\{\delta \in S \cup N \mid \varphi(\delta) \geqslant \psi(\delta)\} \in \mathrm{NS}_\lambda$ iff $\{\delta \in S \mid \varphi(\delta) \geqslant \psi(\delta)\} \in \mathrm{NS}_\lambda$ iff $\varphi <_S \psi$.

*e*) follows directly from *d*). □

Note that $\|\psi\| = \|\psi\|_\lambda \leqslant \|\psi\|_S$. This motivates to exclude stationary sets $S$ which do not compute the "correct" rank of $\psi$.

**Definition 159.** *Let*

$$I_\psi = \mathrm{NS}_\lambda \cup \{S \mid S \textit{ is stationary and } \|\psi\| < \|\psi\|_S\}.$$

**Lemma 160.** *$I_\psi$ is an ideal on $\lambda$.*

**Proof.** $I_\psi$ is a non-empty family of subsets of $\lambda$ and $\lambda \notin I_\psi$.

$I_\psi$ is closed under subsets: let $A \in I_\psi$ and $B \subseteq A$. If $B$ is nonstationary then $B \in I_\psi$. If $B$ is stationary then $\|\psi\|_B \geqslant \|\psi\|_A > \|\psi\|$ and $B \in I_\psi$.

$I_\psi$ is closed under unions: let $A \in I_\psi$ and $B \in I_\psi$. If $A$ and $B$ are nonstationary then $A \cup B \in \mathrm{NS}_\lambda \subseteq I_\psi$. If $A$ is stationary and $B$ is nonstationary then by *e*) of Lemma 158 $\|\psi\|_{A\cup B} = \|\psi\|_A > \|\psi\|$ and $A \cup B \in I_\psi$. If $A$ is stationary and $B$ is stationary then by *c*) of Lemma 158

$$\|\psi\|_{A\cup B} = \min(\|\psi\|_A, \|\psi\|_B) > \|\psi\|$$

and $A \cup B \in I_\psi$. □

**Lemma 161.**

*a*) *If $\|\psi\| = 0$ then $\{\delta < \lambda \mid \psi(\delta) = 0\}$ is stationary in $\lambda$.*

*b*) *If $\|\psi\|$ is a successor ordinal then*

$$\{\delta < \lambda \mid \psi(\delta) \textit{ is a successor ordinal}\} \notin I_\psi.$$

*c*) *If $\|\psi\|$ is a limit ordinal then*

$$\{\delta < \lambda \mid \psi(\delta) \textit{ is a limit ordinal}\} \notin I_\psi.$$

**Proof.** *a*) Let $\|\psi\| = 0$. If $\{\delta < \lambda \mid \psi(\delta) = 0\}$ is nonstationary there is a cub $C \subseteq \lambda$ such that $\forall \delta \in C\ \psi(\delta) > 0$. But then $\psi >_\lambda \mathrm{const}_0$ and $\|\psi\| \geqslant 1$. Contradiction.

*b*) Let $\|\psi\|$ be a successor ordinal but assume that

$$\{\delta < \lambda \mid \psi(\delta) \text{ is a successor ordinal}\} \in I_\psi.$$

Then

$$S = \{\delta < \lambda \mid \psi(\delta) \text{ is a limit ordinal}\} \notin I_\psi.$$

By the definition of $I_\psi$ we get that $\|\psi\|_S = \|\psi\|$ is also a successor ordinal. By the definition of $\|\psi\|_S$ take $\varphi <_S \psi$ such that $\|\psi\|_S = \|\varphi\|_S + 1$. Take a cub set $C \subseteq \lambda$ such that $\forall \delta \in S \cap C\ \varphi(\delta) < \psi(\delta)$. Define $\varphi^+: \lambda \to \alpha$ by $\varphi^+(\delta) = \varphi(\delta) + 1$. Since $\psi(\delta)$ is a limit ordinal for $\delta \in S$:

$$\forall \delta \in S \cap C\ \varphi(\delta) < \varphi^+(\delta) < \psi(\delta).$$

Then $\varphi <_S \varphi^+ <_S \psi$ and $\|\varphi\|_S < \|\varphi^+\|_S < \|\psi\|_S$, contradicting $\|\psi\|_S = \|\varphi\|_S + 1$.
*c*) Let $\|\psi\|$ be a limit ordinal but assume that

$$\{\delta < \lambda \mid \psi(\delta) \text{ is a limit ordinal}\} \in I_\psi .$$

Then

$$S = \{\delta < \lambda \mid \psi(\delta) \text{ is a successor ordinal}\} \notin I_\psi .$$

By the definition of $I_\psi$ we get that $\|\psi\|_S = \|\psi\|$ is also a limit ordinal. Define $\psi^-: \lambda \to \alpha$ by

$$\psi^-(\delta) = \begin{cases} \psi(\delta) - 1, \text{ if } \delta \in S \\ 0, \text{ else} \end{cases}$$

Consider a function $\varphi <_S \psi$. Take a cub set $C \subseteq \lambda$ such that $\forall \delta \in S \cap C\ \varphi(\delta) < \psi(\delta)$ and thus $\forall \delta \in S \cap C\ \varphi(\delta) \leqslant \psi^-(\delta)$. Then

$$\|\varphi\|_S = \|\varphi\|_{S \cap C} \leqslant \|\psi^-\|_{S \cap C} = \|\psi^-\|_S < \|\psi\|_S$$

