THE INFLUENCE OF CLOSED MAXIMALITY PRINCIPLES ON $\mathbf{\Sigma}^1_1$-SUBSETS OF GENERALIZED BAIRE SPACES

PHILIPP LÜCKE

Abstract. Let κ be an infinite cardinal. The *generalized Baire space* of κ is the set ${}^\kappa\kappa$ of all functions $f : \kappa \to \kappa$ equipped with the topology whose basic open sets are of the form $U_s = \{f \in {}^\kappa\kappa \mid s \subseteq f\}$ for some partial function $s : \kappa \xrightarrow{part} \kappa$ of cardinality less than $\mathrm{cof}(\kappa)$. A subset of $({}^\kappa\kappa)^n$ is a $\mathbf{\Sigma}^1_1$-subset if it is the projection of a closed subset of $({}^\kappa\kappa)^{n+1}$ and it is a $\mathbf{\Pi}^1_1$-subset if its complement is a $\mathbf{\Sigma}^1_1$-subset.

It is a well-known phenomenon that many basic and interesting questions about $\mathbf{\Sigma}^1_1$- and $\mathbf{\Pi}^1_1$-subsets of generalized Baire spaces of regular uncountable cardinals κ with $\kappa = \kappa^{<\kappa}$ are independent from the axioms of set theory plus large cardinal axioms. We will consider the following examples of such questions.

(1) We call a well-ordering $\langle A, \prec\rangle$ a $\mathbf{\Sigma}^1_1$-*well-ordering of a subset of* ${}^\kappa\kappa$ if $\prec$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$. Is the least upper bound of the order-types of $\mathbf{\Sigma}^1_1$-well-orderings of subsets of ${}^\kappa\kappa$ greater than 2^κ?

(2) Given a regular cardinal $\lambda < \kappa$, we define $\mathsf{Club}(S^\kappa_\lambda)$ to be the set of all characteristic functions of subsets $X \subseteq \kappa$ such that $S^\kappa_\lambda \cap (\kappa \setminus X)$ is not a stationary subset of κ, where $S^\kappa_\lambda = \{\alpha < \kappa \mid \mathrm{cof}(\alpha) = \lambda\}$. Clearly, $\mathsf{Club}(S^\kappa_\lambda)$ is a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa$. Is $\mathsf{Club}(S^\kappa_\lambda)$ also a $\mathbf{\Pi}^1_1$-subset?

(3) Let $\mathcal{TO}_\kappa$ be the class of all trees of cardinality and height κ without cofinal branches and let $\preceq$ denote the ordering of trees in $\mathcal{TO}_\kappa$ under order-preserving embeddability. We can identify $\mathcal{TO}_\kappa$ with a $\mathbf{\Pi}^1_1$-subset of ${}^\kappa\kappa$ and $\preceq$ is equal to the intersection of a $\mathbf{\Sigma}^1_1$-subset of ${}^\kappa\kappa \times {}^\kappa\kappa$ with $\mathcal{TO}_\kappa \times \mathcal{TO}_\kappa$. What is the bounding and the dominating number of the resulting partial order $\langle \mathcal{TO}_\kappa, \preceq\rangle$?

In my talk, I want to introduce a class of forcing axioms that settle the above questions. These axioms are variants of the *maximality principle* introduced by Jonathan Stavi and Jouko Väänänen and rediscovered by Joel Hamkins.

Mathematisches Institut, Rheinische Friedrich-Wilhelms-Universität Bonn, Endenicher Allee 60, 53115 Bonn, Germany

E-mail address: `pluecke@math.uni-bonn.de`