Free groups and automorphism groups of infinite fields

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Theorem (De Bruijn, 1957)

If λ is an infinite cardinal, then the group $\mathrm{Fin}(\lambda^+)$ consisting of all finite permutations of λ^+ cannot be embedded into the group $\mathrm{Sym}(\lambda)$ consisting of all permutations of λ.

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In particular, every group of cardinality at most λ is isomorphic to the automorphism group of a field of cardinality λ.

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The above question was first asked by David Evans for the case $\lambda = \aleph_0$. Results of Winfried Just, Saharon Shelah and Simon Thomas motivate its generalization to uncountable cardinalities.

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Let λ be a cardinal with $\lambda = \lambda^{\aleph_0}$ and p be either 0 or a prime number. Then there is a field K of characteristic p and cardinality λ whose automorphism group is a free group of cardinality 2^λ.

In particular, there always is a field K whose automorphism group is a free group of cardinality greater than the cardinality of K.

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- *There is a field of cardinality λ whose automorphism group is a free group of cardinality greater than λ.*
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The methods developed in the proof of the above theorem also allow us to derive several other interesting statements.

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Let $\mathcal{L}$ be a countable first-order language and $\mathcal{M}$ be an $\mathcal{L}$-model. Then $\mathrm{Aut}(\mathcal{M})$ is not an uncountable free abelian group.

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Finally, our constructions show that, if it is consistent to have a cardinal λ of uncountable cofinality such that there is no field of cardinality λ whose automorphism group is a free group of cardinality greater than λ, then it is necessary to use large cardinals to construct a model of this statement.

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Theorem (L. & Shelah)

Let λ be a singular cardinal of uncountable cofinality such that there is no field of cardinality λ whose automorphism group is a free group of cardinality greater than λ. Then there is an inner model with a Woodin cardinal.

Representing inverse limits as automorphism groups of fields

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Given a directed set $\mathbb{D} = \langle D, \leq_{\mathbb{D}} \rangle$, a pair

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an *inverse system of groups over* $\mathbb{D}$ if the following statements hold for all $p, q, r \in D$ with $p \leq_{\mathbb{D}} q \leq_{\mathbb{D}} r$.

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Given such an inverse system $\mathbb{I}$, we call the subgroup

$$G_{\mathbb{I}} = \{(g_p)_{p \in D} \mid h_{p,q}(g_q) = g_p \textit{ for all } p, q \in D \textit{ with } p \leq_{\mathbb{D}} q\}$$

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Free groups as inverse limits

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Good inverse systems of sets

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Define $\mathbb{D} = \langle [\lambda]^{\aleph_0}, \subseteq \rangle$. Given $u, v \in [\lambda]^{\aleph_0}$ with $u \subseteq v$, set $A_u = {}^u 2$ and define $f_{u,v} : A_v \longrightarrow A_u$ by $f_{u,v}(s) = s \restriction u$ for all $s \in {}^v 2$.

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Thank you for listening!