This means that

$$\begin{aligned} \|\psi\|_S &= \bigcup \{\|\varphi\|_S + 1 \mid \varphi <_S \psi\} \\ &\leqslant \|\psi^-\|_S + 1 \end{aligned}$$

Since $\|\psi^-\|_S < \|\psi\|_S$ this implies

$$\|\psi\| = \|\psi\|_S = \|\psi^-\|_S + 1$$

is a successor ordinal, contradiction. □

**Lemma 162.** *Let $\psi: \lambda \to \alpha$ and let $\mathcal{F} \subseteq \prod_{\delta < \lambda} A_\delta$ be an almost disjoint family of functions such that*

$$\operatorname{card}(A_\delta) \leqslant \kappa_\delta^{+\psi(\delta)}$$

*for $\delta < \lambda$. Then* $\operatorname{card}(\mathcal{F}) \leqslant \aleph_{\alpha + \|\psi\|}$.

**Proof.** By induction on $\|\psi\| \in \mathrm{Ord}$.
$\|\psi\| = 0$: Then by Lemma 161(*a*) $\psi$ vanishes on a stationary set $S \subseteq \lambda$ and Lemma 152 proves the case.
$\|\psi\|$ is the successor ordinal $\gamma + 1$: by Lemma 161(*b*)

$$S_0 = \{\delta < \lambda \mid \psi(\delta) \text{ is a successor ordinal}\} \notin I_\psi .$$

We may assume that $A_\delta \subseteq \kappa_\delta^{+\psi(\delta)}$ for $\delta < \lambda$.
(1) Let $f \in \mathcal{F}$ and $S \subseteq S_0$, $S \notin I_\psi$. Then the set

$$\mathcal{F}_{f,S} = \{g \in \mathcal{F} \mid \forall \delta \in S\ g(\delta) \leqslant f(\delta)\}$$

has cardinality $\leqslant \aleph_{\alpha + \gamma}$.
*Proof.* Define $\varphi: \lambda \to \alpha$ by

$$\varphi(\delta) = \begin{cases} \psi(\delta) - 1, \text{ if } \delta \in S \\ \psi(\delta), \text{ else} \end{cases}$$

Since $S \notin I_\psi$,

$$\|\varphi\| \leqslant \|\varphi\|_S < \|\psi\|_S = \|\psi\| = \gamma + 1$$

and $\|\varphi\| \leqslant \gamma$. Now

$$\mathcal{F}_{f,S} \subseteq \prod_{\delta < \lambda} f(\delta),$$

and since $f(\delta) < \kappa_\delta^{+\psi(\delta)}$ we have

$$\operatorname{card}(f(\delta) + 1) \leqslant \kappa_\delta^{+\varphi(\delta)}.$$

By the induction hypothesis, $\operatorname{card}(\mathcal{F}_{f,S}) \leqslant \aleph_{\alpha + \|\varphi\|} \leqslant \aleph_{\alpha + \gamma}$. *qed*(1)
(2) Let $f \in \mathcal{F}$. Then the set

$$\mathcal{F}_f = \{g \in \mathcal{F} \mid \exists S \subseteq S_0\, (S \notin I_\psi \wedge \forall \delta \in S\ g(\delta) \leqslant f(\delta))\}$$

has cardinality $\leqslant\aleph_{\alpha+\gamma}$.
*Proof.* Since $\aleph_\alpha$ is a strong limit cardinal, there are at most $2^\lambda<\aleph_\alpha$ many $S\subseteq S_0$. Hence

$$\mathcal{F}_f=\bigcup_{S\subseteq S_0, S\notin I_\psi}\mathcal{F}_{f,S}$$

is a union of less than $\aleph_\alpha$ many sets of size $\leqslant\aleph_{\alpha+\gamma}$. *qed*(2)

Like in the proof of Lemma 153 we construct a sequence $(f_\xi|\ \xi<\zeta)$ of functions in $\mathcal{F}$ by induction such that $\mathcal{F}=\bigcup\ \{\mathcal{F}_{f_\xi}|\ \xi<\zeta\}$.

Take an arbitrary $f_0\in\mathcal{F}$.

If $(f_\nu|\ \nu<\xi)$ is already defined, choose $f_\xi\notin\bigcup\ \{\mathcal{F}_{f_\nu}|\ \nu<\xi\}$ if possible. If there is no such $f_\xi$, set $\zeta=\xi$ and stop.
(3) $\zeta\leqslant\aleph_{\alpha+\gamma+1}$.
*Proof*. Assume that $f_{\aleph_{\alpha+\gamma+1}}$ is defined. Set $\eta=\aleph_{\alpha+\gamma+1}$ . Consider $\nu<\eta$ . $f_\eta\notin\mathcal{F}_{f_\nu}$ and so there is no $S\subseteq S_0$, $S\notin I_\psi$ such that $\forall\delta\in S\ f_\eta(\delta)\leqslant f_\nu(\delta)$. This means that

$$\{\delta\in S\mid f_\eta(\delta)\leqslant f_\nu(\delta)\}\in I_\psi$$

and

$$\{\delta\in S\mid f_\nu(\delta)<f_\eta(\delta)\}\notin I_\psi.$$

This implies that $f_\nu\in\mathcal{F}_{f_\eta}$ and thus $\{f_\nu\mid\nu<\eta\}\subseteq\mathcal{F}_{f_\eta}$. Hence $\operatorname{card}(\mathcal{F}_{f_\eta})\geqslant\eta=\aleph_{\alpha+\gamma+1}$, which contradicts (2). *qed*(3)

