On the Continuum Function in Zermelo-Fraenkel Set Theory

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A) Regular Cardinals

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Let V be a ground model of $ZFC + GCH$ with a function F whose domain consists of regular cardinals and whose range consists of cardinals, such that for all $\kappa, \lambda \in \mathrm{dom}\, F$:

- $F(\kappa) > \kappa$
- $\kappa \leq \lambda \rightarrow F(\kappa) \leq F(\lambda)$
- $\mathrm{cf}\, F(\kappa) > \kappa$.

Then there is a cardinal-preserving model $V[G] \supseteq V$ of the theory

$$ZFC \; + \; \forall \kappa \in \mathrm{dom}\, F \;\; 2^{\kappa} = F(\kappa).$$

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If $\aleph_\omega$ is a strong limit cardinal, then $2^{\aleph_\omega} < \aleph_{\omega_4}$.

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Theorem (*Gitik-Koepke*, 2012)

Let V be a ground model of $ZFC + GCH$ and λ some cardinal in V. There is a cardinal-preserving model $N \supseteq V$ of the theory

$$ZF + \forall n < \omega \;\; \theta(\aleph_n) = \aleph_n^{++} + \theta(\aleph_\omega) \geq \lambda^+.$$

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- In N, there exists a surjection $f : \wp(\aleph_\omega) \to \lambda$: Set $f(X) := i$ whenever $X \in \widetilde{G}_i$.

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There is a cardinal-preserving extension $N \supseteq V$ with $N \vDash ZF$ such that $\forall \eta < \gamma \;\; \theta^N(\kappa_\eta) = \alpha_\eta^+$, i.e. for all $\eta < \gamma$, there exists in N a surjective function $f : \wp(\kappa_\eta) \rightarrow \alpha_\eta$, but no surjective function $f : \wp(\kappa_\eta) \rightarrow \alpha_\eta^+$.

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The forcings P^η and P^η_* preserve cardinals and the *GCH*.

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  $$\forall \kappa_{\nu,j} < \kappa_\eta \quad |a_i^\eta \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1})| \leq 1$$
- $\forall (\eta, i) \neq (\eta', i')$ $a_i^\eta \cap a_{i'}^{\eta'} = \varnothing$ (**Independence Property**).

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- $\forall(\eta, i) \ \ q_i^\eta \supseteq p_i^\eta, \ b_i^\eta \supseteq a_i^\eta$
- **Linking Property:** Let $\zeta \in \operatorname{dom}\left(q_i^\eta \setminus p_i^\eta\right)$, $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $a_i^\eta \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) = \{\xi\}$.

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  Then $q_i^\eta(\zeta) = q_*(\xi, \zeta)$.

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For all (η, i), let $F_G(\eta, i) := \widetilde{G}_i^\eta := \{X \subseteq \kappa_\eta \mid X \simeq_\eta G_i^\eta\}$, and

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$i \neq i' \rightarrow G_i^\eta \not\simeq_\eta G_{i'}^\eta$

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The Symmetric Submodel

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Corollary 6

The Symmetric Submodel

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where for all $\zeta \in [\kappa_{\nu,j}, \kappa_{\nu,j+1})$ with $a_0 \cap [\kappa_{\nu,j}, \kappa_{\nu,j+1}) =: \{\xi\}$, we define $G_*(a_0)(\zeta) := G_*(\xi, \zeta)$.

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A straightforward generalization of our construction to sequences $(\kappa_\eta \mid \eta \in \mathrm{Ord})$ and $(\alpha_\eta \mid \eta \in \mathrm{Ord})$ of class length does not give a ZF-model N.

Theorem C

Let V be a ground model of $ZFC + GCH$ with sequences $(\kappa_\eta \mid \eta \in \mathrm{Ord})$ and $(\alpha_\eta \mid \eta \in \mathrm{Ord})$, where $(\kappa_\eta \mid \eta \in \mathrm{Ord})$ is strictly increasing and closed, and $(\alpha_\eta \mid \eta \in \mathrm{Ord})$ is increasing with $\alpha_\eta \geq \kappa_{\eta+1}^{++}$ for all $\eta \in \mathrm{Ord}$.

There is a cardinal-preserving extension $N \supseteq V$ with $N \vDash ZF$ such that $\theta^N(\kappa_{\eta+1}) = \alpha_\eta$ for all $\eta \in \mathrm{Ord}$.

Thank you!