Now

$$\operatorname{card}(\mathcal{F})=\operatorname{card}\Big(\bigcup\ \{\mathcal{F}_{f_\xi}|\ \xi<\zeta\}\Big)\leqslant\sum_{\xi<\zeta}\operatorname{card}(\mathcal{F}_{f_\xi})\leqslant\sum_{\xi<\zeta}\aleph_{\alpha+\gamma}\leqslant\sum_{\xi<\aleph_{\alpha+\gamma+1}}\aleph_{\alpha+\gamma}=\aleph_{\alpha+\gamma+1}.$$

Finally consider the case that $\|\psi\|$ is a limit cardinal. By Lemma 161($c$),

$$S=\{\delta<\lambda\mid\psi(\delta)\text{ is a limit ordinal}\}\notin I_\psi.$$

Again we may assume that $A_\delta\subseteq\kappa_\delta^{+\psi(\delta)}$ for $\delta<\lambda$.

For functions $\varphi\colon\lambda\to\alpha$ define

$$\mathcal{F}_\varphi=\Big\{f\in\mathcal{F}\mid\forall\delta<\lambda\ f(\delta)\in\kappa_\delta^{+\varphi(\delta)}\Big\}.$$

Consider $f\in\mathcal{F}$. Define $\varphi\colon\lambda\to\alpha$ by taking $\varphi(\delta)$ minimal such that

$$f(\delta)\in\kappa_\delta^{+\varphi(\delta)}.$$

Obviously $f\in\mathcal{F}_\varphi$. Moreover $\varphi<_S\psi$. Since $S\notin I_\psi$ we have

$$\|\varphi\|\leqslant\|\varphi\|_S<\|\psi\|_S=\|\psi\|.$$

By the induction hypothesis

$$\operatorname{card}(\mathcal{F}_\varphi)\leqslant\aleph_{\alpha+\|\varphi\|}<\aleph_{\alpha+\|\psi\|}.$$

Thus

$$\mathcal{F}\subseteq\bigcup\ \{\mathcal{F}_\varphi\mid\exists\varphi\,(\varphi\colon\lambda\to\alpha\wedge\operatorname{card}(\mathcal{F}_\varphi)<\aleph_{\alpha+\|\psi\|})\}$$

and

$$\operatorname{card}(\mathcal{F})\leqslant\sum_{\varphi:\lambda\to\alpha}\aleph_{\alpha+\|\psi\|}\leqslant\operatorname{card}({}^\lambda\alpha)\cdot\aleph_{\alpha+\|\psi\|}\leqslant\aleph_\alpha\cdot\aleph_{\alpha+\|\psi\|}=\aleph_{\alpha+\|\psi\|}.$$

□

We are now able to prove

**Theorem.** Let $\kappa=\aleph_\alpha$ be a singular strong limit cardinal (i.e., $\mu<\aleph_\alpha\to2^\mu<\aleph_\alpha$) with $\omega<\lambda=\operatorname{cof}(\aleph_\alpha)<\aleph_\alpha$. Then

$$2^{\aleph_\alpha}<\aleph_\gamma$$

where $\gamma=\big(2^{\operatorname{card}(\alpha)}\big)^+$.

**Proof.** Define an injective map from $\mathcal{P}(\kappa)$ into $\prod_{\delta<\lambda} \wp(\kappa_\delta)$ by

$$X \mapsto f_X = (X \cap \kappa_\delta \mid \delta < \lambda).$$

If $X \neq Y$ then $f_X$ and $F_Y$ are almost disjoint. So

$$\mathcal{F} = \{f_X \mid X \in \mathcal{P}(\kappa)\} \subseteq \prod_{\delta<\lambda} \wp(\kappa_\delta)$$

is an almost disjoint family of functions. Since $\kappa$ is a strong limit cardinal, there is a function $\psi$: $\lambda \to \alpha$ such that

$$\mathrm{card}(\wp(\kappa_\delta)) \leqslant \kappa_\delta^{+\psi(\delta)}$$

for $\delta < \lambda$. By the previous lemma

$$\mathrm{card}(\mathcal{P}(\kappa)) \leqslant \mathrm{card}(\mathcal{F}) \leqslant \aleph_{\alpha+\|\psi\|}.$$

Since $\mathrm{card}({}^\lambda\alpha) \leqslant \mathrm{card}({}^\alpha\alpha) = 2^{\mathrm{card}(\alpha)}$, the rank function $\|.\|$ on ${}^\lambda\alpha$ is bounded by $\gamma = \left(2^{\mathrm{card}(\alpha)}\right)^+$. Hence $\|\psi\| < \gamma$ and

$$2^{\aleph_\alpha} = \mathrm{card}(\mathcal{P}(\kappa)) \leqslant \aleph_{\alpha+\|\psi\|} < \aleph_{\alpha+\gamma} = \aleph_\gamma.$$

□

If, e.g., $\aleph_{\omega_1}$ is a strong limit cardinal then by this theorem

$$2^{\aleph_{\omega_1}} < \aleph_{(2^{\aleph_1})^+} < \aleph_{\aleph_{\omega_1}}.$$

So the continuum function at singular cardinals can be influenced by the behaviour below that cardinal. In particular instances the bounds for the continuum function can be improved. With considerably more effort one can also deal with singular cardinals of *countable* cofinality and prove, e.g.: if $\aleph_\omega$ is a strong limit cardinal then (Shelah)

$$2^{\aleph_\omega} < \aleph_{\aleph_4}.$$

## 20 Measurable cardinals

The results of the previous section used filters and ideals to express that certain sets are large or small respectively. There are also intermediate notions of size: a set $X$ is of "positive measure" if it is not in the ideal under consideration. One may imagine that the measure of $X$ is some positive real number. This poses the question, which kinds of "measures" do or can exist. Ideally every set should be given some non-negative number as a measure.

This approach is also motivated by the classical theory of Lebesgue measure on the real line and related spaces. Recall that the 1-dimensional Lebesgue measure on $\mathbb{R}$ is a function $l$: $\Omega \to \mathbb{R} \cup \{\infty\}$ taking values in the extended real line with the properties:

a) $\Omega \subseteq \mathcal{P}(\mathbb{R})$ contains all intervals and is closed under complements and countable unions;

b) $l([0,1]) = 1$;

c) $l$ is *countably additive* (*$\sigma$-additive*): if $\{X_i \mid i < \omega\} \subseteq \Omega$ is a pairwise disjoint family then

$$l\left(\bigcup_{i<\omega} X_i\right) = \sum_{i<\omega} l(X_i);$$

d) $l$ is *translation invariant*: if $X \in \Omega$ and $d \in \mathbb{R}$ then $X + d = \{x + d \mid x \in X\} \in \Omega$ and

$$l(X + d) = l(X).$$

**Theorem 163.** *There is a set $Z \subseteq [0,1]$ which is not Lebesgue-measurable, i.e., $Z \notin \Omega$.*

**Proof.** Let

$$A = \{\mathbb{Q} + d \mid d \in \mathbb{R}\}.$$

(1) $A$ consists of pairwise disjoint nonempty sets which intersect the interval $[0, 1)$.
*Proof*. Assume that $x \in (\mathbb{Q} + d) \cap (\mathbb{Q} + e)$. Take rational numbers $r, s \in \mathbb{Q}$ such that

$$x = r + d = s + e.$$

Then $d = (s - r) + e \in \mathbb{Q} + e$ and

$$\mathbb{Q} + d = \{t + d \mid t \in \mathbb{Q}\} = \{t + (s - r) + e \mid t \in \mathbb{Q}\} \subseteq \mathbb{Q} + e.$$

Similarly $\mathbb{Q} + e \subseteq \mathbb{Q} + d$ and so $\mathbb{Q} + e = \mathbb{Q} + d$.

Consider $d \in \mathbb{R}$. Take an integer $z \in \mathbb{Z}$ such that $z \leqslant d < z + 1$. Then

$$-z + d \in (\mathbb{Q} + d) \cap [0, 1)$$

*qed*(1)

By the axiom of choice let $Z$ be a choice set for the set

$$\{(\mathbb{Q} + d) \cap [0, 1) \mid d \in \mathbb{R}\}.$$

(2) If $q, r \in \mathbb{Q}$ and $q \neq r$ then $(Z + q) \cap (Z + r) = \emptyset$.
*Proof*. Assume not, and take $z_0, z_1 \in Z$ such that $z_0 + q = z_1 + r$. Then $z_0 \in \mathbb{Q} + z_0$ and $z_1 \in \mathbb{Q} + z_0$. Since $Z$ is a choice set, $z_0 = z_1$. But then $q = r$. Contradiction. *qed*(2)
(3) $[0, 1] \subseteq \bigcup_{q \in [-1,1] \cap \mathbb{Q}} Z + q$.
*Proof*. Let $d \in [0, 1]$. Let $z \in Z \cap (\mathbb{Q} + d) \cap [0, 1)$. Take $q \in \mathbb{Q}$ such that $z = q + d$. Then $d = z + (-q)$ where $|q| \leqslant 1$. *qed*(3)

Assume now that $Z \in \Omega$. Since $Z \subseteq [0, 1]$ we have $l(Z) \leqslant 1$.
*Case 1*: $l(Z) = 0$. Then

$$1 = l([0, 1]) \leqslant l\left(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} Z + q\right) = \sum_{q \in [-1,1] \cap \mathbb{Q}} l(Z + q) = \sum_{q \in [-1,1] \cap \mathbb{Q}} l(Z) = \sum_{q \in [-1,1] \cap \mathbb{Q}} 0 = 0,$$

contradiction.
*Case 2*: $l(Z) = \varepsilon > 0$. Then

$$l\left(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} Z + q\right) \leqslant l([0, 2]) = 2$$

but on the other hand

$$l\left(\bigcup_{q \in [-1,1] \cap \mathbb{Q}} Z + q\right) = \sum_{q \in [-1,1] \cap \mathbb{Q}} l(Z + q) = \sum_{q \in [-1,1] \cap \mathbb{Q}} l(Z) = \sum_{q \in [-1,1] \cap \mathbb{Q}} \varepsilon = \infty.$$

□

We shall now consider measures which are defined on *all* subsets of a given set, but we do not require a geometric structure on the set and in particular no translation invariance. We also restrict our consideration to *finite* measures.

**Definition 164.** *A* measure *on a set $X$ is a function $\mu: \mathcal{P}(X) \to [0, 1]$ such that*

- *a)* $\mu(\emptyset) = 0$ *and* $\mu(X) = 1$ *;*
- *b)* $l$ *is* countably additive *(*$\sigma$-additive*): if* $\{X_i \mid i < \omega\} \subseteq \mathcal{P}(X)$ *is a pairwise disjoint family then*

$$\mu\left(\bigcup_{i<\omega} X_i\right) = \sum_{i<\omega} \mu(X_i).$$

$\mu$ *is called* non-trivial *if* $\mu(\{x\}) = 0$ *for every* $x \in X$. $\mu$ *is* 2-valued *if* $\operatorname{ran}(\mu) = \{0, 1\}$*, otherwise* $\mu$ *is* real-valued.

Note that if $f: \kappa \leftrightarrow X$ is a bijection then a measure $\mu$ on $X$ immediately induces a measure $\mu'$ on $\kappa$ by

$$\mu'(A) = \mu(f[A]).$$

So we can focus our attention on measures on *cardinals*.

**Lemma 165.** *Every 2-valued measure $\mu$ on $\mathbb{R}$ is trivial, i.e., there is some $x\in\mathbb{R}$ such that*

$$\mu(A)=1 \textit{ iff } x_0\in A.$$

**Proof.** Identify $\mathbb{R}$ with ${}^{\omega}2$. Define $x_0{:}\,\omega\to 2$ by

$$x_0(n)=1 \text{ iff } \mu(\{x\in {}^{\omega}2\,|\,x(n)=1\})=1.$$

By the $\sigma$-additivity of $\mu$,

$$\mu({}^{\omega}2\setminus\{x_0\})=\mu\left(\bigcup_{n<\omega}\{x\in {}^{\omega}2\,|\,x(n)\neq x_0(n)\}\right)\leqslant\sum_{n<\omega}\mu(\{x\in {}^{\omega}2\,|\,x(n)\neq x_0(n)\})=\sum_{n<\omega}0=0.$$

So $\mu(\{x_0\})=1$. □

Assume that $\kappa$ is the smallest cardinal which has a non-trivial measure $\mu$. A set $A\subseteq\kappa$ with $\mu(A)>0$ *splits* if there is a *partition* $A_1,A_2\subseteq A$ such that $A_1\cup A_2=A$, $A_1\cap A_2=\emptyset$, $0<\mu(A_1)<\mu(A)$ and $0<\mu(A_2)<\mu(A)$.

*Case1*. There is a set $A_0\subseteq\kappa$ with $\mu(A_0)>0$ which does *not* split.

Then define $\nu{:}\,\mathcal{P}(A_0)\to 2$ by

$$\nu(A)=1 \text{ iff } \mu(A)=\mu(A_0).$$

(1) $\nu$ is a 2-valued non-trivial measure on $A_0$.

*Proof*. We have to check $\sigma$-additivity. Let $\{X_i\,|\,i<\omega\}\subseteq\mathcal{P}(A_0)$ be a pairwise disjoint family. By the $\sigma$-additivity of $\mu$

$$\mu\left(\bigcup_{i<\omega}X_i\right)=\sum_{i<\omega}\mu(X_i).$$

Then

$$\nu\left(\bigcup_{i<\omega}X_i\right)=1 \text{ iff } \mu\left(\bigcup_{i<\omega}X_i\right)=\mu(A_0) \text{ iff } \exists i<\omega\ \mu(X_i)=\mu(A_0) \text{ iff } \exists i<\omega\ \nu(X_i)=1.$$

Thus

$$\nu\left(\bigcup_{i<\omega}X_i\right)=\sum_{i<\omega}\nu(X_i).$$

*qed*(1)

By the minimality of $\kappa$ we have $\operatorname{card}(A_0)=\kappa$.

*Case 2*. Every set $A\subseteq\kappa$ with $\mu(A)>0$ splits. We first show that indeed $A$ splits into relatively large subsets:

(2) Every set $A\subseteq\kappa$ with $\mu(A)>0$ possesses a subset $B\subseteq A$ such that $\frac{1}{3}\,\mu(A)\leqslant\mu(B)\leqslant\frac{2}{3}\,\mu(A)$.

*Proof*. Assume not. Then

$$\delta=\sup\left\{\mu(B)\,|\,B\subseteq A\wedge\mu(B)\leqslant\frac{1}{2}\,\mu(A)\right\}\leqslant\frac{1}{3}\,\mu(A).$$

For $n\in\omega\setminus\{0\}$ choose $B_n\subseteq A$ such that $\delta-\frac{1}{n}<\mu(B_n)\leqslant\delta$.

We show by induction on $n$ that

$$\mu(B_1\cup B_2\cup\ldots\cup B_n)\leqslant\delta.$$

Assume that $\mu(B_1\cup B_2\cup\ldots\cup B_n)\leqslant\delta$.

Assume for a contradiction that $\mu(B_1\cup B_2\cup\ldots\cup B_n\cup B_{n+1})>\delta$. Then

$$\mu(B_1\cup B_2\cup\ldots\cup B_n\cup B_{n+1})\leqslant\mu(B_1\cup B_2\cup\ldots\cup B_n)+\mu(B_{n+1})\leqslant\delta+\delta\leqslant\frac{2}{3}\,\mu(A).$$

By the initial assumption we cannot have $\mu(B_1\cup B_2\cup\ldots\cup B_n\cup B_{n+1})=\frac{2}{3}\,\mu(A)$. Hence

$$\delta<\mu(B_1\cup B_2\cup\ldots\cup B_n\cup B_{n+1})<\frac{2}{3}\,\mu(A)$$

and

$$\mu(A\setminus(B_1\cup B_2\cup\ldots\cup B_n\cup B_{n+1}))>\frac{1}{3}\,\mu(A)\geqslant\delta.$$

But then $B_1 \cup B_2 \cup \ldots \cup B_n \cup B_{n+1}$ or its relative complement $A \setminus (B_1 \cup B_2 \cup \ldots \cup B_n \cup B_{n+1})$ would be a counterexample to the definition of $\delta$. Thus $\mu(B_1 \cup B_2 \cup \ldots \cup B_n \cup B_{n+1}) \leqslant \delta$. The $\sigma$-additivity of $\mu$ implies

$$\begin{aligned}
\mu\Big(\bigcup_{1\leqslant n<\omega} B_n\Big) &= \mu\Big(\bigcup_{1\leqslant n<\omega} (B_n \setminus (B_1 \cup \ldots \cup B_{n-1}))\Big)\\
&= \sum_{1\leqslant n<\omega} \mu(B_n \setminus (B_1 \cup \ldots \cup B_{n-1}))\\
&= \lim_{m\to\infty} \sum_{n=1}^{m} \mu(B_n \setminus (B_1 \cup \ldots \cup B_{n-1}))\\
&= \lim_{m\to\infty} \mu\Big(\bigcup_{n=1}^{m} (B_n \setminus (B_1 \cup \ldots \cup B_{n-1}))\Big)\\
&= \lim_{m\to\infty} \mu(B_1 \cup \ldots \cup B_m)\\
&= \delta.
\end{aligned}$$

Set $B_* = \bigcup_{1\leqslant n<\omega} B_n$. Then $\mu(B_*) = \delta$. $A \setminus B_*$ splits, so take a partition $A \setminus B_* = C \dot{\cup} D$ such that $0 < \mu(C) \leqslant \mu(D) < \mu(A \setminus B_*) = \mu(A) - \delta$. By the initial assumption we have $\mu(C) < \frac{1}{3}\mu(A)$ or $\mu(C) > \frac{2}{3}\mu(A)$, and $\mu(D) < \frac{1}{3}\mu(A)$ or $\mu(D) > \frac{2}{3}\mu(A)$.

If $\mu(D) < \frac{1}{3}\mu(A)$ then

$$\mu(A) = \mu(B_* \cup C \cup D) = \mu(B_*) + \mu(C) + \mu(D) < \frac{1}{3}\mu(A) + \frac{1}{3}\mu(A) + \frac{1}{3}\mu(A) = \mu(A),$$

contradiction. Hence $\mu(C) < \frac{1}{3}\mu(A)$ and $\mu(D) > \frac{2}{3}\mu(A)$. But then

$$\delta < \mu(B_* \cup C) = \mu(A \setminus D) = \mu(A) - \mu(D) < \mu(A) - \frac{2}{3}\mu(A) < \frac{1}{2}\mu(A),$$

contradicting the definition of $\delta$. *qed*(2)

Recall the binary tree

$${}^{<\omega}2 = \{s \mid \exists n < \omega \ s: n \to 2\}.$$

We construct a *binary splitting* $A: {}^{<\omega}2 \to \mathcal{P}(\kappa)$ of the underlying set $\kappa$ by recursion on the length of the binary sequences. Put $A(\emptyset) = A_0 = \kappa$. If $A(s) = A_s \subseteq \kappa$ is constructed, use (2) to choose a splitting $A_s = A_{s0} \dot{\cup} A_{s1}$ of $A_s$ such that $\frac{1}{3}\mu(A_s) \leqslant \mu(A_{s0}) \leqslant \mu(A_{s1}) \leqslant \frac{2}{3}\mu(A_s)$.

We shall pull the measure $\mu$ back to a measure $\nu$ on the reals. For $X \subseteq \mathbb{R}$ define

$$X' = \bigcup_{x\in X} \bigcap_{n<\omega} A_{x\upharpoonright n}.$$

Define $\nu: \mathcal{P}({}^{\omega}2) \to [0,1]$ by

$$\nu(X) = \mu(X').$$

First we show that the assignment $X \mapsto X'$ preserves some set theoretic properties.
(3) $\emptyset' = \emptyset$.
(4) $({}^{\omega}2)' = \kappa$.
*Proof*. Let $\alpha \in \kappa$. Define $x: \omega \to 2$ recursively by

$$x(n) = 1 \text{ iff } \alpha \in A_{(x\upharpoonright n)1}.$$

Then $\alpha \in \bigcap_{n<\omega} A_{x\upharpoonright n}$ and

$$\kappa = \bigcup_{x\in {}^{\omega}2} \bigcap_{n<\omega} A_{x\upharpoonright n}.$$

*qed*(4)
(5) $(X \cap Y)' = X' \cap Y'$.
*Proof*. Let $\alpha \in (X \cap Y)'$. Take $x \in X \cap Y$ such that $\alpha \in \bigcap_{n<\omega} A_{x\upharpoonright n}$. Then

$$\alpha \in \bigcup_{x\in X} \bigcap_{n<\omega} A_{x\upharpoonright n} = X'$$

and also $\alpha \in Y'$.

Conversely consider $\alpha \in X' \cap Y'$. Take $x \in X$ such that $\alpha \in \bigcap_{n<\omega} A_{x \upharpoonright n}$ and take $y \in Y$ such that $\alpha \in \bigcap_{n<\omega} A_{y \upharpoonright n}$. Assume for a contradiction that $x \neq y$. Take $n \in \omega$ such that $x \upharpoonright n = y \upharpoonright n$ and $x(n) \neq y(n)$. Then $\alpha \in A_{x \upharpoonright (n+1)} \cap A_{y \upharpoonright (n+1)}$ although $A_{x \upharpoonright (n+1)} \cap A_{y \upharpoonright (n+1)} = \emptyset$ by construction.

Thus $x = y \in X \cap Y$ and

$$\alpha \in \bigcup_{x \in X \cap Y} \bigcap_{n<\omega} A_{x \upharpoonright n} = (X \cap Y)'.$$

*qed*(5)

(6) $\left(\bigcup_{i \in I} X_i\right)' = \bigcup_{i \in I} X_i'$.

*Proof*.

$$\begin{aligned} \left(\bigcup_{i \in I} X_i\right)' &= \bigcup_{x \in \bigcup_{i \in I} X_i} \bigcap_{n<\omega} A_{x \upharpoonright n} \\ &= \bigcup_{i \in I} \bigcup_{x \in X_i} \bigcap_{n<\omega} A_{x \upharpoonright n} \\ &= \bigcup_{i \in I} X_i'. \end{aligned}$$

*qed*(6)

(7) $\nu$ is a non-trivial measure on ${}^{\omega}2$.

*Proof*. $\nu(\emptyset) = \mu(\emptyset') = \mu(\emptyset) = 0$ and $\nu({}^{\omega}2) = \mu(({}^{\omega}2)') = \mu(\kappa) = 1$.

To check $\sigma$-additivity consider a pairwise disjoint family $\{X_i \mid i < \omega\} \subseteq \mathcal{P}({}^{\omega}2)$. Then $\{X_i' \mid i < \omega\} \subseteq \mathcal{P}(\kappa)$ is pairwise disjoint by (5) and (3). By the $\sigma$-additivity of $\mu$ and by (6),

$$\nu\left(\bigcup_{i<\omega} X_i\right) = \mu\left(\bigcup_{i<\omega} X_i'\right) = \sum_{i<\omega} \mu(X_i') = \sum_{i<\omega} \nu(X_i).$$

To check non-triviality consider $x \in {}^{\omega}2$. Then

$$\{x\}' = \bigcap_{n<\omega} A_{x \upharpoonright n}.$$

One can show inductively that

$$\mu(A_{x \upharpoonright n}) \leqslant \left(\frac{2}{3}\right)^n$$

for all $n < \omega$. Thus

$$\mu(\{x\}') \leqslant \mu(A_{x \upharpoonright n}) \leqslant \left(\frac{2}{3}\right)^n$$

for all $n$ and so $\mu(\{x\}') = 0$. This implies

$$\nu(\{x\}) = \mu(\{x\}') = 0.$$

*qed*(7)

Since $\kappa$ was assumed to be minimal carrying a non-trivial measure, we have $\kappa \leqslant 2^{\aleph_0}$ in *Case 2*.

In both cases we can show a strong additivity property:

(8) The measure $\mu$ on $\kappa$ is *$\kappa$-additive*, i.e., for every pairwise disjoint family $\{X_i \mid i < \gamma\} \subseteq \mathcal{P}(\kappa)$ with $\gamma < \kappa$ we have

$$\mu\left(\bigcup_{i<\gamma} X_i\right) = \sum_{i<\gamma} \mu(X_i),$$

where the right hand side is defined as

$$\sum_{i<\gamma} \mu(X_i) = \sup\left\{\sum_{i \in I_0} \mu(X_i) \mid I_0 \text{ is a finite subset of } \gamma\right\}.$$

*Proof*. Assume that $\mu$ is not $\kappa$-additive and let $\gamma < \kappa$ be least such that there is a family $\{X_i \mid i < \gamma\} \subseteq \mathcal{P}(\kappa)$ with

$$\mu\left(\bigcup_{i<\gamma} X_i\right) \neq \sum_{i<\gamma} \mu(X_i).$$

Then $\gamma$ is an uncountable cardinal and

$$\mu\left(\bigcup_{i<\gamma} X_i\right) > \sum_{i<\gamma} \mu(X_i)\,.$$

Let

$$J = \{i < \gamma \mid \mu(X_i) > 0\} \subseteq \gamma\,.$$

We claim that $J$ is at most countable. If $J$ were uncountable, there must be some rational number $\frac{1}{n}$ such that $\mu(X_i) > \frac{1}{n}$ for uncountably many $i \in J$. But then

$$\mu\left(\bigcup_{i<\gamma} X_i\right) = \infty\,,$$

contradiction.

The $\sigma$-additivity of $\mu$ entails

$$\begin{aligned}
\mu\left(\bigcup_{i\in\gamma\setminus J} X_i\right) &= \mu\left(\bigcup_{i<\gamma} X_i\right) - \mu\left(\bigcup_{i\in J} X_i\right)\\
&> \sum_{i<\gamma} \mu(X_i) - \sum_{i\in J} \mu(X_i)\\
&= \sum_{i<\gamma} \mu(X_i) - \sum_{i<\gamma} \mu(X_i)\\
&= 0.
\end{aligned}$$

So we may assume that *all* elements of the disjoint family $\{X_i \mid i < \gamma\} \subseteq \mathcal{P}(\kappa)$ have $\mu$-measure $0$.

We shall pull the measure $\mu$ back to a measure $\nu$ on $\gamma$. Let $\mu_0 = \mu\big(\bigcup_{i\in\gamma} X_i\big)$. Define $\nu$: $\mathcal{P}(\gamma) \to [0,1]$ by

$$\nu(Y) = \frac{1}{\mu_0}\,\mu\left(\bigcup_{i\in Y} X_i\right)$$

We show that $\nu$ is a non-trivial measure on $\gamma$.

$\nu(\emptyset) = \frac{1}{\mu_0}\,\mu(\emptyset) = 0$ and $\nu(\gamma) = \frac{1}{\mu_0}\,\mu\big(\bigcup_{i\in\gamma} X_i\big) = 1$.

To check $\sigma$-additivity consider a pairwise disjoint family $\{Y_j \mid j < \omega\} \subseteq \mathcal{P}(\gamma)$. Then

$$\left\{\bigcup_{i\in Y_j} X_i \mid j < \omega\right\} \subseteq \mathcal{P}(\kappa)$$

is pairwise disjoint. By the $\sigma$-additivity of $\mu$

$$\begin{aligned}
\nu\left(\bigcup_{j<\omega} Y_j\right) &= \frac{1}{\mu_0}\,\mu\left(\bigcup_{i\in\bigcup_{j<\omega} Y_j} X_i\right)\\
&= \frac{1}{\mu_0}\,\mu\left(\bigcup_{j<\omega}\bigcup_{i\in Y_j} X_i\right)\\
&= \frac{1}{\mu_0}\sum_{j<\omega} \mu\left(\bigcup_{i\in Y_j} X_i\right)\\
&= \sum_{j<\omega} \frac{1}{\mu_0}\,\mu\left(\bigcup_{i\in Y_j} X_i\right)\\
&= \sum_{j<\omega} \nu(Y_j)\,.
\end{aligned}$$

Finally, $\nu$ is non-trivial since for every $i < \gamma$

$$\nu(\{i\}) = \frac{1}{\mu_0}\,\mu\left(\bigcup_{j\in\{i\}} X_j\right) = \frac{1}{\mu_0}\,\mu(X_i) = 0\,.$$

But the existence of $\nu$ contradicts the minimality of $\kappa$. *qed*(8)

We can now draw conclusions from the previous arguments.

**Definition 166.** *A cardinal $\kappa$ is* measurable *if $\kappa$ is uncountable and possesses a 2-valued $\kappa$-additive non-trivial measure. A cardinal $\kappa$ is* real-valued measurable *is $\kappa$ is uncountable and possesses a real-valued $\kappa$-additive non-trivial measure.*

**Theorem 167.** *Let $\kappa$ be minimal such that $\kappa$ carries a non-trivial measure. Then either $\kappa$ is a measurable cardinal, or $\kappa \leqslant 2^{\aleph_0}$ and $\kappa$ is a real-valued measurable cardinal.*

**Proof.** If we are in *Case 1* above, then $\kappa$ is measurable. In *Case 2*, $\kappa \leqslant 2^{\aleph_0}$. By Lemma 165 there is no 2-valued non-trivial measure on $\kappa$. Hence $\kappa$ is real-valued measurable. □

**Lemma 168.** *Let $\kappa$ be measurable or real-valued measurable. Then $\kappa$ is regular.*

**Proof.** Let $\mu$ be a non-trivial $\kappa$-additive measure on $\kappa$ . Assume for a contradiction that $\mathrm{cof}(\kappa) = \gamma < \kappa$. Let $(\kappa_i \mid i < \gamma) \subseteq \kappa$ be cofinal in $\kappa$. For $i < \gamma$ be have

$$\mu(\kappa_i) = \mu\left(\bigcup_{\alpha < \kappa_i} \{\alpha\}\right) = \sum_{\alpha < \kappa_i} \mu(\{\alpha\}) = \sum_{\alpha < \kappa_i} 0 = 0,$$

using non-triviality and $\kappa$-additivity. Similarly

$$\mu(\kappa) = \mu\left(\bigcup_{i < \gamma} \kappa_i\right) \leqslant \sum_{i < \gamma} \mu(\kappa_i) = \sum_{i < \gamma} 0 = 0,$$

contradiction. □

**Lemma 169.** *Let $\kappa$ be a measurable cardinal. Then $\kappa$ is strongly inaccessible.*

**Proof.** Let $\mu$ be a non-trivial $\kappa$-additive 2-valued measure on $\kappa$ . Assume for a contradiction that $\lambda < \kappa$ but $2^\lambda \geqslant \kappa$. We may assume that $\mu$ is a measure on ${}^\lambda 2$. Define $x_0 \colon \lambda \to 2$ by

$$x_0(i) = 1 \text{ iff } \mu(\{y \in {}^\lambda 2 \mid y(i) = 1\}) = 1.$$

The $\kappa$-additivity of $\mu$ implies

$$\begin{aligned} \mu({}^\lambda 2 \setminus \{x_0\}) &= \mu\left(\bigcup_{i < \lambda} \{y \in {}^\lambda 2 \mid y(i) = 1 - x_0(i)\}\right) \\ &\leqslant \sum_{i < \lambda} \mu(\{y \in {}^\lambda 2 \mid y(i) = 1 - x_0(i)\}) \\ &= \sum_{i < \lambda} 0 \\ &= 0. \end{aligned}$$

But then $\mu(\{x_0\}) = 1$ which contradicts the non-triviality of $\mu$. □